Homework Assignment 8 - Solution

Hopf algebras - Spring Semester 2018

Exercise 1

Let G be a group.

- Let $N \leq G$ be a subgroup. Show that $k[G]$ is free as a left and right $k[N]$-module.
- If $N \trianglelefteq G$ is a normal subgroup, let $p : G \mapsto G/N$ denote the natural projection and $\pi : k[G] \to k[G/N]$ the induced algebra morphism. Recall that $k[N]^+$ denotes the augmentation ideal, that is the kernel of the counit ϵ. Let $k[G]^{co\, k[G/N]}$ be the space of $k[N]$-coinvariant elements of $k[G]$ with respect to the right $k[G/N]$ - comodule algebra structure given by $(\mathrm{id} \otimes \pi)\Delta$ (see last exercise sheet).

  Show that $\ker \pi = k[G](k[N])^+$ and $k[G]^{\mathrm{co}\, k[G/N]} = k[N]$.

Solution. The left and right $k[N]$-module algebra structures on $k[G]$ are given, respectively for $g \in G, n \in N$, by $e_n e_g = e_{ng}$ and $e_g e_n = e_{gn}$.

The free generators of $k[G]$ as a left $k[N]$-module algebra are, simply, $\{e_{r_i}\}_{i \in I}$, where r_i are representatives of the left cosets of N. The free generators of $k[G]$ as a right $k[N]$-module algebra are, simply, $\{e_{l_i}\}_{i \in I}$, where l_i are representatives of the right cosets of N.

Now suppose that N is a normal subgroup. It is a direct computation to see that $\ker \pi = k[G](k[N])^+$. First, since $\epsilon(e_g) = 1$ for any $g \in G$, we note that $(k[N])^+ = \{\sum_g \in N a_g e_g | \sum_{g \in N} a_g = 0\}$ Indeed, suppose that $v \in k[G](k[N])^+$, so we can write $v = \sum_{g \in G} a_g e_g$ such that $\sum_{g \in hN} a_g = 0$. Hence, it is immediate that

$$v \in \ker \pi \Leftrightarrow \sum_{g \in G} a_g e_{gN} = 0 \Leftrightarrow \sum_{hN \text{ cosets}} \left(\sum_{g \in hN} a_g \right) e_{gN} = 0 \Leftrightarrow v \in k[G](k[N])^+ .$$

To compute $k[G]^{co\, k[G/N]} = \{v \in k[G] | \delta v = v \otimes 1\}$, note that if $v = \sum_{g \in G} a_g e_g$ then

$$\delta v = \sum_{g \in G} a_g (\mathrm{id} \otimes \pi) \circ \Delta e_g = \sum_{g \in G} a_g e_g \otimes e_{gN} ,$$

consequently, as $1 = e_N$,

$$\delta v - v \otimes 1 = \sum_{g \in N \backslash G} e_g \otimes (a_g e_{gN} - a_g e_N) \Rightarrow a_g = 0 \forall g \notin N .$$

This conclues that $v \in k[G]^{co\, k[G/N]} \Leftrightarrow v \in k[N]$, as desired. $\square$

Exercise 2

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a} \subset \mathfrak{g}$ a Lie subalgebra.

- Show that the map $U(\iota) : U(\mathfrak{a}) \to U(\mathfrak{g})$ induced by the inclusion $\iota : \mathfrak{a} \to \mathfrak{g}$ is injective. Show as well that $U(\mathfrak{g})$ is free as a left and right $U(\mathfrak{a})$-module.
- Suppose that $\mathfrak{a}$ is a Lie ideal. Let $p : \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ be the canonical map $\pi : U(\mathfrak{g}) \to U(\mathfrak{g}/\mathfrak{a})$ the induced algebra homomorphism. Let $U(\mathfrak{a})^+$ be the augmentation ideal and let $U(\mathfrak{g})^{\mathrm{co}\,U(\mathfrak{g}/\mathfrak{a})}$ be the space of $U(\mathfrak{g}/\mathfrak{a})$-coinvariant elements of $U(\mathfrak{g})$ with respect to the right $U(\mathfrak{g}/\mathfrak{a})$ - comodule algebra structure given by $(\mathrm{id} \otimes \pi)\Delta$ (see last exercise sheet). Show that $\ker \pi = U(\mathfrak{g})U(\mathfrak{a})^+$ and describe $U(\mathfrak{g})^{\mathrm{co}\,U(\mathfrak{g}/\mathfrak{a})}$.

