Homework Assignment 7 -Solution

Hopf algebras - Spring Semester 2018

Exercise 1

Let k be a field with characteristic 0. Consider the Weyl alebra

$$A = k < x, y | xy - yx = 1 > .$$

a) Show that A is a simple algebra. That is, the only two-sided ideals of A are 0 and A.

b) Let $k[t]$ be the polynomial algebra with indeterminate t. We define the endomorphisms $\hat{t}, d \in \mathrm{End}_k(k[t])$ by

$$\hat{t}(t^n) = t^{n+1}, \ n \geq 0 .$$
$$d(t^n) = nt^{n-1}, \ n \geq 0 \text{ and } d(1) = 0 .$$

Consider the subalgebra $k[\hat{t}, d] \subseteq \mathrm{End}(k[t])$. Show that $A \simeq k[\hat{t}, d]$.

Proof of first item. First we note some simple properties of A. It is easy to see that A has a k-basis given by $\{y^i x^j\}_{i,j \geq 0}$. Also note that we have the following equations, that can be easily shown by induction:

$$xy^i x^j = iy^{i-1}x^j + y^i x^{j+1} . \tag{1}$$

$$x^j y = jyx^{j-1} + yx^n . \tag{2}$$

To show that A is simple, suppose that $I \subseteq A$ is a non-zero ideal. Our goal is to show that $k \cap I \neq 0$.

Consider the following total order in $\mathbb{N}_0^2$: we say that $(i, j) \leq (i', j')$ if $j < j'$ or if $j = j', i \leq i'$. This is the dictionary order after switching the coordinates. For instance, we have that $(3, 4) \leq (5, 4) \not\leq (6, 1)$.

Take $v \in I$ such that $v = \sum_{(i,j) \leq (i',j')} v_{i,j} y^i x^j$ for minimal (i', j'). Note that if $i' > 0$, from (1) we have

$$xv - vx = \sum_{(i,j) \leq (i'-1,j')} v_{i+1,j}(i + 1)y^i x^j \in I ,$$

is non-zero, contradicting the minimality of v. Similarly, if $i' = 0$ and $j' > 0$, note that from (2) we have

$$yv - vy = \sum_{(i,j) < (0,j')} v'_{i,j} y^i x^j + -nyx^{j'-1} \neq 0 ,$$

hence the minimal $v \neq 0$ is in k, and we are done. ☐

Proof of second item. We just note that $\hat{t}d - d\hat{t} = 1$, so we have a map $\phi : A \to k < \hat{t}, d >$ that sends $x \mapsto \hat{t}$ and $y \mapsto d$. Since A is simple, and $\ker\phi \neq A$, ϕ is an isomorphism. □

Exercise 2

Compute the algebras Lie(SL_n) and Lie(O_n).

Solutions. The resulting groups are, respectively, isomorphic to $\{M \in M_n(k) | tr(M) = 0\}$ and $\{M \in M_n(k) | M = -M^T\}$.

Indeed, recall that $\text{Lie}(A) = \ker A(\pi : A(k < \tau | \tau^2 = 0 >) \to A(k))$.

So $M \; in \ker SL_n(\pi)$ is a matrix $M = M_0 + \tau M_1 = SL_n(k < \tau | \tau^2 = 0 >)$, where $M_0, M_1 \in M_n(k)$ that satisfies

$$M\Big|_{\tau=0} = Id\,,$$
$$\det(M) = 1\,.$$

Note that the first equality is equivalent to $M_0 = Id$. Let $p_{M_1}(x) = \det(M - xId) = \sum_{k=0}^{n} b_k x^k$ be the characteristic polynomial of the matrix M_1.

Then $\det(M) = \det(\tau(M_1 - (-\tau^{-1})Id)) = \tau^n p_{M_1}(-\tau^{-1})$. With the fact that $\tau^2 = 0$ we have

$$\det(M) = b_n(-1)^n + b_{n-1}\tau(-1)^{n-1}\,.$$

It is well known that $b_0 = (-1)^n$ and $b_{n-1} = (-1)^{n-1} tr(M_1)$. Hence $\det(M) = 1$ if and only if $trM_1 = 0$.

Note that the product structure behaves as $(Id + M_1\tau)(Id + M_1'\tau) = Id\tau(M_1 + M_1')$.

Now to compute $\text{Lie}(O_n) = \ker O_n(\pi)$ we consider the matrices $M \in M_n(k < \tau | \tau^2 = 0 >)$ that satisfy both

$$M\Big|_{\tau=0} = Id\,,$$
$$MM^T = Id\,.$$

So we obtain again that $M = Id + \tau M_1$, from the first equation. Additionally, we have that $MM^T = Id \Leftrightarrow M_1 + M_1^T = 0_n$, finally one notes that as a a group, it holds

$$\text{Lie}(O_n) = \{M \in M_n(k) | M = -M^T\}\,.$$

□

Exercise 3

Consider a group G and let $k[G]$ denote the corresponding group algebra. Let A be an algebra over k and $(A_g)_{g\in G}$ a family of linear subspaces $A_g \subset A$. We say $(A, (A_g)_{g\in G})$ is a graded algebra if the following conditions hold:

- If 1_G is the identity of G and 1_A the unit of the algebra, then $1_A \in A_{1_G}$.
- We have that $A = \oplus_g A_g$

- For any $g, h \in G$, we have $A_g A_h \subset A_{gh}$.

