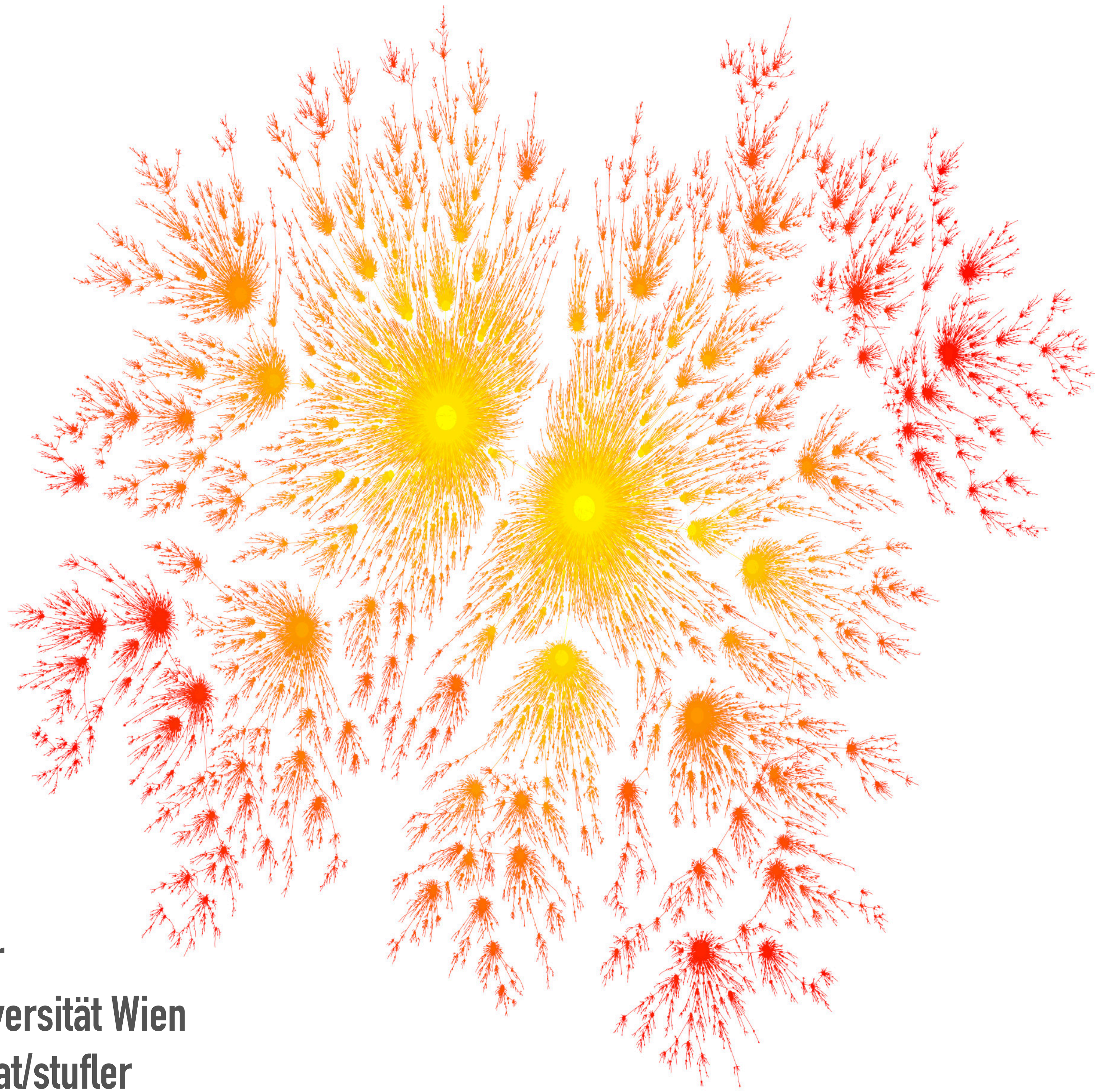
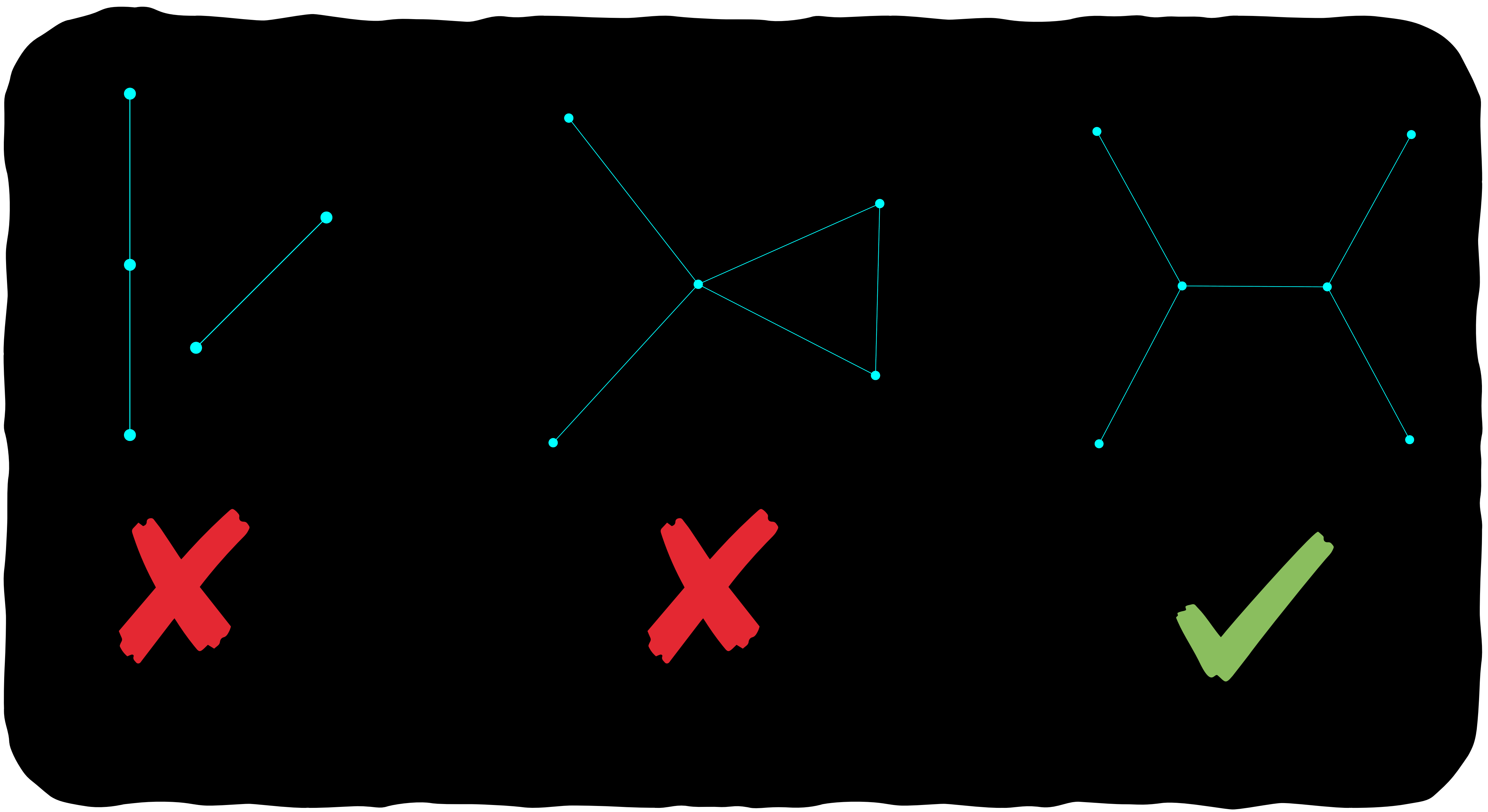
INTRODUCTION TO RANDOM TREES AND THEIR LOCAL LIMITS



Benedikt Stufler
Technische Universität Wien
dmg.tuwien.ac.at/stufler

TREES ARE CONNECTED (SIMPLE) GRAPHS WITHOUT CYCLES



MODELS

- Combinatorial classes of trees
- Trees with given degree sequences
- Branching processes
- Spanning trees
- ...

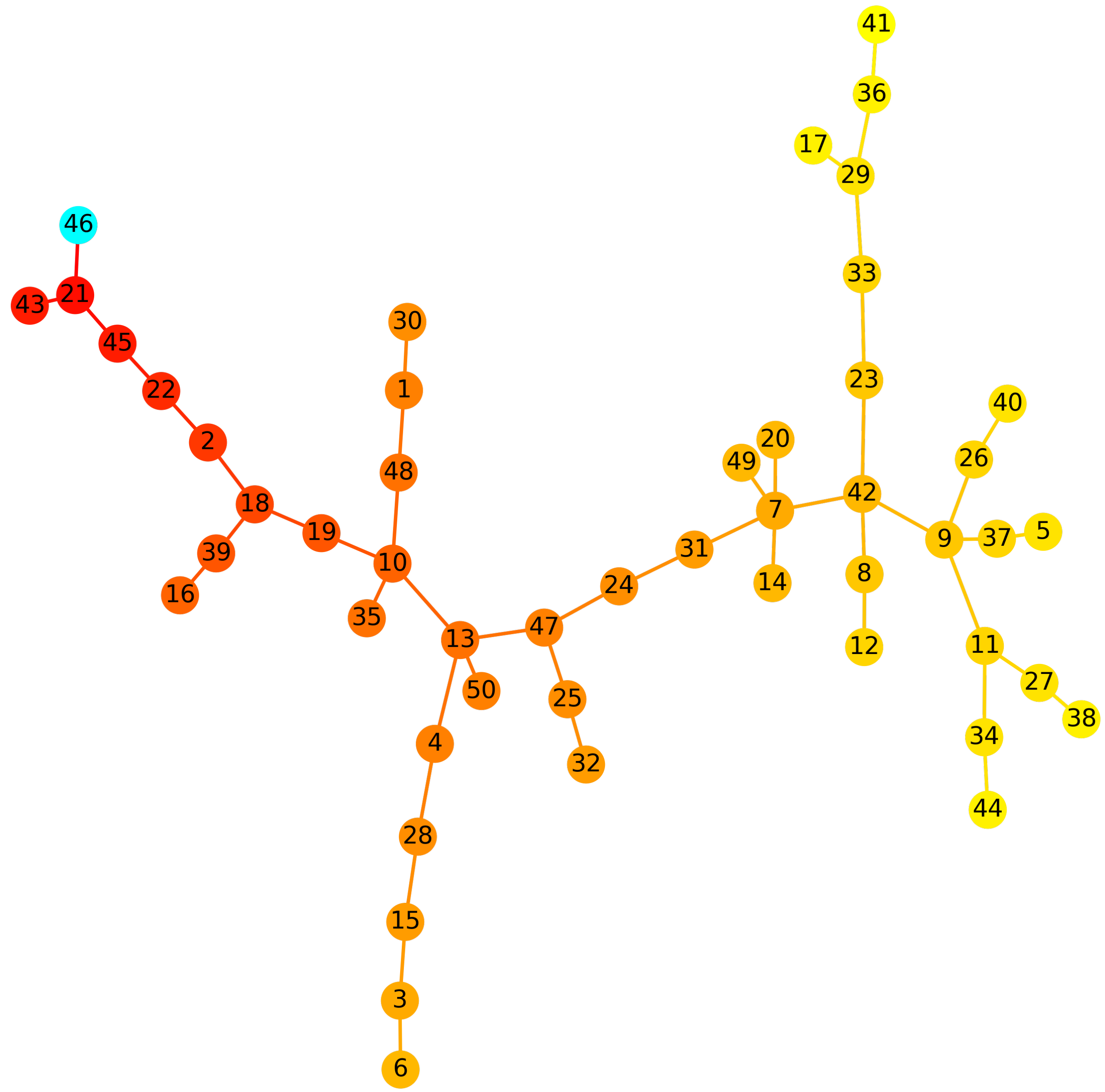
SAMPLING

- Various Open Source Projects
  github.com/BenediktStufler/
- Visualizations and interactive 3d models:
  dmg.tuwien.ac.at/stufler/gal.html

