Homework Assignment 5 - Comodules

Hopf algebras - Spring Semester 2018

Exercise 1

a) Let C be a coalgebra, (V, δ) a C right comodule, W a vector space. The tensor product $W \otimes C$ is a C right comodule via $\mathrm{id} \otimes \Delta$. Prove that

$$\mathrm{Hom}_k(V, W) \simeq \mathcal{M}^C(V, W \otimes C)$$

as vector spaces.

Proof. Given a linear map $f : V \to W$ we may define a k-linear map

$$V \xrightarrow{(f \otimes \mathrm{id})\delta} W \otimes C.$$

The condition

$$(\delta \otimes \mathrm{id})\delta = (\mathrm{id} \otimes \Delta)\delta$$

ensures that this map is colinear. Conversly, given $\varphi \in \mathcal{M}^C(V, W \otimes C)$ we may define a k-linear map

$$V \xrightarrow{(\mathrm{id} \odot \epsilon)\varphi} W.$$

The two constructions are inverse to each other, since

$$\begin{aligned} ((\mathrm{id} \odot \epsilon)\varphi \otimes \mathrm{id})\delta &= ((\mathrm{id} \odot \epsilon) \otimes \mathrm{id})(\varphi \otimes \mathrm{id})\delta \\ &= ((\mathrm{id} \odot \epsilon) \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\varphi \\ &= \varphi \end{aligned}$$

and

$$\begin{aligned} (\mathrm{id} \odot \epsilon)(f \otimes \mathrm{id})\delta &= f(\mathrm{id} \odot \epsilon)\delta \\ &= f. \end{aligned}$$

□

b) Let A be an algebra, M an A left module, W a vector space. The tensor product $A \otimes W$ is an A left module via $\mu_A \otimes \mathrm{id}$. Prove that

$$\mathrm{Hom}_k(W, M) \simeq {}_A\mathcal{M}(A \otimes W, M)$$

as vector spaces.

Proof. Given a linear map $f : W \to M$ we may define the map

$$A \otimes W \to M, \qquad a \otimes w \mapsto a.f(w).$$

Conversely, given $\varphi \in \mathrm{Hom}_A(A \otimes W, M)$ we may define the map

$$W \to M, \qquad w \mapsto \varphi(1 \otimes w).$$

Clearly the two constructions are inverse to each other. □

Exercise 2

a) Let G be a monoid. Show that G is a group if and only if the map

$$\varphi : G \times G \to G \times G, \qquad (g, h) \mapsto (gh, h)$$

is bijective.

Proof. Suppose that G is a group. Then the map

$$\psi : G \times G \to G \times G, \qquad (g, h) \mapsto (gh^{-1}, h)$$

is inverse to φ. Note that with $\epsilon : G \to \{e_G\}$ it holds that

$$h^{-1} = (\mathrm{id} \odot \epsilon)\psi^{-1}(1, h) \tag{1}$$

for all $h \in G$.

Conversely, suppose that ψ is a bijection. Then for each $g \in H$ there exist $x, y \in G$ with

$$(xy, y) = \varphi(x, y) = (e_G, g)$$

Hence $xg = e_G$. This shows that each element of G has a left inverse. In particular, x also has a left inverse $y \in G$. It follows that

$$y = y(xg) = (yx)g = g.$$

That is, $xg = gx = e$. Hence G is a group. □

b) Let H be a bialgebra. Show that H is a Hopf algebra if and only if the linear map

$$\varphi : H \otimes_k H \to H \otimes_k H, \qquad x \otimes y \mapsto xy_1 \otimes y_2$$

is bijective.

