Homework Assignment 1 - Tensor products

Hopf algebras - Spring Semester 2018

Exercise 1 - Tensor product

Consider a module $X \in \mathcal{M}_R$ and two-sided ideals $I, J \subset R$.

a) Show that $X \otimes_R R/I \simeq X/XI$ in $\mathcal{M}_{R/I}$.

b) Show that $R/I \otimes_R R/J \simeq R/(I+J)$ in $\mathcal{M}_R$.

Exercise 2 - Tensor product of maps

a) Let $\iota : \mathbb{Z}/(2) \to \mathbb{Z}/(4)$ be the unique injective group homomorphism. Compute $\mathrm{id} \otimes_{\mathbb{Z}} \iota$ that maps $\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(2) \to \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(4)$.

b) For two integers m, n, compute $\mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n)$.

c) Compute, for an abelian group G and an integer n, the tensor $G \otimes_{\mathbb{Z}} \mathbb{Z}/(n)$.

d) An abelian group G is a torsion abelian group if for every element $g \in G$ there is a natural number n such that $ng = 0$. Show that for any torsion group G we have $G \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

Exercise 3 - Tensor product of maps

Let X, Y be vector spaces over k, and $U \subseteq X$, $V \subseteq Y$ be subspaces. Let $p_U : U \to X$ and $p_V : V \to Y$ be the canonical inclusions. Show that

$$\ker p_U \otimes_k p_V = X \otimes_k V + U \otimes_k Y\ .$$

Exercise 4 - Examples

Find modules M, N over a ring R such that $M \otimes_{\mathbb{Z}} N \not\simeq M \otimes_R N$ as $\mathbb{Z}$-modules.

Exercise 5 - Dual spaces

Let X, Y be k-vector spaces

a) Show that the map $(x, f) \mapsto (y \mapsto f(y)x)$ defines a linear map from $X \times Y^*$, which gives rise to a linear map $\phi_{X,Y} : X \otimes_k Y^* \to \mathrm{Hom}_k(Y, X)$. Additionally, show that if either X or Y are finite dimensional, then $\phi_{X,Y}$ is an isomorphism.

b) Show that the map $(x, f) \mapsto f(x)$ defines a bilinear map from $X \times X^*$, which gives rise to a linear map $e_X : X \otimes_k X^* \to k$.

c) Define $\mathrm{Tr}_X = e_X \circ \phi_{X,X}^{-1} : \mathrm{End}_k(X) \to k$, for X finite dimensional. Show that if we take $F \in \mathrm{End}_k(X)$ and $G \in \mathrm{End}_k(Y)$ then

$$\mathrm{Tr}_{X \otimes Y}(F \otimes G) = \mathrm{Tr}_X(F)\,\mathrm{Tr}_Y(G)\,.$$

Exercise 6 - Exact sequences

Given M, N left modules over a ring R, show that the functors $\mathrm{Hom}(-, M)$ and $\mathrm{Hom}(N, -)$ are both left exact. I.e. whenever $X \to Y \to Z \to 0$ and $0 \to X' \to Y' \to Z'$ are exact sequences of left R-modules, then the following are exact:

$$0 \to \mathrm{Hom}_R(Z, M) \to \mathrm{Hom}_R(Y, M) \to \mathrm{Hom}_R(X, M)\,,$$

$$0 \to \mathrm{Hom}_R(N, X') \to \mathrm{Hom}_R(N, Y') \to \mathrm{Hom}_R(N, Z')\,.$$