Reinhard Winkler: What is mathematics? – A subjective approach

1. One question, many answers

Questions such as ‘What is man?’ or, particularly, ‘What is the world?’ elicit such a plethora of possible answers as to defy any practicable framework. The same applies, if to a lesser extent, to the question ‘What is mathematics?’ – at least if the answer we are looking for is to meet the criteria of both precision and comprehensiveness. The number of aspects one can focus on is simply overwhelming.

If, on a more modest level, one were to ask ‘What areas does mathematics consist of?’, this would elicit as an answer the approximately 4,000 special areas listed by the American Mathematical Society in its internationally accepted classification of mathematics. There is no doubt that such a detailed classification is both impressive and indispensable for scientific purposes, yet in terms of a question aiming at the nature of mathematics it does not contribute much that is either exciting or helpful.

Another approach, which might appeal particularly to non-mathematicians, would be the question ‘What role does mathematics play?’. There is much to be said about the practical applicability of a grounding in mathematics, about the desirability and indeed the indispensability of mathematical idioms and concepts for our understanding of nature, economy etc. and about the countless links that tie mathematics to philosophy, the humanities and the arts. Yet, however many contexts we might add to this list in which mathematics appears as a tool, we would not be learning much about its nature per se.

More promising is a methodological approach based on the question, ‘What is admissible as a building block for the system of mathematics?’ This will yield an astonishingly precise answer: anything that can (in principle) be described in terms of first order predicate logic and set theory and can be dealt with within the framework of a deductive methodology. From this admittedly formalistic perspective deep insights of modern mathematics come into view. If I wanted to give instances for this claim, so much space would have to be given over to results of mathematical logic (in particular to Kurt Gödel’s Completeness and Incompleteness Theorems) that aspects less frequently dealt with in literature, which appear to me to be at least as attractive in the present context, would have to be neglected.

For me the best way to consider the nature is subjectively: ‘What does mathematical activity consist in?’ I believe that the most straightforward way of dealing with this question consists in making one’s personal experience of what it means to be a mathematician accessible to the reader. This will also create an understanding of the fact that there are actually people who are fired by an intense passion for mathematics.

A few distinctions are required first to clear the path for discourse.

2. Three types of mathematics – and some common misunderstandings

The ideas people entertain regarding what happens in mathematicians' heads when they are engaged in practicing their science originate no doubt from their own personal mathematical experiences. For non-mathematicians these will mostly be confined to math classes in school or at university, where mathematics appears wearing the hat of an ancillary science. This type of experience is unfortunately prone to lead to fundamental misunderstandings that give rise to completely mistaken ideas as to what mathematics is all about: it is most emphatically not a machine-translatable aptitude for calculating according to formulae and rigid precepts that do not allow space for individual freedom. The reason for the wide prevalence of this travestied image of mathematics is arguably the fact that exams cast in this ostensibly 'objective' form are easier to implement both for preparation and assessment.

Lack of experience on the part of teachers and examiners will often lead to an aggravation of the misunderstanding.

The insights gained by those who are fortunate enough to become acquainted with the world of mathematical research results are different in kind and much more interesting. They are put in a position to appreciate great individual achievements – mostly in the guise of mathematical proofs – of mathematicians of genius, who managed to solve problems that had resisted the efforts, in many cases, of generation upon generation.

It is in this world and mostly at university level that thousands of researchers are active in contexts that are similar, if usually humbler, to the ones that the mathematical geniuses deal with; they too regard it as their main task in professional terms to discover and prove new theorems. In this there is an additional factor at work, over and above their personal interest, which one may safely say has hypertrophied during the past few decades beyond desirable limits, namely the pressure to get into print. There are many arguments in favour of these mechanisms, yet however much they may spur the ambition of researchers, they also detract from what is even more important.

Even the discovery of new theorems should not be an end in itself in mathematics; it ought to be subservient to making visible wider contexts of the kind that are to be found in the 'realm of ideas'.

We can therefore distinguish three types of mathematical activity:

1. Calculating and carrying out machine-translatable algorithms
2. Devising mathematical proofs, using a logical-deductive method
3. Developing and systematically integrating ideas and concepts and improving our understanding of them

In this hierarchy of activities the first stage is the precondition for the second, sometimes in a humdrum, sometimes in a subtle manner. The second stage in its turn is an indispensable pool of explications for the third and its uses do not often extend far beyond this function.

As becomes apparent from studying truly great mathematicians, the real motivation to get involved with mathematics is the third stage. Only those who have some knowledge of this allure of mathematics at least from hearsay can hope to do justice to the science.

3. Mathematics as experienced from the inside

At its core, mathematical activity and creativity has little to do with devising sophisticated new sequences of operations – even though this too may sometimes lead to unhoped-for breakthroughs. What matters more is the imaginative feat that allows the inner eye to take up a new point of view. Ideas usually appear in the form of novel configurations of well-known mathematical entities bathed in the light of our imagination.

