Abstract

For a prime $p$ and nonnegative integers $j$ and $n$ let $\vartheta_p(j, n)$ be the number of entries in the $n$-th row of Pascal’s triangle that are exactly divisible by $p^j$. Moreover, for a finite sequence $w = w_{r-1} \cdots w_0 \neq 0 \cdots 0$ in $\{0, \ldots, p-1\}$ we denote by $|n|_w$ the number of times that $w$ appears as a factor (contiguous subsequence) of the base-$p$ expansion $n_{r-1} \cdots n_0$ of $n$.

It follows from the work of Barat and Grabner (Distribution of binomial coefficients and digital functions, J. London Math. Soc. (2) 64(3), 2001), that $\vartheta_p(j, n)/\vartheta_p(0, n)$ is given by a polynomial $P_j$ in the variables $X_w$, where $w$ are certain finite words in $\{0, \ldots, p-1\}$, and each variable $X_w$ is set to $|n|_w$. This was later made explicit by Rowland (The number of nonzero binomial coefficients modulo $p^\alpha$, J. Comb. Number Theory 3(1), 2011), independently from Barat and Grabner’s work, and Rowland described and implemented an algorithm computing these polynomials $P_j$.

In this paper, we express the coefficients of $P_j$ using generating functions, and we prove that these generating functions can be determined explicitly by means of a recurrence relation. Moreover, we prove that $P_j$ is uniquely determined, and we note that the proof of our main theorem also provides a new proof of its existence. Besides providing insight into the structure of the polynomials $P_j$, our results allow us to compute them in a very efficient way.