SHAPE ANALYSIS

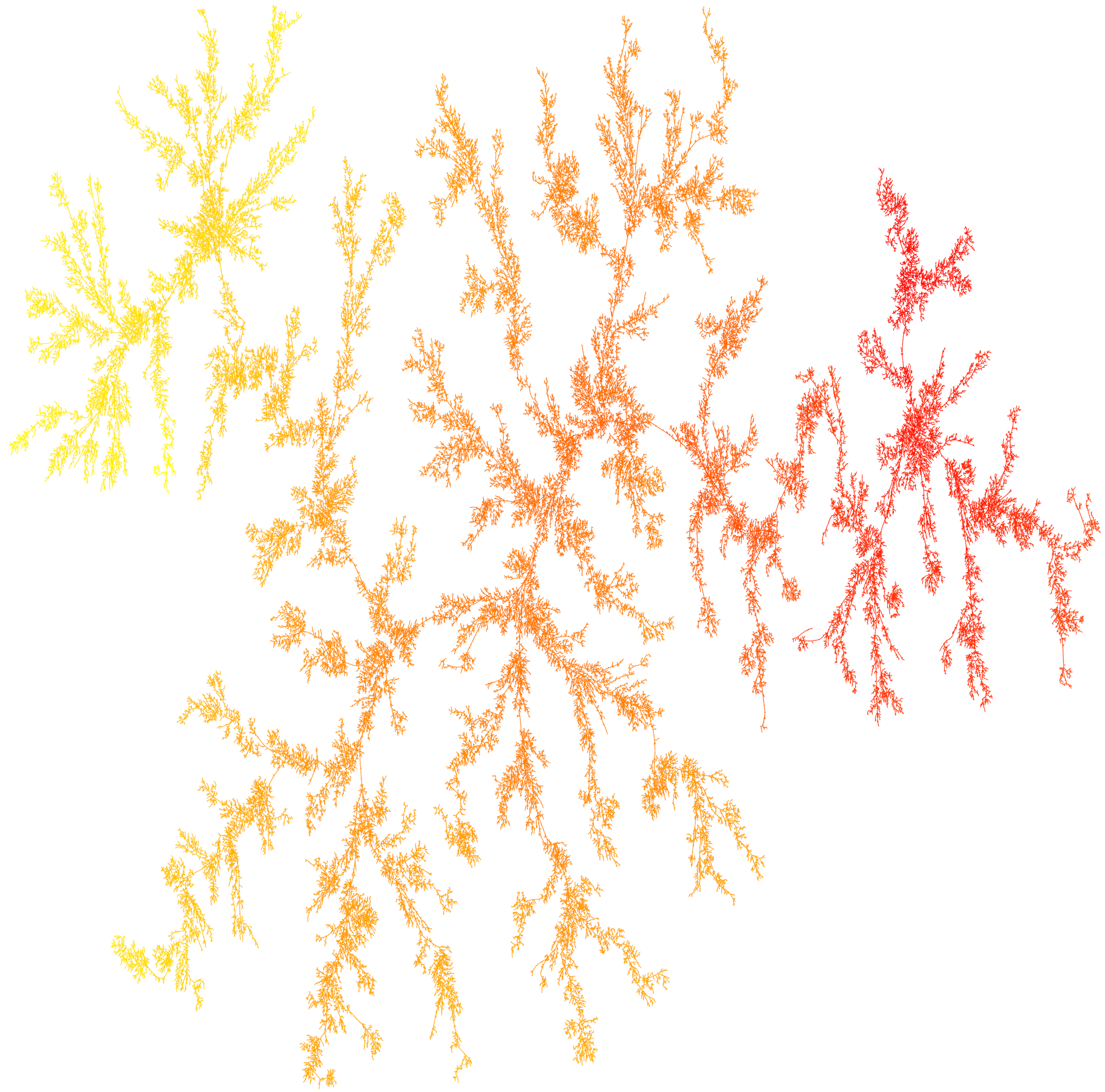
- Additive parameters (subtree counts, degree sequence, ...)
- Extremal parameters (diameter, maximal degree, ...)
- Limits (local limits, scaling limits)

UNIFORM ROOTED TREE ON A FIXED N-ELEMENT VERTEX SET: N=50



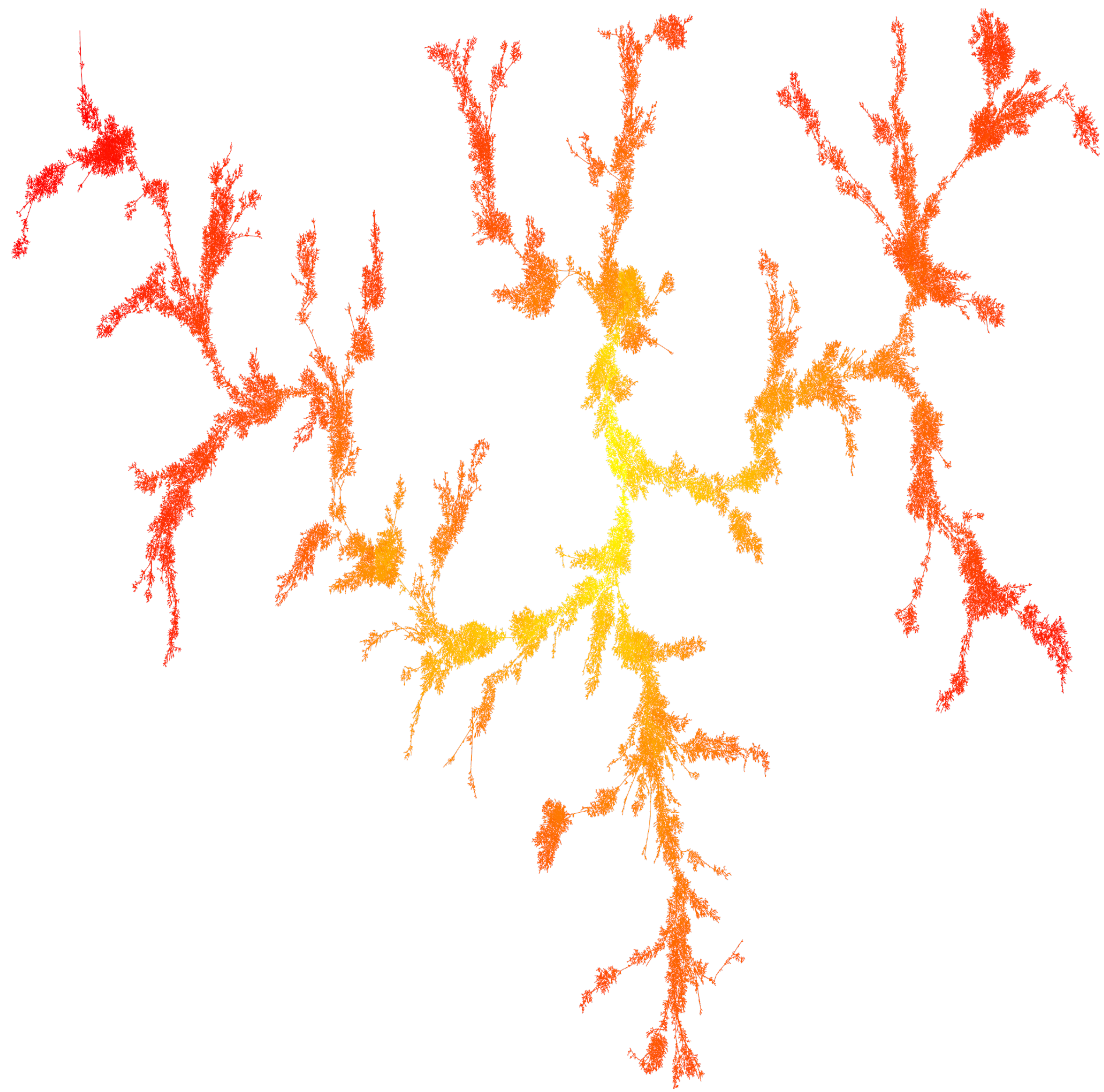
Simulation: GRANT (Generate RANdom Trees), available here: http://github.com/BenediktStufler/grant

UNIFORM ROOTED TREE ON A FIXED N-ELEMENT VERTEX SET: N=500K



Simulation: GRANT (Generate RANdom Trees), available here: http://github.com/BenediktStufler/grant

UNIFORM ROOTED TREE ON A FIXED N-ELEMENT VERTEX SET: N =1M



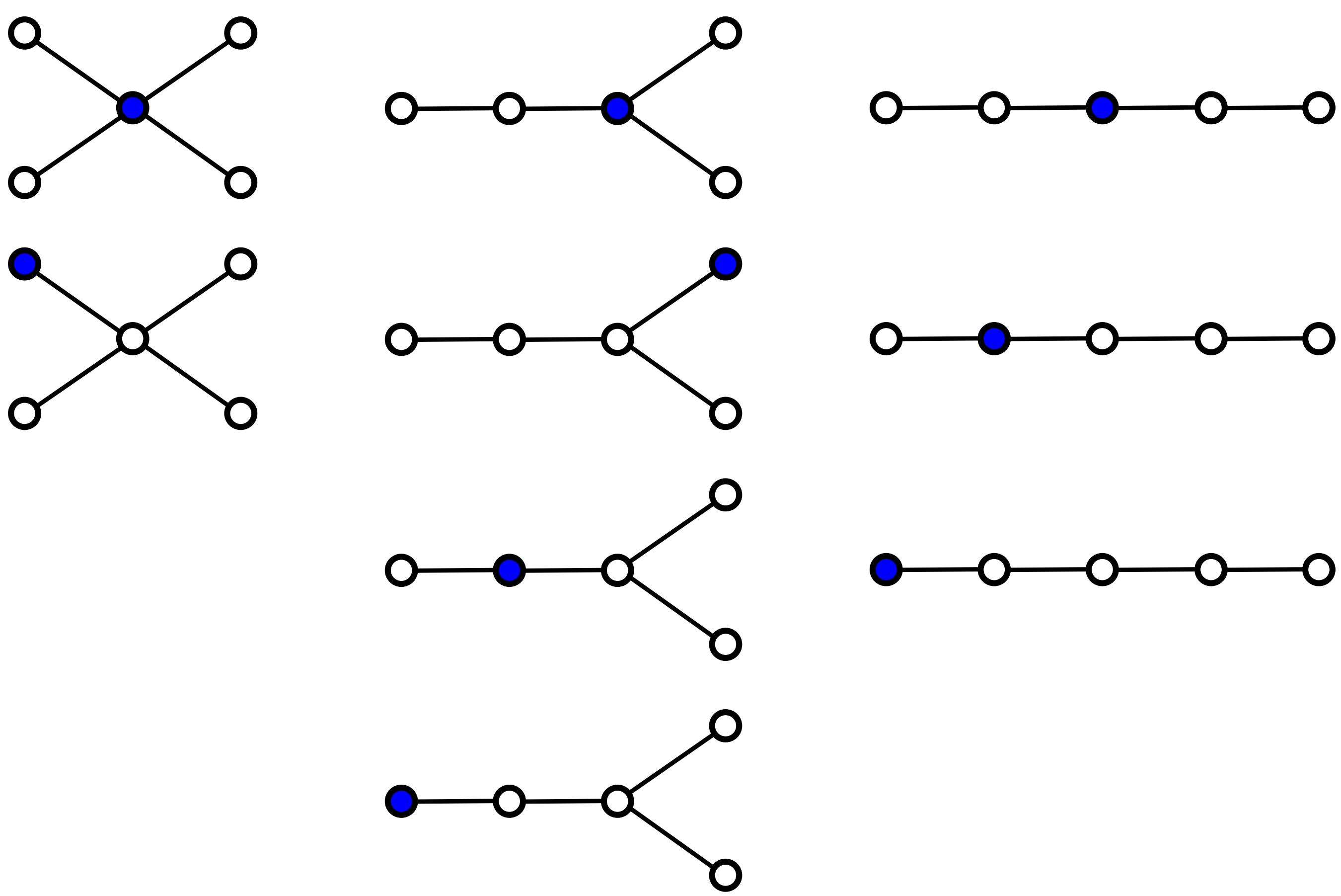
Simulation: GRANT (Generate RANdom Trees), available here: http://github.com/BenediktStufler/grant