Proof. Suppose that H is a Hopf algebra. Then the map

$$\psi : H \otimes_k H \to H \otimes_k H, \qquad x \otimes y \mapsto xS(y_1) \otimes y_2$$

is inverse to ϕ. Indeed,

$$\begin{aligned}\psi(\phi(x \otimes y)) &= \psi(xy_1 \otimes y_2) \\ &= xy_1 S(y_2) \otimes y_2 \\ &= x\epsilon(y_1) \otimes y_2 \\ &= x \otimes y\end{aligned}$$

and

$$\begin{aligned}\phi(\psi(x \otimes y)) &= \phi(xy_1 \otimes y_2) \\ &= xy_1 S(y_2) \otimes y_3 \\ &= x\epsilon(y_1) \otimes y_2 \\ &= x \otimes y.\end{aligned}$$

Conversely, suppose that φ is bijective. Clearly H is an H left module via μ. Applying $1, b)$ with $W = H$ and $M = H$ yields an isomorphism

$$\mathrm{Hom}_k(H, H) \simeq {}_H\mathcal{M}(H \otimes H, H), \qquad f \mapsto \mu(\mathrm{id} \otimes f), \qquad \phi(1 \otimes -) \leftarrow\!\shortmid \phi$$

The tensor product $H \otimes H$ is an H right comodule via $\mathrm{id} \otimes \Delta$. Applying $1, a)$ with $V = H \otimes H$ and $W = H$ yields an isomorphism

$$\mathrm{Hom}_k(H \otimes H, H) \simeq \mathcal{M}^H(H \otimes H, H \otimes H), \qquad g \mapsto (g \otimes \mathrm{id})(\mathrm{id} \otimes \Delta), \qquad (\mathrm{id} \odot \epsilon)\psi \leftarrow\!\shortmid \psi$$

Here H-linear maps on the left side correspond precisely to H linear maps on the right side. Hence the isomorphism restricts to an isomorphism

$${}_H\mathcal{M}(H \otimes H, H) \simeq {}_H\mathcal{M}^H(H \otimes H, H \otimes H)$$

with ${}_H\mathcal{M}^H(-, -)$ denoting maps that are both H left linear and H right colinear. Thus,

$$\begin{aligned}&T : \mathrm{Hom}_k(H, H) \simeq {}_H\mathcal{M}^H(H \otimes H, H \otimes H) \\ &(\mathrm{id} \odot \epsilon)\alpha(1 \otimes x) \leftarrow\!\shortmid \alpha \\ &\qquad f \mapsto (\mu(\mathrm{id} \otimes f) \otimes \mathrm{id})(\mathrm{id} \otimes \Delta) : (x \otimes y \mapsto xf(y_1) \otimes y_2)\end{aligned}$$

Note that

$$\varphi = (\mu \otimes \mathrm{id})(\mathrm{id} \otimes \Delta) \in {}_H\mathcal{M}^H(H \otimes H, H \otimes H)$$

corresponds to

$$\mathrm{id} \in \mathrm{Hom}_k(H, H).$$

There is a monoid structure on $\mathrm{Hom}_k(H, H)$ given by the $*$-product and a monoid structure on ${}_H\mathcal{M}^H(H \otimes H, H \otimes H)$ given by the composition of morphisms. We assumed that φ is bijective, so it has an inverse with respect to this monoid structure. In order to show that its image $\mathrm{id} \in \mathrm{Hom}_k(H, H)$ has a $*$-inverse it suffices to verify that the correspondence is an anti monoid homomorphism.

To this end, note that

$$\begin{aligned} T(g)(T(f)(x \otimes y)) &= T(g)(xf(y_1) \otimes y_2) \\ &= xf(y_1)g(y_2) \otimes y_3 \\ &= x(f * g)(y_1) \otimes y_2 \\ &= T(f * g)(x \otimes y) \end{aligned}$$

and

$$\begin{aligned} T(\eta\epsilon)(x \otimes y) &= x\epsilon(y_1) \otimes y_2 \\ &= x \otimes \epsilon(y_1)y_2 \\ &= x \otimes y. \end{aligned}$$

□

Exercise 3

Let H be a Hopf algebra. Which condition do we have to impose on H such that the canonical monomorphism

$$\varphi : V \to V^{**}, \qquad v \mapsto (f \mapsto f(v))$$

is H-linear for each H left module V?

