Homework Assignment 5 - Comodules

Hopf algebras - Spring Semester 2018

Exercise 1

a) Let C be a coalgebra, (V, δ) a C right comodule, W a vector space. The tensor product $W \otimes C$ is a C right comodule via $\mathrm{id} \otimes \Delta$. Prove that

$$\mathrm{Hom}_k(V, W) \simeq \mathcal{M}^C(V, W \otimes C)$$

as vector spaces.

b) Let A be an algebra, M an A left module, W a vector space. The tensor product $A \otimes W$ is an A left module via $\mu_A \otimes \mathrm{id}$. Prove that

$$\mathrm{Hom}_k(W, M) \simeq {}_A\mathcal{M}(A \otimes W, M)$$

as vector spaces.

Exercise 2

a) Let G be a monoid. Show that G is a group if and only if the map

$$\varphi : G \times G \to G \times G, \qquad (g, h) \mapsto (gh, h)$$

is bijective.

b) Let H be a bialgebra. Show that H is a Hopf algebra if and only if the linear map

$$\varphi : H \otimes_k H \to H \otimes_k H, \qquad x \otimes y \mapsto xy_1 \otimes y_2$$

is bijective.

Exercise 3

Let H be a Hopf algebra. Which condition do we have to impose on H such that the canonical monomorphism

$$\varphi : V \to V^{**}, \qquad v \mapsto (f \mapsto f(v))$$

is H-linear for each H left module V?

Exercise 4

Suppose that $\operatorname{char} k = p > 0$ and let $H = k < t \mid t^p = 0 >$ be the Hopf algebra with t primitive. Show that

$$H \simeq H^*$$

as Hopf algebras.

Exercise 5

Let $q \in k^\times$ be a primitive root of unity. Show that the Taft Hopf algebra

$$H = k < g, x \mid g^n = 1, x^n = 0, gx = qxg >$$

with g group-like and x $(g, 1)$-primitive has dimension n^2.