UNIFORM ROOTED TREE ON A FIXED N-ELEMENT VERTEX SET

- **Enumeration** $n^{n-1} \sim (2\pi)^{-\frac{1}{2}} n^{-\frac{3}{2}} e^n n!$
- **Efficient sampler** expected time $O(n)$
- **Additive parameters** $\left(d_k(T_n) - \frac{n}{(k-1)!e}\right) n^{-\frac{1}{2}} \to \mathcal{N}(0, \sigma_k)$
- **Extremal parameters** $\Delta(T_n) = \frac{\log n}{\log \log n}(1 + o_p(1))$, $H(T_n)/\sqrt{n} \to 2 \max_{0 \le t \le 1} e(t)$
- **Structural limits** **Local: Kesten's tree** **Global: Brownian continuum random tree**
- **...**

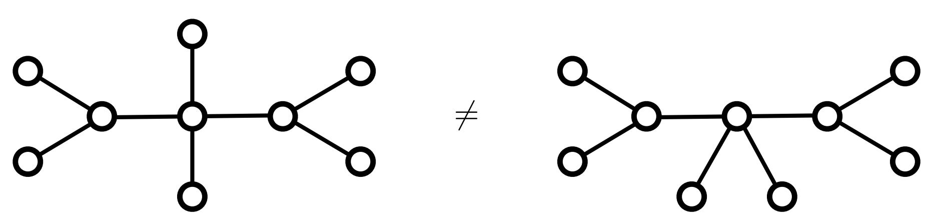
MODELS OF UNORDERED TREES

- **Trees: labelled rooted/unrooted unordered**
- **Pólya trees: unlabelled rooted unordered**
- **Free trees: unlabelled unrooted unordered**

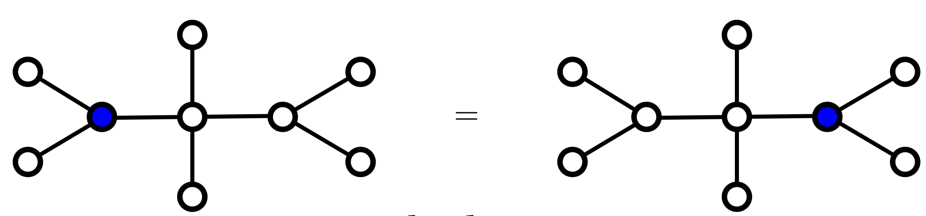


MODELS OF ORDERED TREES

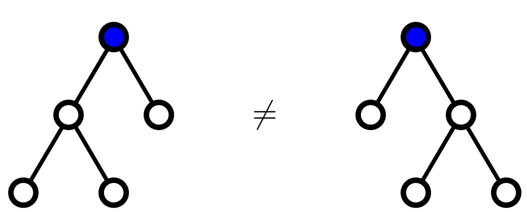
- **Planted plane trees: unlabelled rooted linearly ordered**
- **Rooted plane trees: unlabelled rooted cyclically ordered**
- **Unrooted plane trees: unlabelled unrooted cyclically ordered**



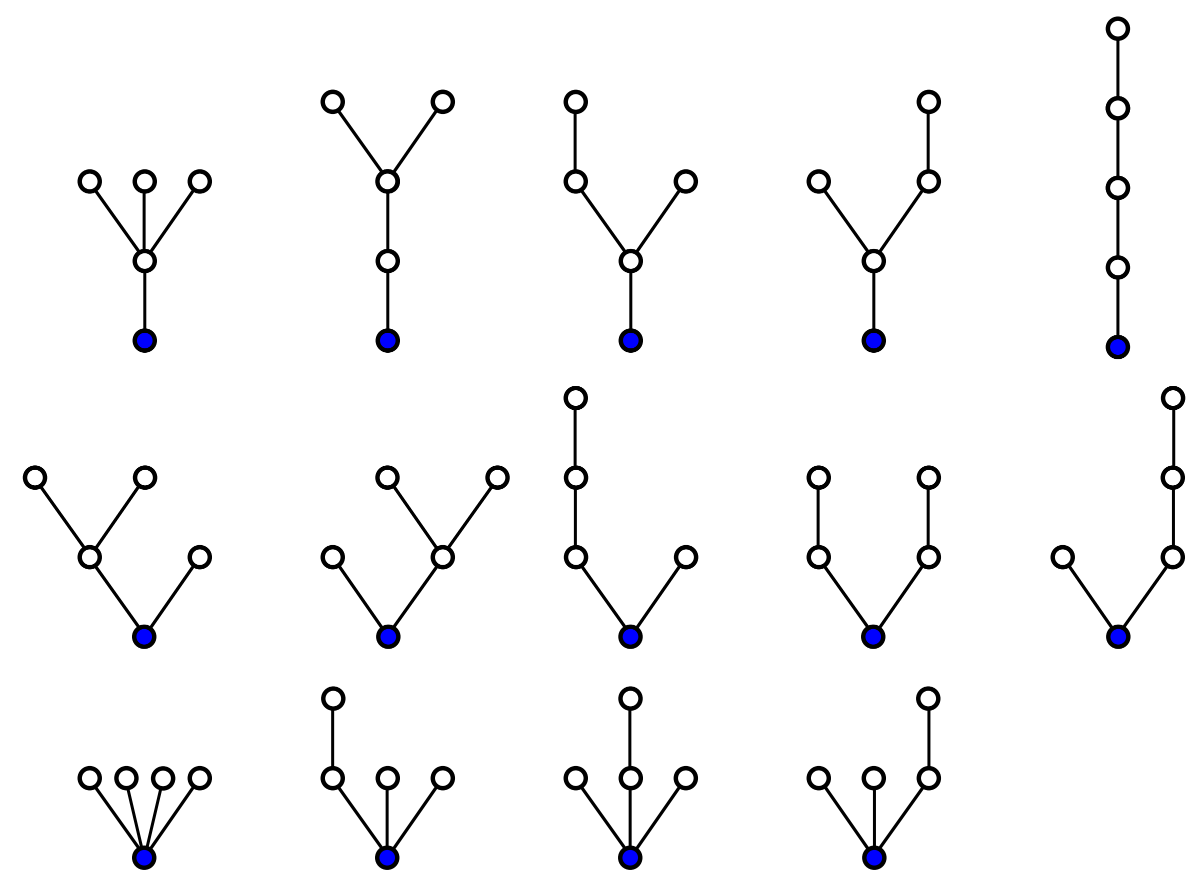
unrooted plane trees



rooted plane trees

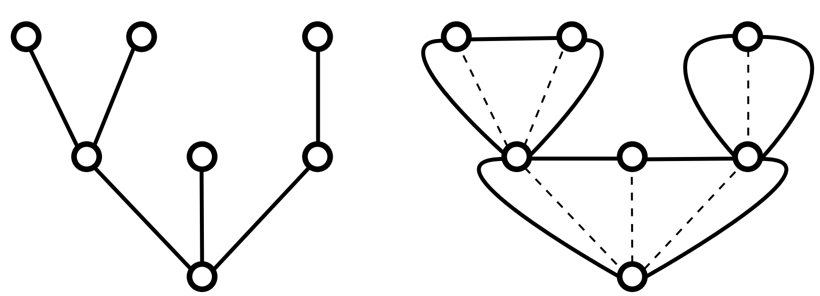


planted plane trees



PROBABILISTIC MODELS OF TREES

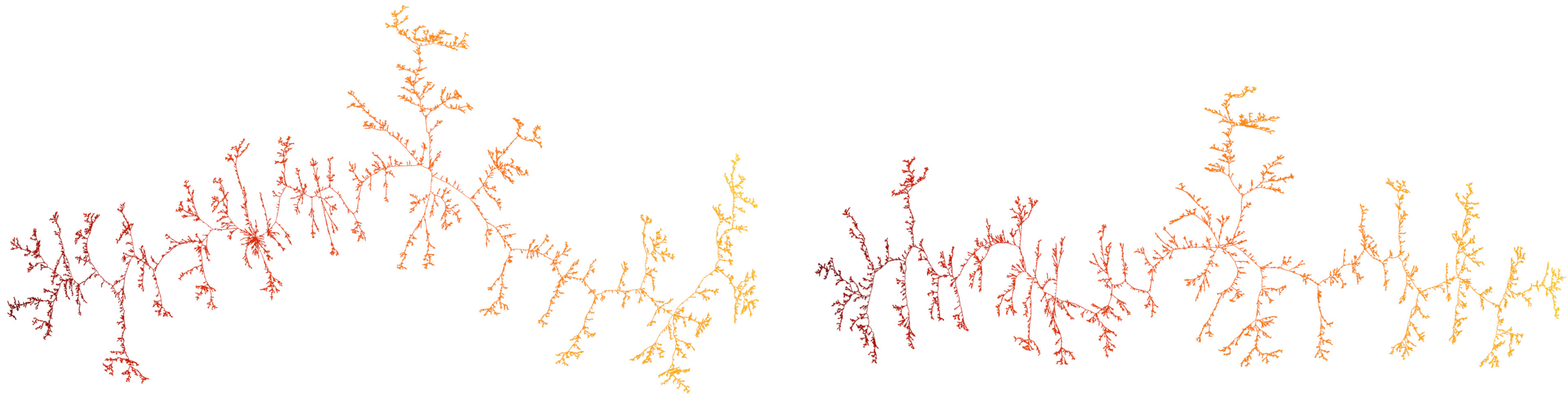
- **Combinatorial trees with weights. For example, simply generated trees**
  - $(\omega_k)_{k\geq 0}$ **sequence of non-negative weights**
  - $\mathbb{P}(\mathscr{T}_n = T) \propto \prod_{v\in T} \omega_{d_T^+(v)}$ **for T an n-vertex plane tree**
- **Associated graphs like looptrees**