Professional mathematicians are expected to formalize these ideas, i.e. to translate them into scientific concepts and theorems susceptible to proofs. An even greater challenge to their creativity and fantasy consists in traversing a similar distance in the reverse direction in order to make ideas come alive that are encrypted in scientific mathematical language.

In the search for a suitable metaphor one might compare mathematical texts to fossilized skeletons that enable the specialist to infer the entire anatomy and the way of life of creatures who have left no other evidence behind or one might draw, perhaps even more to the point, a comparison with the musical score of, say, a symphony. The composer works from an idea of an acoustic event, which he attempts to trace with the help of musical notation. In this way he is able to pass on a precise description of the event to other musicians. Mathematicians and musicians use symbolic notations to recreate the mathematical or the acoustic ideas they describe.

The latter analogy suffers from the defect that in the case of music it is possible for the listener to enjoy the symphony without being able to read the score. The communication of mathematical ideas without the use of formalistic stepping-stones is much more difficult.

Mathematical sensuality demands great sophistication, and its mastery presupposes a great deal of preparatory work. However, once that imaginative groundwork has been done and ideas have been equipped with quasi-sensual qualities, it will be possible in future to make direct use of these modified ideas in preference to the symbols representing them and to resort to these symbols only for the technical elaboration.

Perhaps it is admissible to extend the analogy between mathematics and art by distinguishing between expressive and everyday mathematics. Mathematical theorems are not appreciated for their logical content alone, but also for the clarity and elegance of their presentation. Mathematicians do not only attempt to extend the reach of existing theorems and to devise variations on them, they strive to present both new and old ideas and concepts with as much persuasive force as possible and to increase their acceptance.

This demanding task belongs primarily to mathematics teachers at all levels of tuition, from primary school to university.

Mathematics is a journey through the realm of ideas, through worlds of the imagination. Nevertheless their description must be undertaken with the utmost clarity and precision in view of the fact that they might be real after all. A case in point is Bernhard Riemann’s achievement of ingeniously placing non-Euclidean geometries, which had only been discovered a short time before, in a startlingly new context. At the time this could have been regarded as an exercise in art for art’s sake remote from any kind of practical applicability. However, less than half a century later Einstein based his description of radically new concepts regarding the nature of time and space in his General Theory of Relativity on precisely these ideas. Only a few years later experiments and observations confirmed that this geometry was more ‘real’ than the Euclidean one that Kant had considered to be inherent to human thought.

The great David Hilbert’s comment when he was told that one of his former pupils had forsaken mathematics for literature makes sense: ‘This is as it should be; he was lacking the fantasy required for mathematics anyway.’

This is also a powerful argument for why aesthetic considerations are so important in mathematics.

Compared to other sciences that are linked to specific aspects of reality, mathematics makes an infinitely richer contribution to our capacity for perceiving abstract connections. Once our perception has been trained, we recognize these connections in concrete everyday situations. Abstraction is in fact a more refined form of visualization. By eliminating the superfluous from known phenomena and by directing our attention to what really matters, abstraction makes the essence of things accessible to us and thereby enlarges our consciousness.

What is required for this process is above all leisure and concentration. Mathematicians tend to be at their most creative when they travel across mathematical landscapes with closed eyes, guided solely by their imagination. Knowledge is relevant of course, but not in the shape of formulae, theorems or isolated facts. It is relevant as appreciation of internal relations, as familiarity with types of mathematical landscape and as the capacity to divine new pathways that will lead to interesting scenarios. It might be a task for brain specialists to investigate whether the switch from formula to idea is to do with different functions of the two brain hemispheres, as the renowned physicist and mathematician Roger Penrose has plausibly suggested in his bestseller ‘The Emperor’s New Mind’.

During the state of highest concentration all sense of time is lost, which is little wonder since the worlds that come into view have all the attributes of eternity; normal articulateness is also lost.

This meditative state is intensely enjoyable and presumably resembles the state of mind of a child at play, who, completely under the spell of his or her toy, cannot even say ‘yes’ or ‘no’ to the parents’ urgent invitation to come down to dinner.

4. Conclusion

The proper stuff of mathematics is ideas and concepts. Mathematicians are called upon to describe these as accurately as possible and to ascertain whether they are categories inherent to the process of thought or whether other options are available. As opposed to empirical sciences that explore the world as it is, mathematics charts the world under the double aspects of necessity and freedom.

This is why mathematics is the least restricted and most universal science. This also accounts for its applicability to many other branches of science.

Mathematics is the highest form of symbiosis between intuition and scientific precision. For it to be taught adequately requires the most holistic form of communication, i.e. communication between individuals in terms of states of mind.

By way of conclusion one might, harking back to Galileo’s famous paradigm for physics, say about mathematics that its methodology is the logically consistent analysis of all possibilities of thought cast into a precise language; its task is to present all possibilities that are conceivable and to approximate the inconceivable ones to the point where they stop being inconceivable. (Translated from the German by Otmar Binder)

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