Proof. The condition is $S^2 = \mathrm{id}$. To see this, note that

$$h.f(v) = f(S(h).v)$$

for $h \in H$, $f \in V^*$, $v \in V$, and thus

$$\begin{aligned} (h.\varphi(v))(f) &= \varphi(v)(S(h).f) \\ &= (S(h).f)(v) \\ &= f(S^2(h)v) \\ &= \varphi(S^2(h).v)(f). \end{aligned}$$

If $S^2 = \mathrm{id}$ then clearly φ is H-linear. Conversely, taking $V = H$, $v = 1_H$ it follows that $f(S^2(h)) = f(h)$ for all $h \in H$ and $f \in H^*$, yielding $S^2 = \mathrm{id}$. □

Exercise 4

Suppose that $\operatorname{char} k = p > 0$ and let $H = k < t \mid t^p = 0 >$ be the Hopf algebra with t primitive. Show that

$$H \simeq H^*$$

as Hopf algebras.

Proof. Let $\varphi \in H^*$ be the map with $\varphi(t^i) = \delta_{1,i}$ for $0 \le i < p$. φ is a primitive element of the Hopf algebra H^*, because for all $0 \le i, j < n$ it holds that

$$\varphi(t^i)\epsilon(t^j) + \epsilon(t^i)\varphi(t^j) = \varphi(t^i)\delta_{j,0} + \delta_{i,0}\varphi(t^j) = \varphi(t^{i+j}).$$

It holds that $\varphi^p = 0$, because

$$\varphi^p(t) = \sum_{i=1}^{p} \varphi(t)\varphi(1)^{p-1} = p = 0.$$

Hence there is an algebra homomorphism

$$H \to H^*, \qquad t \mapsto \varphi.$$

As t and φ are both primitive, it follows that this is also a coalgebra homomorphism.

We know that for a primitive element x in a bialgebra over a field with characteristic 0 the powers $1, x, x^2, \dots$ are linear independent. Since $\binom{n}{k} \neq 0$ for $n < p$, $0 \le k \le n$ the exact same argument yields that $1, x, x^2, \dots, x^{p-1}$ are linear independent if the field has characteristic p. Hence the morphism $H \to H^*$ with $t \mapsto \varphi$ is an isomorphism. □

Exercise 5

Let $q \in k^\times$ be a primitive root of unity. Show that the Taft Hopf algebra

$$H = k < g, x \mid g^n = 1, x^n = 0, gx = qxg >$$

with g group-like and x $(g, 1)$-primitive has dimension n^2.

Proof. We may check that

$$H \simeq k[X]/(X^n)\#k[G]/(G^n - 1)$$

with G group-like, $G.X = qX$. To this end, note that

$$\sigma : k[X]/(X^n) \to k[X]/(X^n), \qquad \bar{X} \mapsto q\bar{X}$$

is a well-defined algebra homomorphism, and $\sigma^n = \mathrm{id}$. Hence

$$\delta : k[G] \to \mathrm{End}_k(k[X]/(X^n)), \qquad G \mapsto \sigma$$

factors over $k[G]/(G^n - 1)$. This makes the algebra $k[X]/(X^n)$ a left module over the Hopf algebra $k[G]/(G^n - 1)$. Since $\bar{G}$ is group-like and acts as an algebra endomorphism, it is clear that $k[X]/(X^n)$ is a left module algebra. Hence we may form the smash product

$$k[X]/(X^n)\#k[G]/(G^n - 1).$$

The algebra homomorphism

$$\varphi : k < g, x > \to k[X]/(X^n)\#k[G]/(G^n - 1)$$

factors over H, since $\bar{G}^n = 1$, $\bar{X}^n = 0$, and

$$\bar{G}\bar{X} = (\bar{G}.\bar{X})\bar{G} = q\bar{X}\bar{G}.$$

The vector space generating set $(\bar{g}^i\bar{x}^j)_{0 \le i,j < n}$ gets mapped to the basis $(\bar{G}^i\bar{X}^j)_{0 \le i,j < n}$, making the induced map an isomorphism.

Hence the Taft Hopf algebra has dimension n^2 and $(\bar{x}^i\bar{g}^j)_{0 \le i,j < n}$ is basis. □