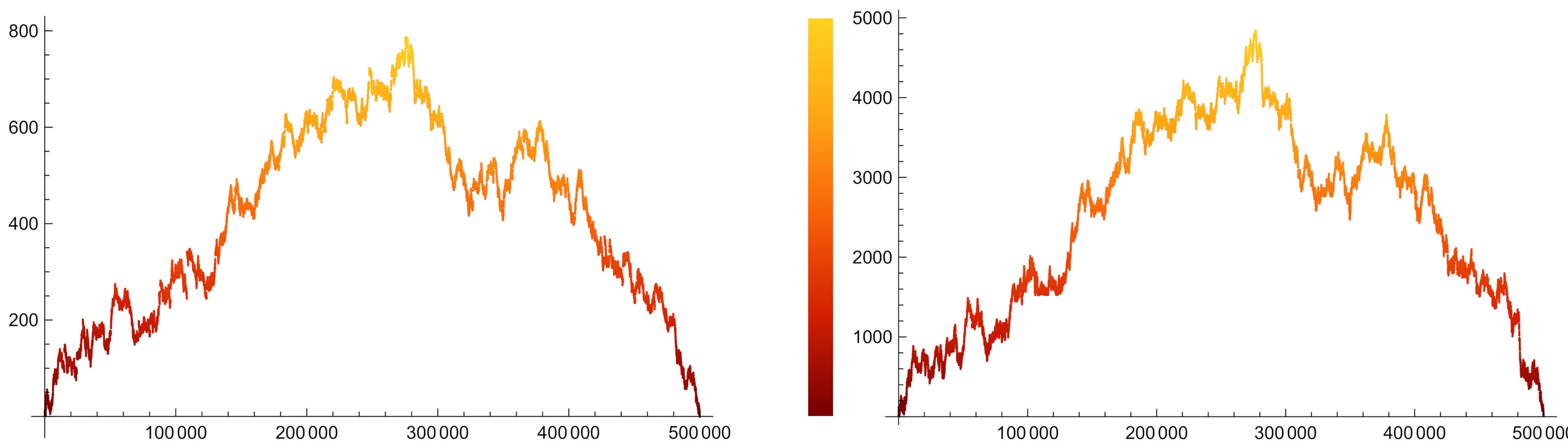


- **Associated processes:**
  - $v_1, v_2, \ldots$ **vertices in depth-first-search order**
  - $H_k =$ **height of v_k, $k = 1,2,\ldots$ the height process**
  - $W_k = \sum_{i=1}^{k} (d_T^+(v_i) - 1),\ k = 1,2,\ldots$ **the Łukaszewicz path**

INTRODUCTION: SIMPLY GENERATED TREES

N=500K, $\sum_{k\geq 0} \omega_k = 1$ (PROBABILITY WEIGHTS), $\sum_{k\geq 0} k\omega_k = 1$ (CRITICALITY), $\omega_k \sim ck^{-4}$

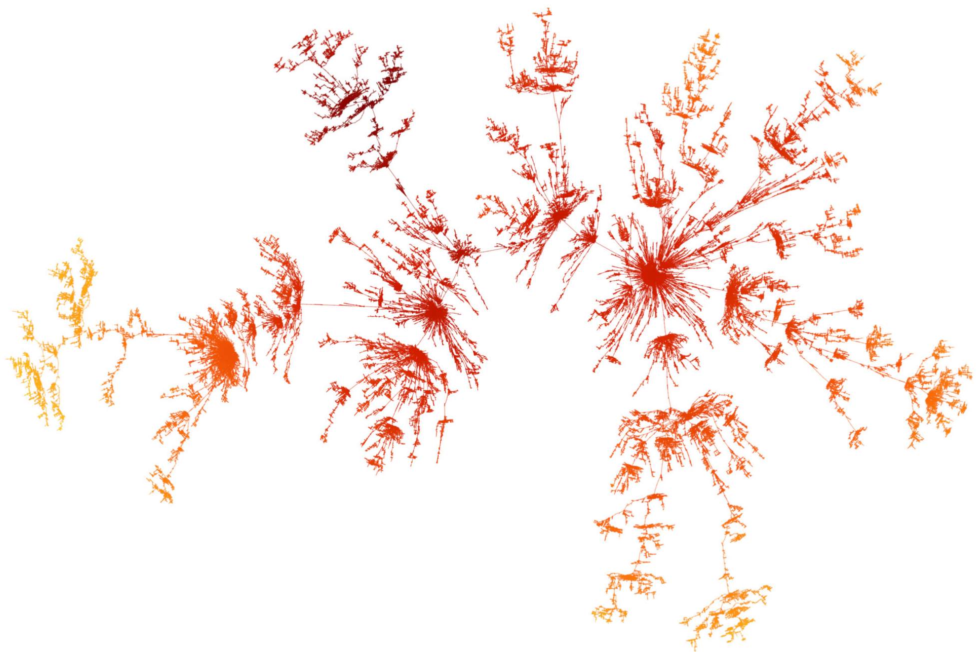


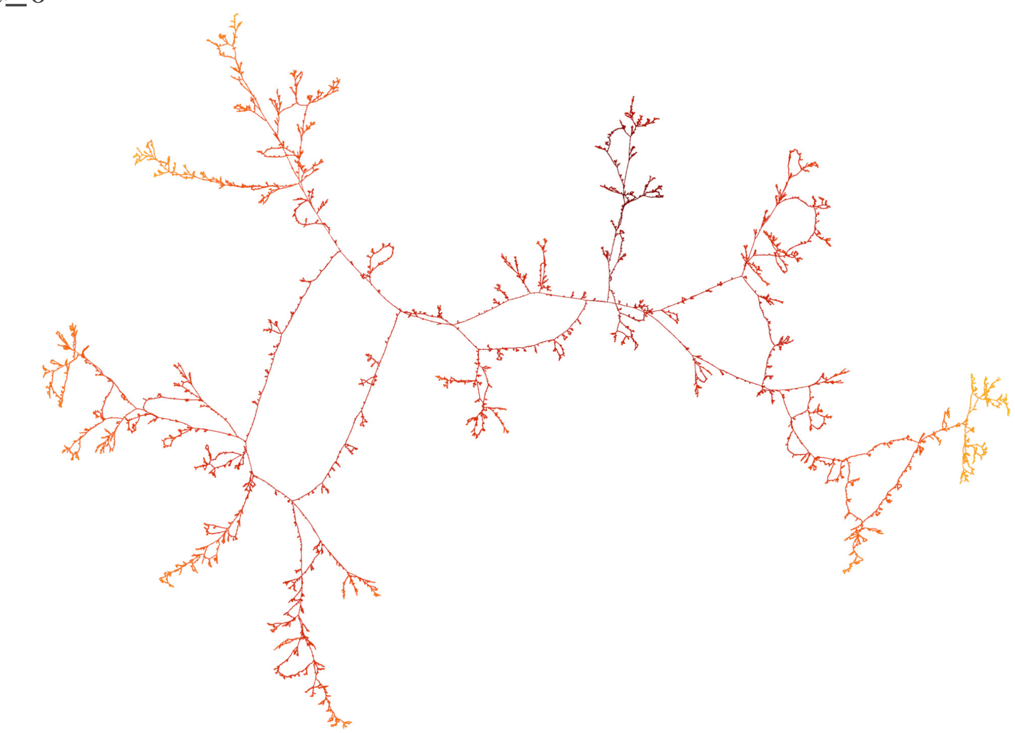


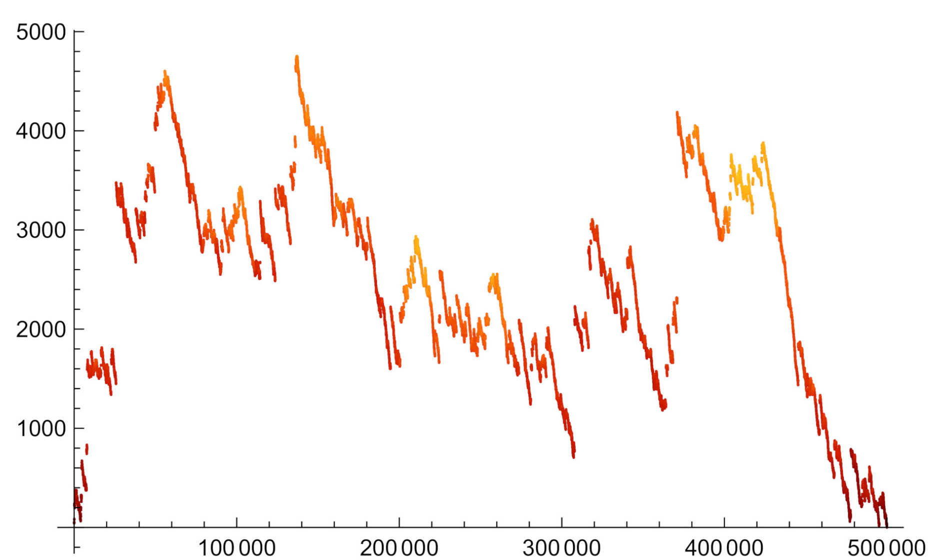
Tree	Looptree
Łukaszewicz path	Height process

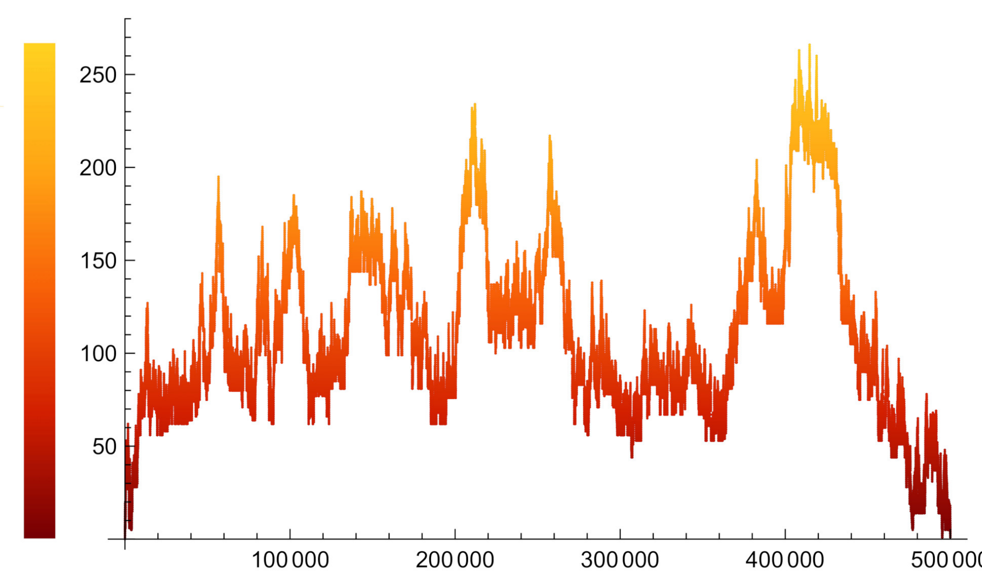
INTRODUCTION: SIMPLY GENERATED TREES

N=500K, $\sum_{k\geq 0} \omega_k = 1$ (PROBABILITY WEIGHTS), $\sum_{k\geq 0} k\omega_k = 1$ (CRITICALITY), $\omega_k \sim ck^{-2.5}$





