Homework Assignment 2 - Algebras, Category Theory

Hopf algebras - Spring Semester 2018

Exercise 1 - Algebras

a) Let V and W be finite dimensional vector spaces over k. Show that there is an algebra isomorphism
$$\mathrm{End}_k(V \otimes_k W) \simeq \mathrm{End}_k(V) \otimes_k \mathrm{End}_k(W).$$
What does that imply for the algebra $M_n(k) \otimes_k M_m(k)$, $m, n \geq 1$?

b) Let M be a finite abelian group. Show that there are integers $n \geq 1$, $m_1, \dots, m_n \geq 1$ such that
$$k[M] \simeq k[X_1, \dots, X_n]/(X_1^{m_1} - 1, \dots, X_n^{m_n} - 1).$$
Hint: Show first that $k[G \times H] \simeq k[G] \otimes k[H]$ for any two monoids G and H.

Exercise 2 - Algebras and field extensions

Let $k \subset L$ be a field extension.

a) Let A be a k-algebra. Show that the L-algebra $A \otimes_k L$ has dimension
$$\dim_L(A \otimes_k L) = \dim_k(A).$$

b) Verify that $k[X] \otimes_k L \simeq L[X]$ as L-algebras.

c) Let $k \subset L$ be a Galois extension. Find an explicit description of the L-algebra $L \otimes_k L$.

Exercise 3 - Morita equivalence

a) Let R be a ring and $S = M_n(R)$ the ring of $n \times n$ matrices with coefficients in R. Let P be the space of all matrices in $M_n(R)$ with the property, that the coefficients in the rows $2, \dots, n$ are equal to zero. Let Q be the space of all matrices in $M_n(R)$ with the property, that the coefficients in the columns $2, \dots, n$ are equal to zero. Show that
$$P \otimes_S Q \simeq R \text{ in } {}_R\mathcal{M}_R \qquad \text{and} \qquad Q \otimes_R P \simeq S \text{ in } {}_S\mathcal{M}_S\ .$$

b) Let R, S be rings, P an (R, S)-bimodule, Q an (S, R)-bimodule. Suppose that
$$P \otimes_S Q \simeq R \text{ in } {}_R\mathcal{M}_R \qquad \text{and} \qquad Q \otimes_R P \simeq S \text{ in } {}_S\mathcal{M}_S\ .$$
Show that the functor $Q \otimes_R - : {}_R\mathcal{M} \to {}_S\mathcal{M}$ and $P \otimes_S - : {}_S\mathcal{M} \to {}_R\mathcal{M}$ are quasi-inverse equivalences of categories.

Applications: Equivalences of categories preserve category theoretic notions such as coproducts. For example, if $R = k$ is a field, then there is exists a module $U \in {}_k\mathcal{M}$ such that for all $V \in {}_k\mathcal{M}$ there is an index set I such that $V \simeq \coprod_{i\in I} U$. Since ${}_k\mathcal{M} \simeq {}_{M_n(k)}\mathcal{M}$ an analogous statement holds for left modules over $M_n(k)$.

Exercise 4 - Exact functors

Let R and S be rings. A covariant functor $F : {}_R\mathcal{M} \to {}_S\mathcal{M}$ is termed left exact, if for any exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ in ${}_R\mathcal{M}$ the sequence

$$0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact as well. It is termed right-exact, if for any exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ the sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$$

is exact as well. We say F is exact, if it is both left- and right-exact. A contravariant functor $F : {}_R\mathcal{M} \to {}_S\mathcal{M}$ is left-exact, if for any exact sequence $0 \to C^{\mathrm{op}} \xrightarrow{g^{\mathrm{op}}} B^{\mathrm{op}} \xrightarrow{f^{\mathrm{op}}} A^{\mathrm{op}}$ in ${}_R\mathcal{M}^{\mathrm{op}}$ the sequence

$$0 \to F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A)$$

is exact. We have seen that for all left R-modules M, N the functors $\mathrm{Hom}(M, -)$ and $\mathrm{Hom}(-, N)$ are left-exact.

a) Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be group homomorphisms of abelian groups. Show that if

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(Z, A) \overset{\mathrm{Hom}(g,\mathrm{id})}{\longrightarrow} \mathrm{Hom}_{\mathbb{Z}}(Y, A) \overset{\mathrm{Hom}(f,\mathrm{id})}{\longrightarrow} \mathrm{Hom}_{\mathbb{Z}}(X, A)$$

is exact for every abelian group A, then

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is exact as well.

b) Let R and S be rings, and let ${}_RX_S$ be an (R, X)-bimodule. Recall that the functor

$${}_RX_S \otimes_S - : {}_S\mathcal{M} \to {}_R\mathcal{M}$$

is left adjoint to

$$\mathrm{Hom}_R({}_RX_S, -) : {}_R\mathcal{M} \to {}_S\mathcal{M}.$$

Combine this fact with a) to deduce that for any right R-module M_R the functor

$$M \otimes_R - : {}_R\mathcal{M} \to {}_{\mathbb{Z}}\mathcal{M}$$

is right-exact.

c) An R-module M is termed free, if there is an index set I with $M \simeq R^{(I)}$. M is termed projective if there is an R-module N such that $M \oplus N$ is free. Show that if M is a projective right R-module, then the functor $M \otimes_R -$ is exact.

Comment: It is clear that the same holds also for the functor $- \otimes_R N$ if N is a projective left R-module. In particular, the tensor product over a skew field (i.e. a division ring) is always exact.