Homework Assignment 4 - Bialgebras and Hopf algebras

Hopf algebras - Spring Semester 2018

Exercise 1 - The orthogonal group

Let $n \geq 1$ we may consider the functor O_n that maps a commutative k-algebra A to the orthogonal group

$$O_n(A) = \{M \in M_n(A) \mid MM^T = I\}.$$

Find a commutative Hopf algebra H such that

$$O_n \simeq \mathrm{Alg}_k(H, -)\,.$$

Exercise 2 - Finite dimensional Hopf algebras

a) Let A be a subalgebra of an algebra B and denote by $A^\times$ and $B^\times$ the set of invertible elements in the respective algebras.

   Show that if A is finite dimensional then $A^\times = B^\times \cap A$.

b) Let B be a finite dimensional subbialgebra of a Hopf algebra H.

   Show that B is a Hopf algebra as well.

c) Let B be a finite dimensional bialgebra and H a Hopf algebra.

   Show that if there is a surjective bialgebra homomorphism $\phi : H \to B$ then B is a Hopf algebra as well.

Exercise 3 - Kernel of counit

Suppose that B is a bialgebra, and denote $B^+ = \ker(\epsilon)$ the augmentation ideal. Show that if $x \in B^+$, then

$$\Delta(x) \in x \otimes 1 + 1 \otimes x + B^+ \otimes B^+\,.$$

Exercise 4 - Primitive elements in characteristic 0

Suppose that the field k has characteristic 0. Let B be a k-bialgebra and $0 \neq x \in P(B)$ a primitive element. Show that $1, x, x^2, \ldots$ are linear independent.

Exercise 5 - Primitive elements

a) If G is a group then the group algebra $k[G]$ is a Hopf algebra with all $g \in G$ being group-like elements.

   Show that $k[G]$ has no non-zero primitive elements, that is $P(H) = 0$.

b) If G is a finite group, then $k^G = k[G]^*$ is a Hopf algebra. If we consider the basis $(e_g)_{g \in G}$ with $e_g(h) = \delta_{g,h}$ for all $h \in H$ then the product is

$$e_g * e_h = \delta_{g,h} e_g ,$$

   and the comultiplication is given by

$$\Delta(e_g) = \sum_{ab=g} e_a \otimes e_b .$$

   Show that $P(k^G) = \mathrm{Gr}(G, (k, +))$ and $G(k^G) = \mathrm{Gr}(G, k^\times)$.

c) The polynomial algebra $k[T]$ is a Hopf algebra with T being a primitive element.

   Describe $P(k[T])$ for both $\operatorname{char} k = 0$ and $\operatorname{char} k = p > 0$.