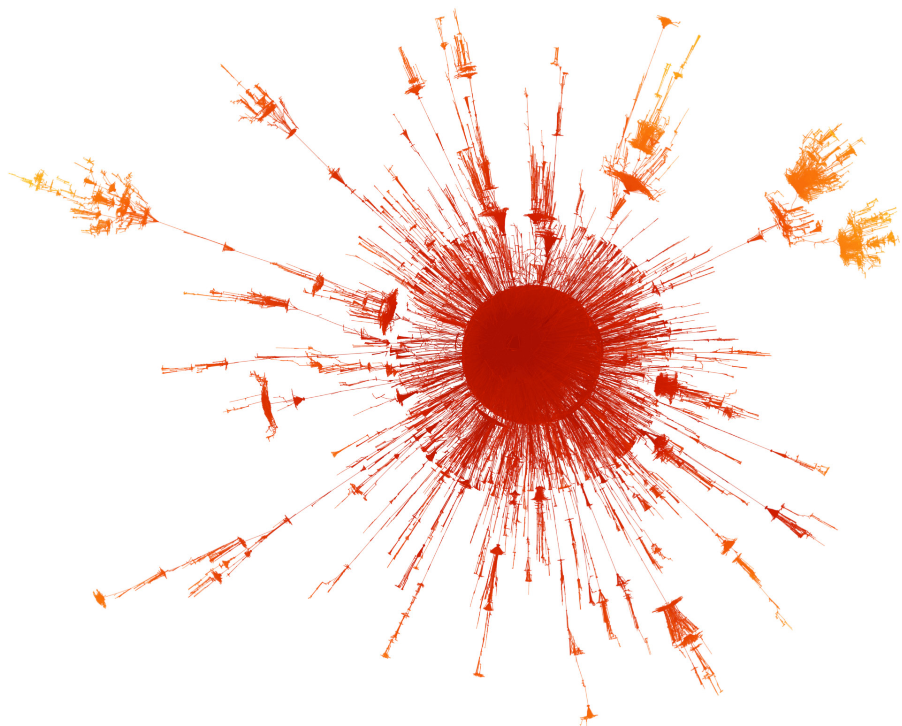


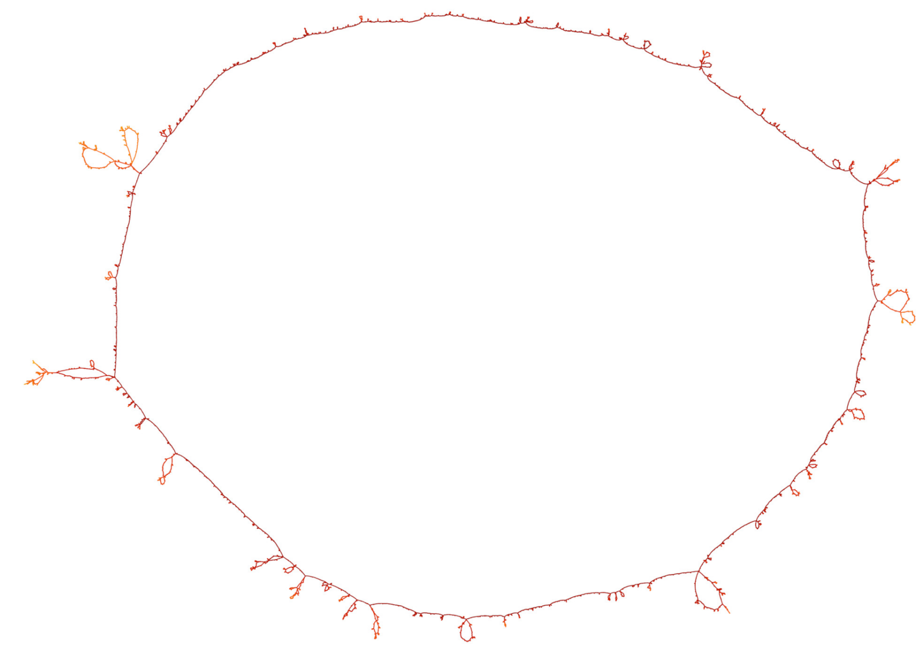


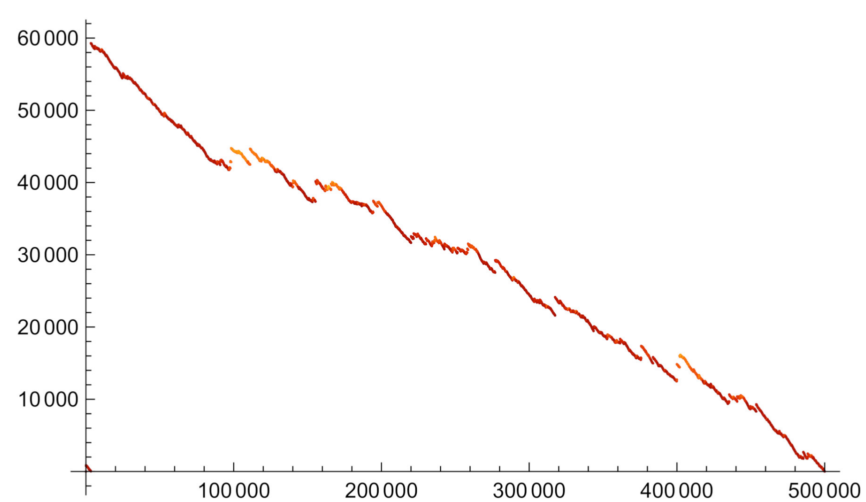
Tree	Looptree
Łukaszewicz path	Height process

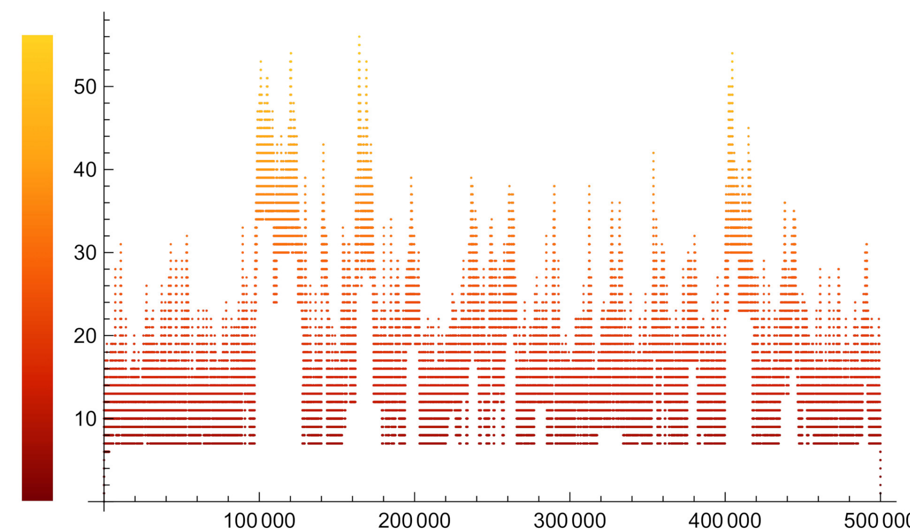
INTRODUCTION: SIMPLY GENERATED TREES

N=500K, $\sum_{k\geq 0} \omega_k = 1$ (PROBABILITY WEIGHTS), $\sum_{k\geq 0} k\omega_k = 1$ (CRITICALITY), $\omega_k \sim c k^{-2} \log^{-2}(k)$





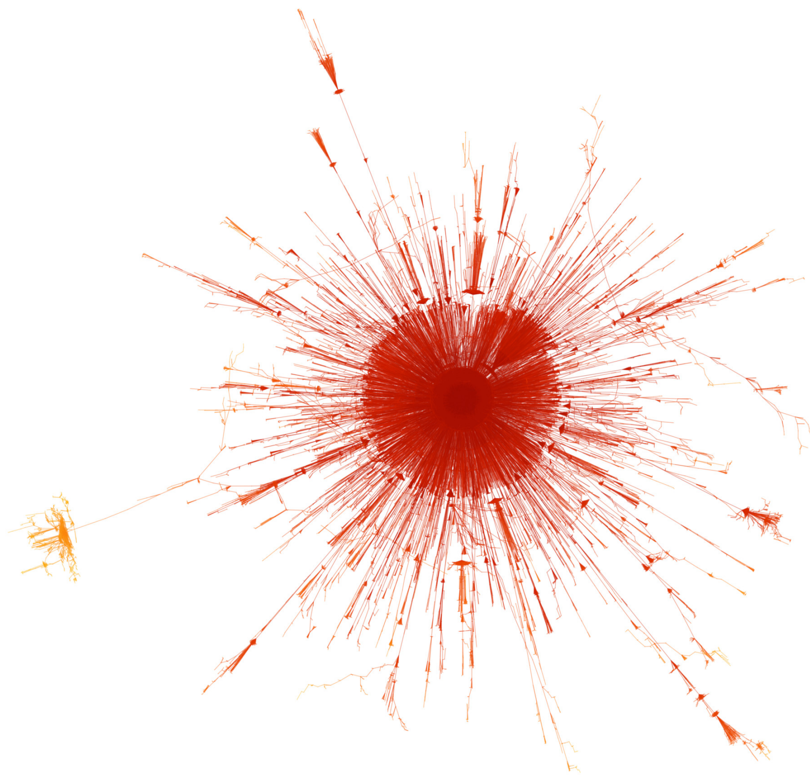


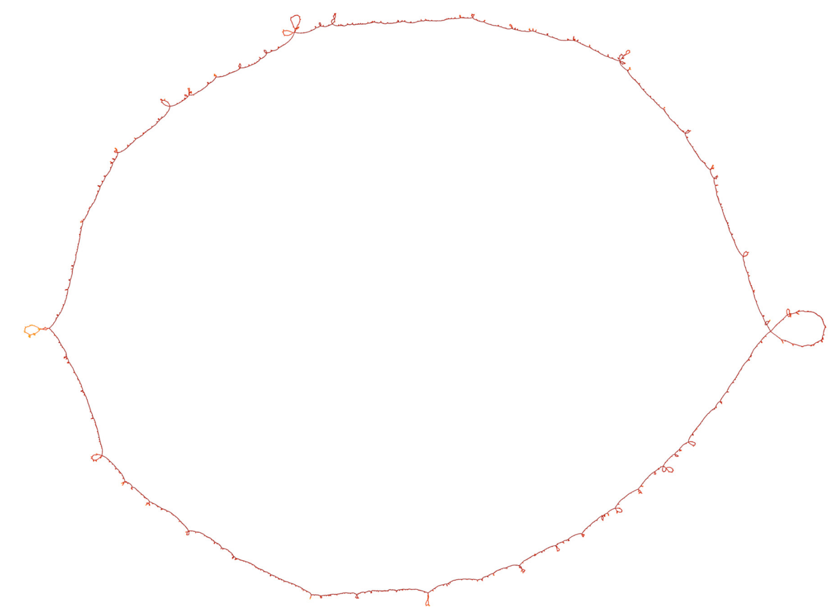


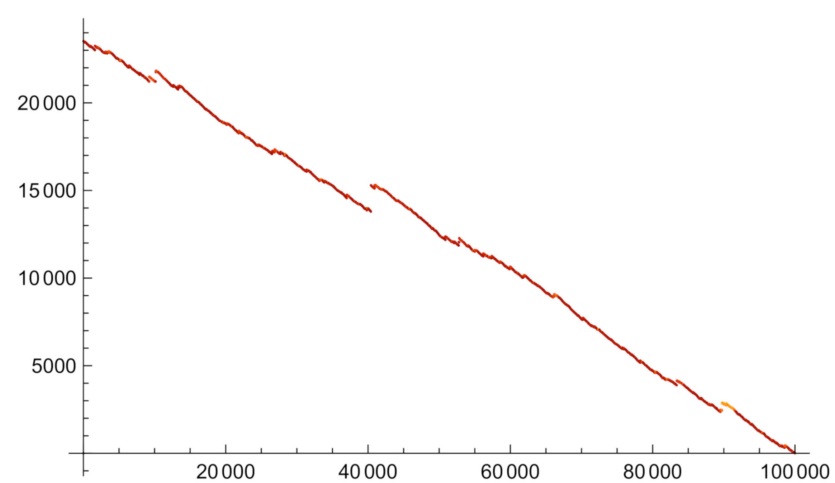
Tree	Looptree
Łukaszewicz path	Height process