For any comodule algebra structure $\delta : A \to A \otimes k[G]$ we may define a family $(A_g)_{g \in G}$

$$A_g = \{a \in A \mid \delta(a) = a \otimes g\}$$

for all $g \in G$. Show that this yields a bijection between $k[G]$-comodule algebra structures on A and gradings $\{A_g \mid g \in G\}$ of G.

Proof. It is easy to see that if $(A, (A_g)_{g \in G}))$ is a G-graded algebra, then δ defined at each A_g via $\delta : a \mapsto a \otimes g$ determines a $k[G]$-comodule structure . The additional requirement that $A_g A_h \subseteq A_{gh}$ tells us that this is also a comodule algebra structure. This is the inverse construction from the one given. It suffices then, to show that if we have a $k[G]$-comodule algebra (A, δ), then $(A, (A_g)_{g \in G})$ is a G-grading.

Since $\delta(1_A) = 1 = 1_A \otimes e_{id}$, $1_A \in A_{id}$. It is also clear that if $a \in A_g, b \in A_h$, then $\delta(ab) = \delta(a)\delta(b) = ab \otimes gh$, so $ab \in A_{gh}$.

It suffices to show that $A = \oplus_{g \in G} A_g$. Note that if $\sum_{g \in G} \lambda_g a_g = 0$ with $a_g \in A_g$, then

$$0 = \delta(\sum_{g \in G} \lambda_g a_g) = \sum_{g \in G} \lambda_g a_g \otimes g \,,$$

it follows by linear independence, that $\lambda_g = 0$ for any $g \in G$. This concludes that $\oplus_{g \in G} A_g \subset A$.

On the other hand, let $a \in A$, and note that there is a unique way of writing $\delta a = \sum_{g \in G} a_g \otimes g$. It follows that $a_g \in A_g$ because $(id \otimes \Delta) \circ \delta = (\delta \otimes id) \circ \delta$. We conclude that $a \in \oplus_{g \in G} A_g$. □

Exercise 4

Consider a group G, $k[G]$ the group algebra and A an algebra over k. Recall from the previous exercise the definition of G-graded algebra. Additionally, if A is an H-comodule algebra let $A^{co\ H} = \{v \in A | \delta(v) = v \otimes 1\}$ denote the H-coinvariants.

Such G-graded algebra is said to be strongly graded if $A_g A_h = A_{gh}$.

Show that $A^{\text{co}\ k[G]} \subset A$ is a $k[G]$ Galois extension if and only if the grading $(A_g)_{g \in G}$ is strong.

Hint: We can take an expression of $1 \in A_g A_{g^{-1}}$. *Use this to show that* $A_g \otimes_{A_{\text{id}_G}} A_h \to A_{gh}$ *is an isomorphism.*

Proof. In the previous exercise, we have seen that $A^{co\ k[G]} = A_{\text{id}_G}$, so $A_{\text{id}_G} \subset A$ is a Galois extension if

$$\text{can} A \otimes_{A_{\text{id}_G}} A \to A \otimes k[G] \,,$$

is bijective. Note that can acts on $A_g \otimes_{A_{\text{id}_G}} A_h \to A_{gh} \otimes e_h \cong A_{gh}$ as the multiplication, so we have that $A_{\text{id}_G} \subset A$ is a Galois extension if and only if each $A_g \otimes_{A_{\text{id}_G}} A_h \to A_{gh}$ is an isomorphism.

To establish one direction of the proof, it is easy to see that if $A_{\text{id}_G} \subset A$ is a Galois extension, then $A_g \otimes_{A_{\text{id}_G}} A_h \to A_{gh}$ is surjective, and so A is G-strongly graded.

On the other hand, if A is strongly graded, then $\mu : A_g \otimes_{A_{\mathrm{id}_G}} A_h \to A_{gh}$ is inverted in the following way: Take $1 \in A_{\mathrm{id}_G} = A_{g^{-1}} A_g$, so that we can write $a = \sum_i v_i \otimes w_i$ where $\delta v_i = v_i \otimes g$ and $\delta w_i = w_i \otimes h$.

Consider the map $\alpha : A_{gh} \to A_g \otimes A_h$ given as

$$x \mapsto \sum_i v_i \otimes w_i x \,,$$

then it is clear that $\mu \circ \alpha(x) = \mu\left(\sum_i v_i \otimes w_i x\right) = \sum_i v_i w_i x = x$.

On the the other hand, for $a \in A_g, b \in A_h$ note that $w_i a \in A_{\mathrm{id}_G}$ so we have that

$$\alpha \circ \mu(a \otimes b) = \alpha(ab) = \sum_i v_i \otimes w_i ab = \sum_i v_i w_i a \otimes b = a \otimes b \,.$$

This conclucludes that μ is an isomorphism. $\square$