Homework Assignment 7

Hopf algebras - Spring Semester 2018

Exercise 1

Let k be a field with characteristic 0. Consider the Weyl alebra

$$A = k < x, y | xy - yx = 1 > .$$

a) Show that A is a simple algebra. That is, the only two-sided ideals of A are 0 and A.

b) Let $k[t]$ be the polynomial algebra with indeterminate t. We define the endomorphisms $\hat{t}, d \in \mathrm{End}_k(k[t])$ by

$$\hat{t}(t^n) = t^{n+1},\ n \geq 0\,.$$
$$d(t^n) = nt^{n-1},\ n \geq 0 \text{ and } d(1) = 0\,.$$

Consider the subalgebra $k[\hat{t}, d] \subseteq \mathrm{End}(k[t])$. Show that $A \simeq k[\hat{t}, d]$.

Exercise 2

Compute the Lie algebras $\mathrm{Lie}(SL_n)$ and $\mathrm{Lie}(O_n)$.

Exercise 3

Consider a group G and let $k[G]$ denote the corresponding group algebra. Let A be an algebra over k and $(A_g)_{g\in G}$ a family of linear subspaces $A_g \subset A$. We say $(A, (A_g)_{g\in G})$ is a graded algebra if the following conditions hold:

- If 1_G is the identity of G and 1_A the unit of the algebra, then $1_A \in A_{1_G}$.
- We have $A = \oplus_g A_g$.
- For any $g, h \in G$, we have $A_g A_h \subset A_{gh}$.

For any comodule algebra structure $\delta : A \to A \otimes k[G]$ we may define a family $(A_g)_{g\in G}$

$$A_g = \{a \in A \mid \delta(a) = a \otimes g\}$$

for all $g \in G$. Show that this yields a bijection between $k[G]$-comodule algebra structures on A and gradings $\{A_g \mid g \in G\}$ of G.

Exercise 4

Consider a group G and let A be an algebra over k. We say a grading $(A_g)_{g\in G}$ of A is strong if $A_g A_h = A_{gh}$ holds for all $g, h \in G$. If (A, δ) is a comodule algebra over some bialgebra H we set

$$A^{\mathrm{co}\ H} = \{v \in A \mid \delta(v) = v \otimes 1\}.$$

Now, suppose that $H = k[G]$ and that the grading $(A_g)_{g\in G}$ is given by $A_g = \{a \in A \mid \delta(a) = a \otimes g\}$ for all $g \in G$ as in Exercise 3.

Show that $A^{\mathrm{co}\ k[G]} \subset A$ is a $k[G]$ Galois extension if and only if the grading $(A_g)_{g\in G}$ is strong.

Hint: We can take an expression of $1 \in A_g A_{g^{-1}}$. *Use this to show that* $A_g \otimes A_h \to A_{gh}$ *is an isomorphism.*