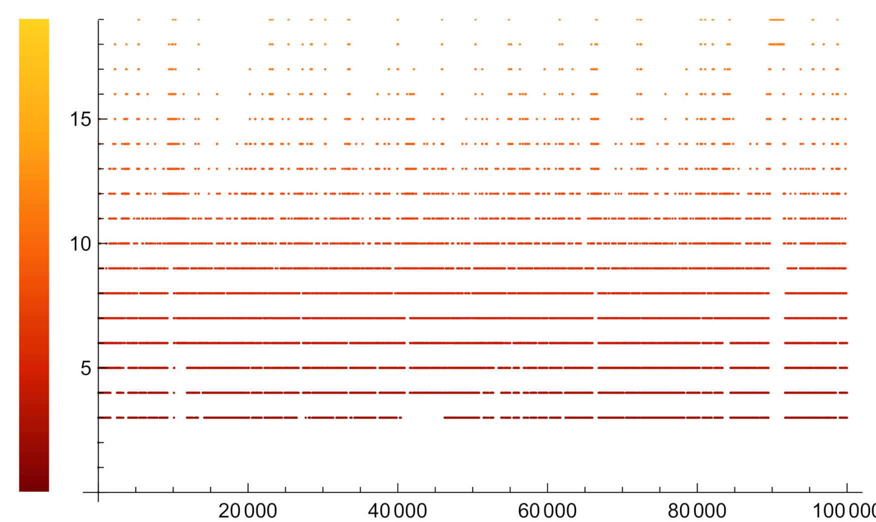
INTRODUCTION: SIMPLY GENERATED TREES

N=100K, $\sum_{k\geq 0} \omega_k = 1$ (PROBABILITY WEIGHTS), $\sum_{k\geq 0} k\omega_k = 0.75$ (SUBCRITICALITY), $\omega_k \sim ck^{-2.5}$





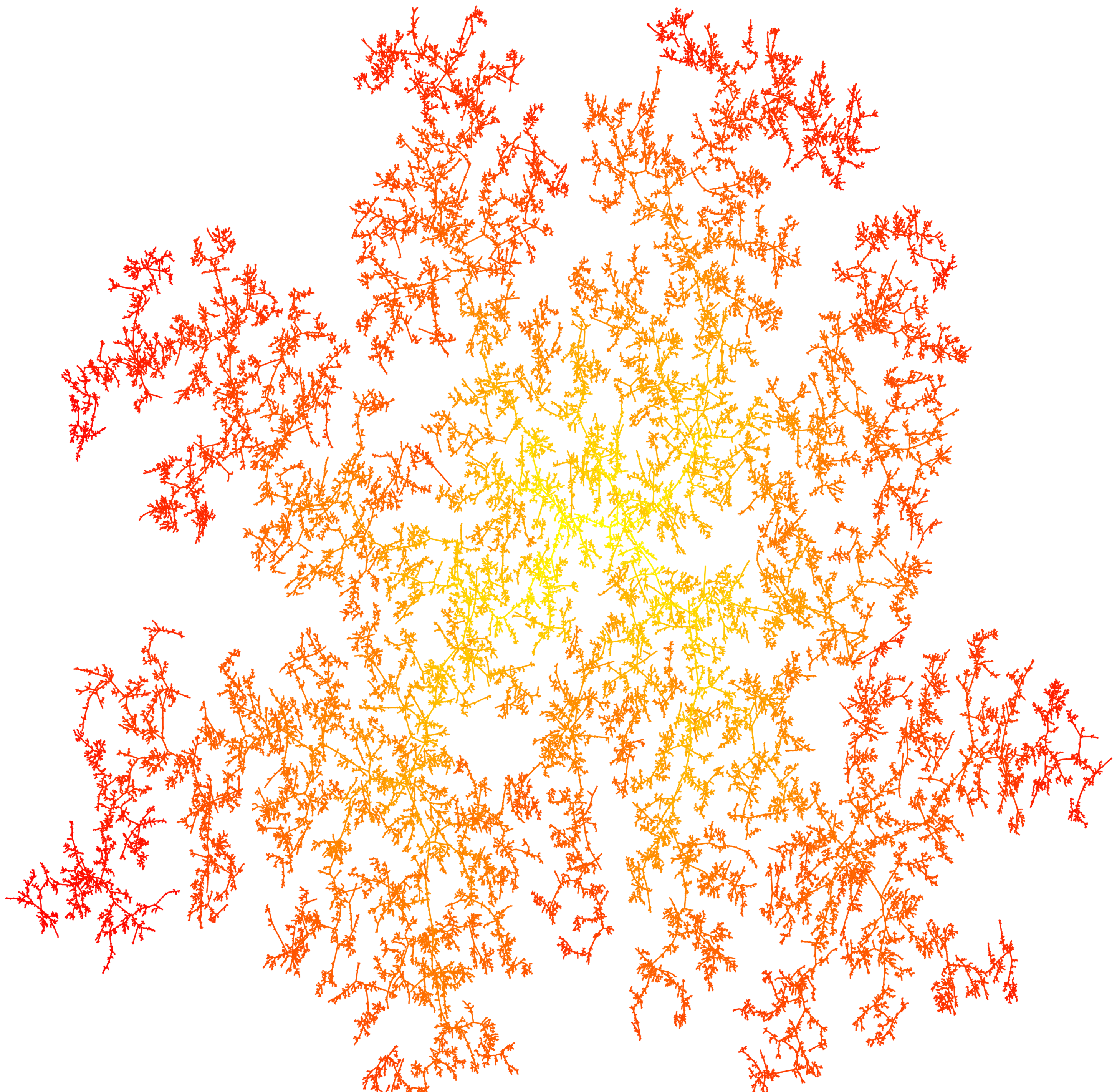




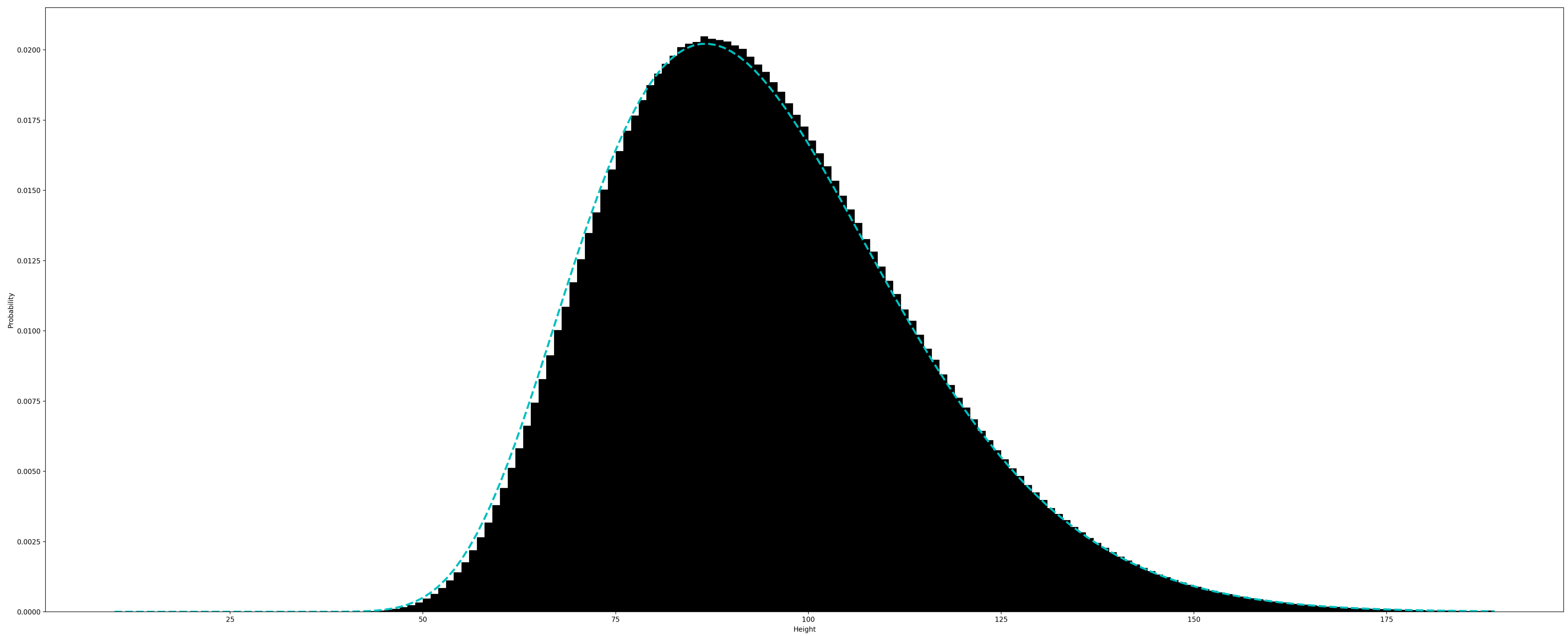
Tree	Looptree
Łukaszewicz path	Height process