Proof. That $U(\iota)$ is injective, choose a basis $\{b_i\}_{i\in I}$ of $\mathfrak{a}$ and extend it to a basis $\{b_j\}_{j\in J}$ of $\mathfrak{g}$, with $I \subset J$ totally ordered. Then the map $U(\iota)$ clearly sends the Poincaré-Birkhoff-Witt basis of $U(\mathfrak{a})$ to a subset of the Poincaré-Birkhoff-Witt basis of $U(\mathfrak{a})$, so this is injective.

Since $U(\mathfrak{g})$ is commutative, the left and right $U(\mathfrak{a})$-module structures are the same. Then $U(\mathfrak{g})$ is a free left $U(\mathfrak{a})$-module spanned by $\{\sigma(b_{j_1}) \cdots \sigma(b_{j_s})\}_{\substack{j_k \in J\setminus I \\ j_1 \leq \cdots \leq l_s}}$.

Now, to compute the kernel of π, we see how π acts on the PBW basis of $U(\mathfrak{g})$, and it follows directly. Consider a basis element $\sigma(x_{i_1}) \ldots, \sigma(x_{i_k})$ and identify interchangeably the index sets $\{i_1, \ldots, i_k\} = \{1, \ldots, k\} = [k]$. We have

$$\pi(\sigma(x_{i_1}) \ldots, \sigma(x_{i_k})) = \begin{cases} \sigma(x_{i_1}) \ldots, \sigma(x_{i_k}) & \text{if all } i_j \notin I \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We conclude that $\ker \pi$ is spanned by all basis elements $\sigma(x_{i_1}) \ldots, \sigma(x_{i_k})$ that contain some $i_j \in I$.

On the other hand, since $\epsilon(\sigma(x_{i_1}) \ldots, \sigma(x_{i_k})) = 0$ if $k > 0$, then $U(\mathfrak{a})^+$ is spanned by all basis elements of degree at least one. Consequently, $\ker \pi = U(\mathfrak{g})U(\mathfrak{a})^+$, as desired.

Finally, recall that each $\sigma(x_i)$ is primitive, so, from (1) we have:

$$\begin{aligned}(\mathrm{id} \otimes \pi) \circ \Delta\, \sigma(x_{i_1}) \ldots, \sigma(x_{i_k}) &= (\mathrm{id} \otimes \pi) \left(\sum_{A \subseteq [k]} \prod_{j\in A} \sigma(x_{i_j}) \otimes \prod_{j \notin A} \sigma(x_{i_j}) \right) \\ &= \sum_{I\cap[k] \subseteq A \subseteq [k]} \prod_{j\in A} \sigma(x_{i_j}) \otimes \prod_{j \notin A} \sigma(x_{i_j}) \\ &= \sigma(x_{i_1}) \ldots, \sigma(x_{i_k}) \otimes 1 + \sum_{I\cap[k] \subseteq A \subsetneq [k]} \prod_{j\in A} \sigma(x_{i_j}) \otimes \prod_{j \notin A} \sigma(x_{i_j}) \,.\end{aligned}$$

Hence, the basis elements that are in $U(\mathfrak{g})^{coU(\mathfrak{g}/\mathfrak{a})}$ are precisely the ones where there is no set A such that $I \cap [k] \subseteq A \subsetneq [k]$, so $[k] \subseteq I$. It follows that $U(\mathfrak{a}) \subset U(\mathfrak{g})^{coU(\mathfrak{g}/\mathfrak{a})}$. On the other hand, it is easy to see that if we write

$$v = \sum_{i_1 \leq \cdots \leq i_k} a_{i_1,\ldots,i_k} \sigma(x_{i_1}) \cdots \sigma(x_{i_k}) \in U(\mathfrak{g})^{coU(\mathfrak{g}/\mathfrak{a})} \,,$$