UNIFORM SPANNING TREES

UNIFORM SPANNING TREE OF UNIFORM PLANAR MAP WITH 1M EDGES

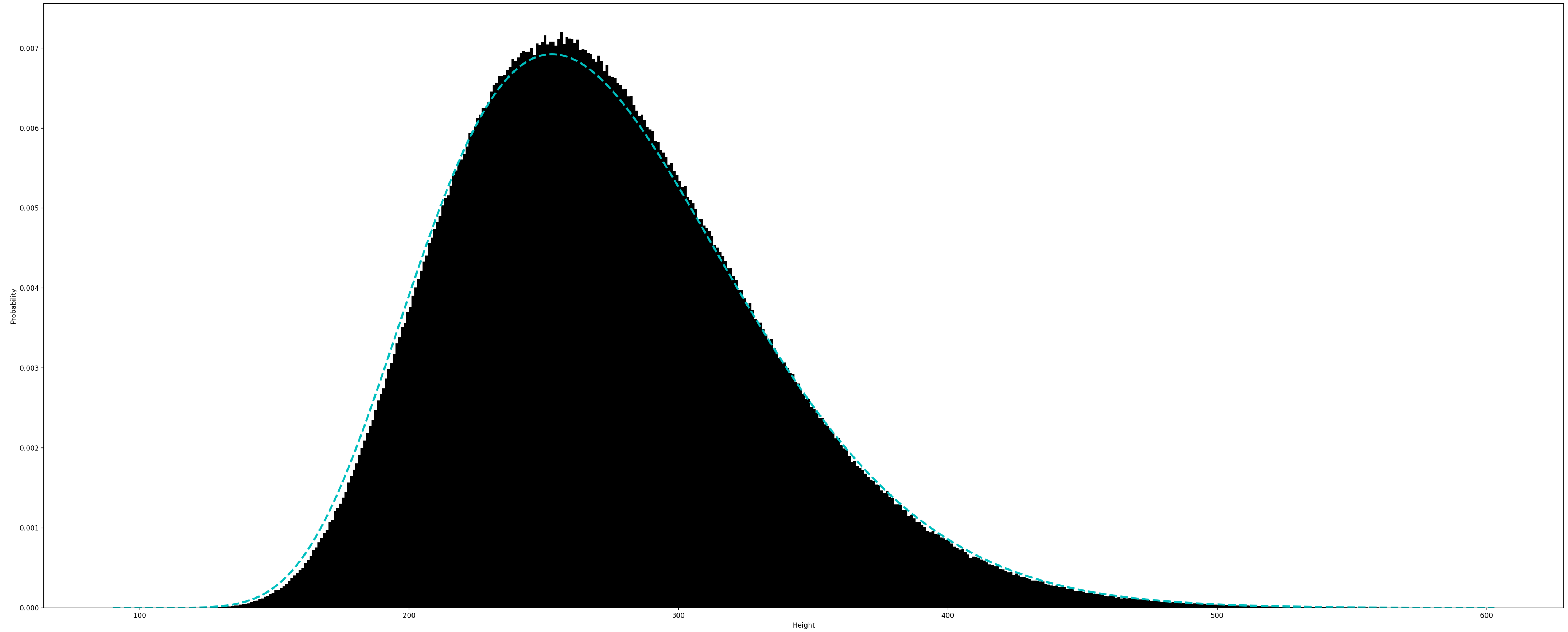


UNIFORM SPANNING TREE OF UNIFORM PLANAR MAP WITH 10K EDGES



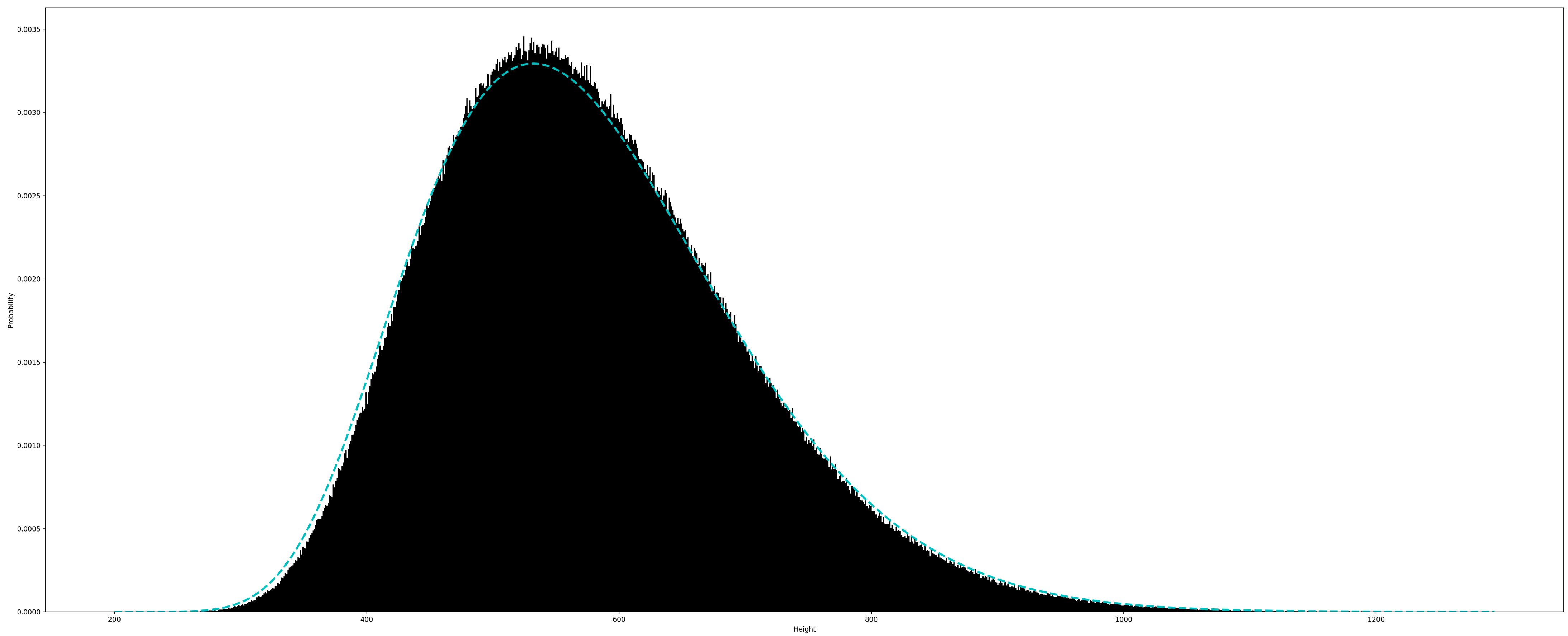
HISTOGRAM FOR THE HEIGHT

UNIFORM SPANNING TREE OF UNIFORM PLANAR MAP WITH 100K EDGES



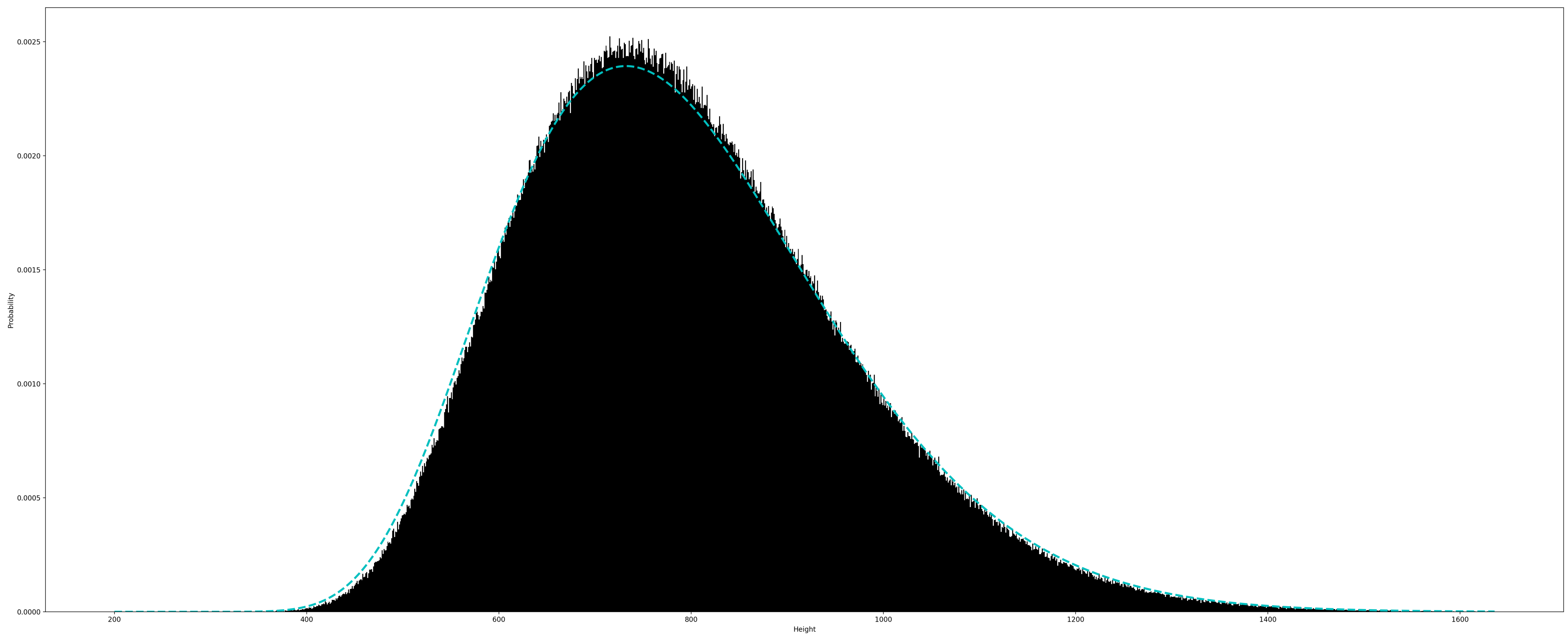
HISTOGRAM FOR THE HEIGHT

UNIFORM SPANNING TREE OF UNIFORM PLANAR MAP WITH 500K EDGES



HISTOGRAM FOR THE HEIGHT

UNIFORM SPANNING TREE OF UNIFORM PLANAR MAP WITH 1M EDGES



HISTOGRAM FOR THE HEIGHT

UNIFORM SPANNING TREE OF UNIFORM PLANAR MAPS

$h(n)$ the average height of simulations of UST of uniform planar map with n edges.

$\alpha(n) = \log(\frac{h(10n)}{h(n)})/\log n$ so that if $h(n) \sim cn^{\alpha}$ then $\alpha(n) \to \alpha$.

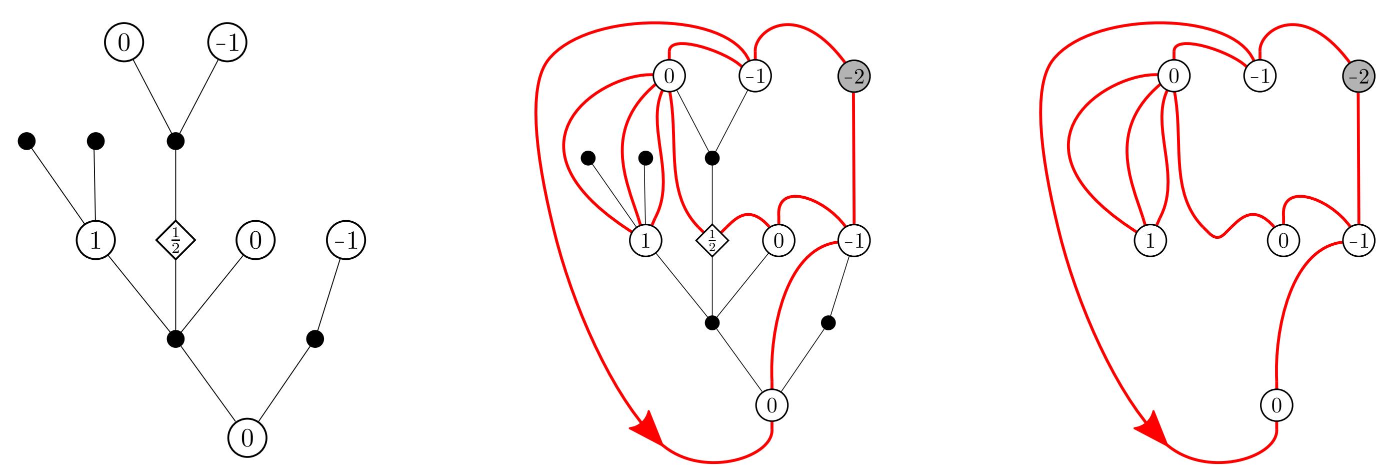
n	10^3	10^4	10^5	10^6	10^7	10^8
h(n)	31.2812	93.8020	273.9275	792.7325	2285.815	6585.556
alpha(n)	**0.4**76927	**0.4**65423	**0.4**61491	**0.459**914	**0.459**551	