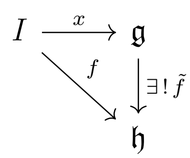
then $\delta v = v \otimes 1$ becomes

$$\sum_{i_1 \leq \cdots \leq i_k} a_{i_1,\ldots,i_k} \sum_{I \cap [k] \subseteq A \subsetneq [k]} \prod_{j \in A} \sigma(x_{i_j}) \otimes \prod_{j \notin A} \sigma(x_{i_j}) = 0 \,.$$

It follows by linear independence that $a_{i_1,\ldots,i_k} = 0$ whenever $\sum_{I \cap [k] \subseteq A \subsetneq [k]} \prod_{j \in A} \sigma(x_{i_j}) \otimes \prod_{j \notin A} \sigma(x_{i_j})$ is non-zero, which is exactly where $[k] \subseteq I$, so $U(\mathfrak{a}) = U(\mathfrak{g})^{coU(\mathfrak{g}/\mathfrak{a})}$. □

Exercise 3

Let $\mathfrak{g}$ be a Lie algebra, I a set, and $x : I \to \mathfrak{g}$ an injective map. We say that $\mathfrak{g}$ is freely generated by I if for every Lie algebra $\mathfrak{h}$ and any map $f : I \to \mathfrak{h}$ there is a unique Lie algebra homomorphism $\bar{f} : \mathfrak{g} \to \mathfrak{h}$ such that the following diagram commutes:



Show that for any set I there is a sub Lie algebra $\mathfrak{g}_I \subset k < x_i \mid i \in I >^-$ that is freely generated by I. Show that $U(\mathfrak{g}_I) \simeq k < x_i \mid i \in I >$.

Proof. Simply take the smallest Lie subalgebra $\mathfrak{g}$ of $k < x_i | i \in I >$ that contain $\{x_i\}_{i \in I}$. For simplicity, we can describe a spanning set of $\mathfrak{g}$ inductively, where $I_0 = \{x_i\}_{i \in I}$ and I_{n+1} are all elements $x = [y,z] = yz - zy$ where $y, z \in I_n$, together with the elements of I_n. We will show that this is a freely generated Lie algebra.

Indeed, if $f : I \to \mathfrak{h}$, any Lie algebra homomorphisms $\bar{f}_1, \bar{f}_2$ that make (**??**) commute coincide in some Lie subalgebra $\mathfrak{g}' = \ker \bar{f}_1 - \bar{f}_2$ that contain $\{1\} \cup \{x_i\}_{i \in I}$, so by minimality it should be $\mathfrak{g}' = \mathfrak{g}$. This concludes uniqueness. The existence follows from the existence of such a map $\bar{f} : k < x_i | i \in I > \to U(\mathfrak{h})$, which when restricted to $\mathfrak{g}$, by minimality, goes to elements of degree one in $U(\mathfrak{h})$, as desired. Indeed, we act inductively on n to show that $\bar{f}(I_n) \in \mathfrak{g}$. For $n = 0$ is simple, as $\bar{f}(x_i) = f(\sigma(x_i)) = \sigma(f(x_i))$ are elements of degree one. Now if $x = [y,z]$, where $\bar{f}(y), \bar{f}(z)$ are elements of $U(\mathfrak{h})$ of degree one, then

$$\bar{f}(x) = \bar{f}([y,z]) = \bar{f}(yz - zy) = \bar{f}(y) \otimes \bar{f}(z) - \bar{f}(z) \otimes \bar{f}(y) = [\bar{f}(y), \bar{f}(z)] \,,$$

and so $\bar{f}(x)$ is also of degree one in $U(\mathfrak{h})$, as desired.

Note that it follows $Alg(k < x_i | i \in I >, A) = Hom(I, A) \cong Lie(\mathfrak{g}, A^-)$, so we have that

$$U(\mathfrak{g}) \cong k < x_i | i \in I > \,,$$

by the universal property of the enveloping algebra. □