Knizhnik-Polyakov-Zamolodchikov (KPZ) formula predicts: $\alpha = \frac{5-\sqrt{10}}{4} = 0.4594305...$

(Many thanks to Nathanaël Berestycki for explaining this to me)

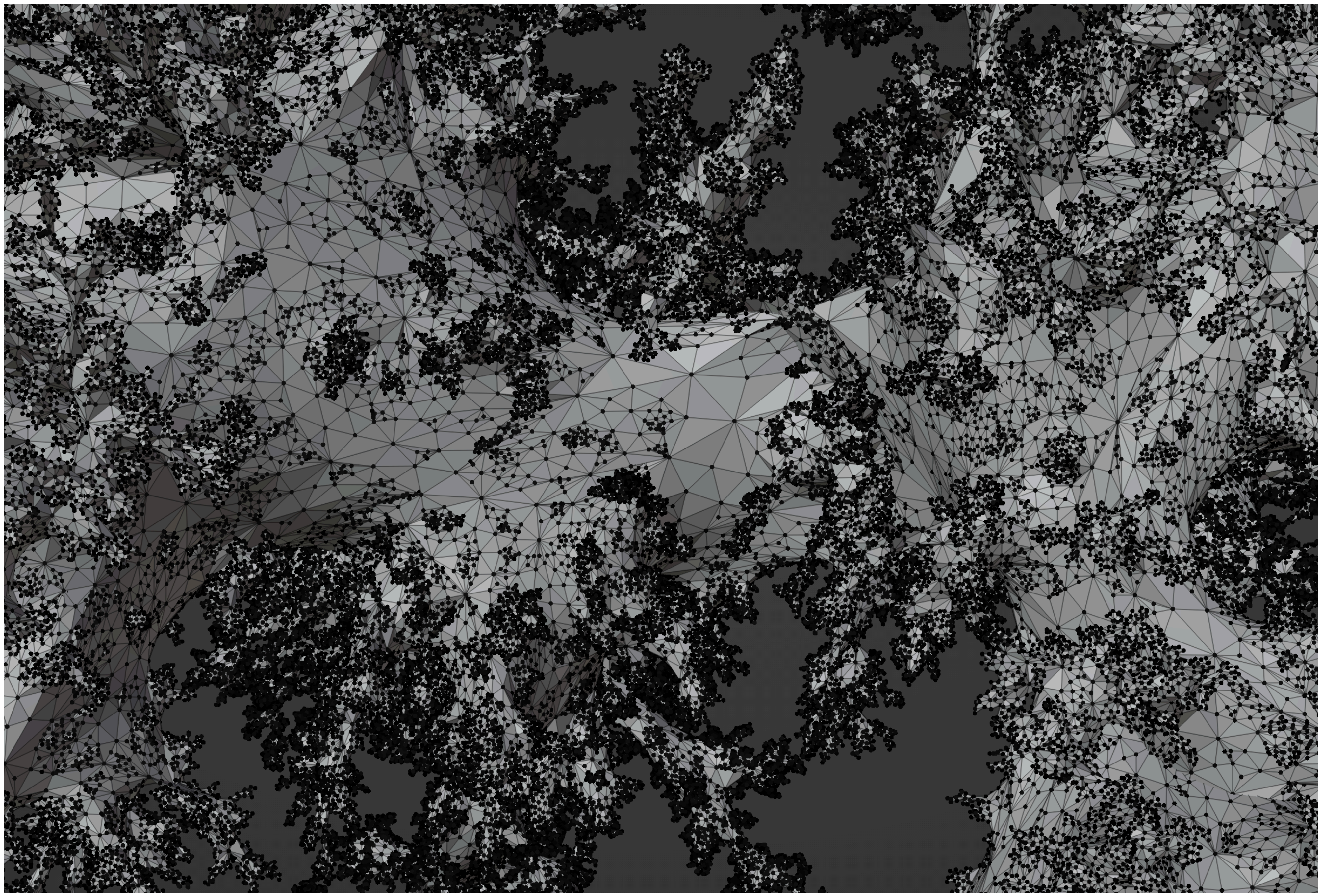
APPLICATIONS OF RANDOM TREES

- **Tree-structures are part of many algorithms in computer science**
- **Trees appear in combinatorial encodings of other discrete structures**



Bouttier-Di Francesco-Guitter bijection for planar maps

APPLICATIONS OF RANDOM TREES



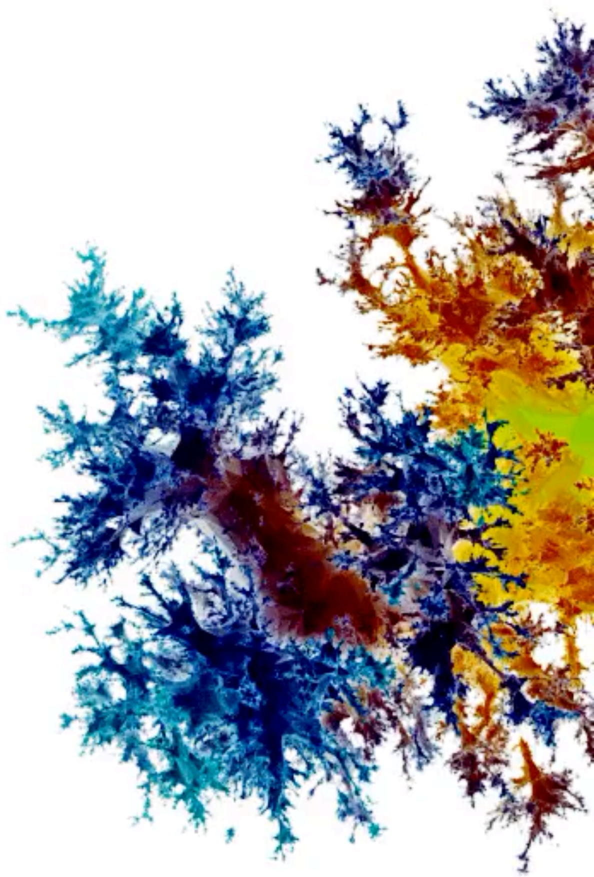
Random simple triangulation with 1M triangles

APPLICATIONS OF RANDOM TREES

Random simple triangulation with 1M triangles

Simulation: SIMTRIA (Generate SIMple TRIAngulations): http://github.com/BenediktStufler/simtria,
SCENT (Calculate closeness centrality): http://github.com/BenediktStufler/scent

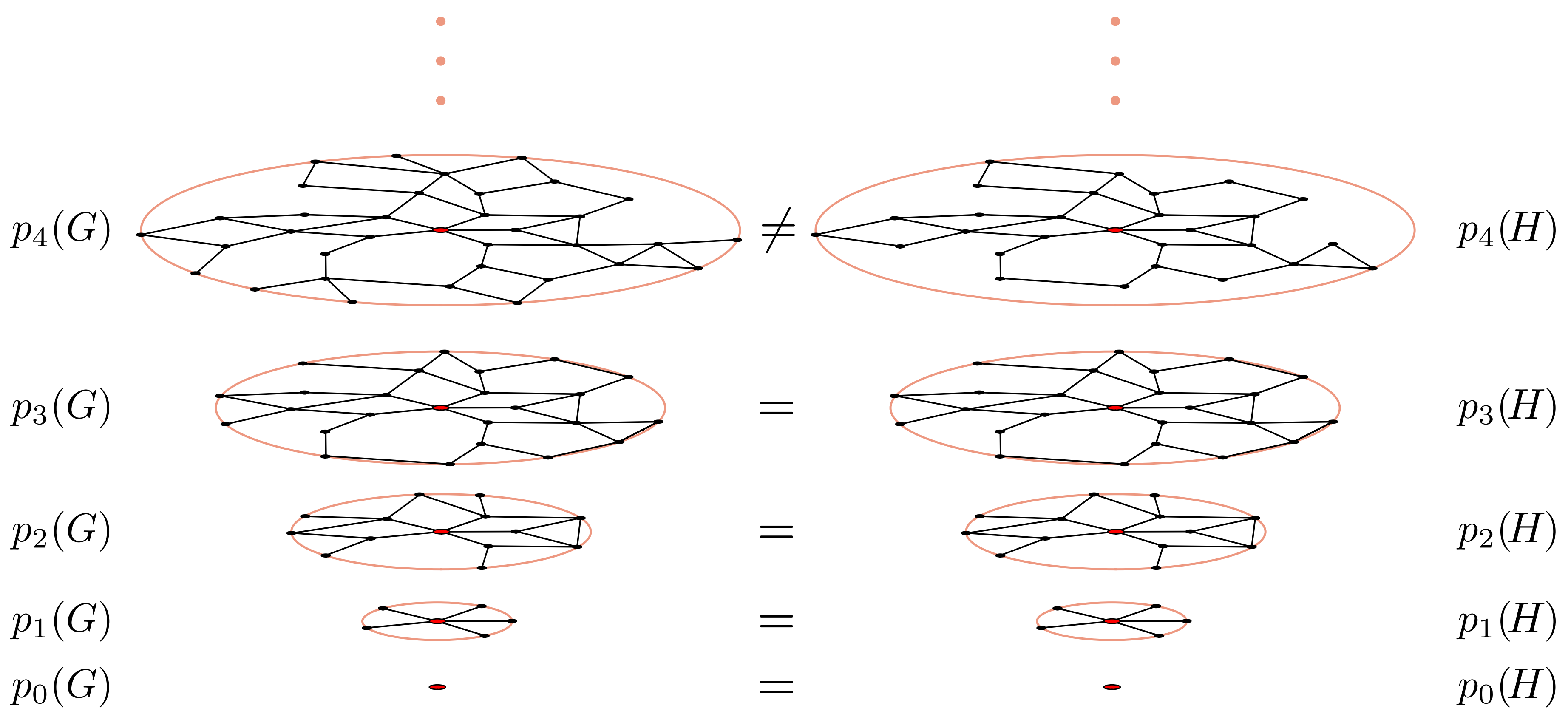
APPLICATIONS OF RANDOM TREES



LOCAL DISTANCE

$\mathfrak{M}$ = collection of vertex-rooted locally finite connected unlabelled graphs

$p_k : \mathfrak{M} \to \mathfrak{M}$ projection to k-neighbourhood of the root vertex



$$d_{\mathrm{loc}}(G,H) = \frac{1}{1+\sup\{k \in \mathbb{N}_0 \mid p_k(G) = p_k(H)\}}$$

$(\mathfrak{M}, d_{\mathrm{loc}})$ is a Polish space

LOCAL CONVERGENCE

View random trees, graphs, or maps rooted at some specified vertex as random elements of this space.

What can we say about convergence and limit objects?

Let's move to the blackboard!

Thanks for your attention.