Mathematik

Pure and Hybrid Mice with Finitely Many Woodin Cardinals from Levels of Determinacy

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Für meinen Opa.

Abstract

Mice are sufficiently iterable canonical models of set theory. Martin and Steel showed in the 1980s that for every natural number n the existence of n Woodin cardinals with a measurable cardinal above them all implies that boldface $\boldsymbol{\Pi}^1_{n+1}$ determinacy holds, where $\boldsymbol{\Pi}^1_{n+1}$ is a pointclass in the projective hierarchy. Neeman and Woodin later proved an exact correspondence between mice and projective determinacy. They showed that boldface $\boldsymbol{\Pi}^1_{n+1}$ determinacy is equivalent to the fact that the mouse $M_n^\#(x)$ exists and is ω_1-iterable for all reals x.

We prove one implication of this result, that is boldface $\boldsymbol{\Pi}^1_{n+1}$ determinacy implies that $M_n^\#(x)$ exists and is ω_1-iterable for all reals x, which is an old, so far unpublished result by W. Hugh Woodin. As a consequence, we can obtain the Determinacy Transfer Theorem for all levels n.

Following this, we will consider pointclasses in the $L(\mathbb{R})$-hierarchy and show that determinacy for them implies the existence and ω_1-iterability of certain hybrid mice with finitely many Woodin cardinals, which we call $M_k^{\Sigma,\#}$. These hybrid mice are like ordinary mice, but equipped with an iteration strategy for a mouse they are containing, and they naturally appear in the core model induction technique.

Preface

The research area of Set Theory goes back to Georg Cantor, who phrased many essential questions and provided a framework for further research in a series of papers in the late 19th century. During the beginning of the 20th century Ernst Zermelo and Abraham Fraenkel developed an axiomatic system for Set Theory called ZF, that is still fundamental today. Together with the Axiom of Choice it constitutes ZFC, the central background theory for modern Mathematics and Set Theory.

Already in 1878 Georg Cantor asked the very natural question of whether there is a set of size strictly between that of the natural numbers and that of the continuum. This question, known as the *Continuum Problem*, motivated lots of research in Set Theory since then and continues to do so, for example in cardinal arithmetic. It was proved to be independent of ZFC by work of Kurt Gödel on his model L in 1940 and Paul Cohen's development of *forcing* in 1963 and hence is an example which shows that the axiomatic system ZFC is not sufficient to solve all interesting questions about sets.

Therefore one of the main goals of research in Set Theory can be phrased as the *search for the "right" axioms for mathematics.* This means in particular studying various extensions of ZFC and their properties. The work in this thesis is related to this overall goal, because we study different extensions of ZFC and their relationships.

In 1953 David Gale and Frank M. Stewart developed a basic theory of infinite games in [GS53]. For every set of reals, that means set of sequences of natural numbers, A they considered a two-player game $G(A)$ of length ω, where player I and player II alternately play natural numbers. They defined that player I wins the game $G(A)$ if and only if the sequence of natural numbers produced during a run of the game $G(A)$ is contained in A and otherwise player II wins. Moreover the game $G(A)$ or the set A itself is called determined if and only if one of the two players has a winning strategy, that means a method by which they can win, no matter what their opponent does, in the game described above.

Already in [GS53] the authors were able to prove that every open and every closed set of reals is determined using ZFC. But furthermore they proved that determinacy for all sets of reals contradicts the Axiom of Choice. This leads to the natural question of what the picture looks like for definable sets of reals which are more complicated than open and closed sets. After some partial results by Philip Wolfe in [Wo55] and Morton Davis in [Da64], Donald A. Martin was finally able to prove in [Ma75] from ZFC that every Borel set of reals is determined.

In the meantime the development of so called *Large Cardinal Axioms* was proceeding in Set Theory. In 1930 Stanisław Ulam first defined measurable cardinals and at the beginning of the 1960's H. Jerome Keisler [Kei62] and Dana S. Scott [Sc61] found a way of making them more useful in Set Theory by reformulating the statements using elementary embeddings.

About the same time, other set theorists were able to prove that *Determinacy Axioms* imply *regularity properties* for sets of reals. More precisely Banach and Mazur showed that under the Axiom of Determinacy (that means every set of reals is determined), every set of reals has the Baire property. Mycielski and Swierczkowski proved in [MySw64] that under the same hypothesis every set of reals is Lebesgue measurable. Furthermore Davis showed in [Da64] that under this hypothesis every set of reals has the perfect set property. Moreover all three results also hold if the Determinacy Axioms and regularity properties are only considered for sets of reals in certain pointclasses. This shows that Determinacy Axioms have a large influence on the structure of sets of reals and therefore have a lot of applications in Set Theory.

In 1965 Robert M. Solovay was able to prove these regularity properties for a specific pointclass, namely $\mathbf{\Sigma}^1_2$, assuming the existence of a measurable cardinal instead of a Determinacy Axiom (see for example [So69] for the perfect set property). Then finally Donald A. Martin was able to prove a direct connection between Large Cardinals and Determinacy Axioms: he showed in 1970 that the existence of a measurable cardinal implies determinacy for every analytic set of reals (see [Ma70]).

Eight years later Leo A. Harrington established that this result is in some sense optimal. In [Ha78] he proved that determinacy for all analytic sets implies that $0^\#$, a countable active mouse which can be obtained from a measurable cardinal, exists. Here a mouse is a fine structural, iterable model. Together with Martin's result mentioned above this yields that the two statements are in fact equivalent.

This of course motivates the question of whether a similar result can be obtained for larger sets of reals, so especially for determinacy in the projective hierarchy. The right large cardinal notion to consider for these sets of reals was introduced by W. Hugh Woodin in 1984 and is nowadays called a Woodin cardinal. Building on this, Donald A. Martin and John R. Steel were able to prove in [MaSt89] almost twenty years after Martin's result about analytic determinacy that, assuming the existence of n Woodin cardinals and a measurable cardinal above them all, every $\mathbf{\Sigma}^1_{n+1}$-definable set of reals is determined.

In the meantime the theory of mice was further developed. At the level of strong cardinals it goes back to Ronald B. Jensen, Robert M. Solovay, Tony Dodd and Ronald B. Jensen, and William J. Mitchell. Then it was further extended to the level of Woodin cardinals by Donald A. Martin and John R. Steel in [MaSt94] and William J. Mitchell and John R. Steel in [MS94], where some errors were later repaired by Ralf Schindler, John R. Steel and Martin Zeman in [SchStZe02]. Moreover Ronald B. Jensen developed another approach to the theory of mice at the level of Woodin cardinals in [Je97].

In 1995 Itay Neeman was able to improve the result from [MaSt89] in [Ne95]. He showed that the existence and ω_1-iterability of $M_n^\#$, the minimal countable active mouse at the level of n Woodin cardinals, is enough to obtain that every $⅁^n(< \omega^2 - \Pi^1_1)$-definable set of reals is determined. Here "$⅁$" is a quantifier which is defined by a game. Neeman's result implies that if for all reals x the premouse $M_n^\#(x)$ exists and is ω_1-iterable, then in particular every $\boldsymbol{\Sigma}^1_{n+1}$-definable set of reals is determined. For odd n this latter result was previously known by Woodin. The converse of this latter result was announced by W. Hugh Woodin in the 1980's but a proof has never been published. The goal of the first part of this thesis will be to finally provide a complete proof of his result.

The next obvious question is if these results can be lifted to even bigger pointclasses. Natural candidates for these bigger pointclasses are pointclasses within the $L(\mathbb{R})$-hierarchy. Building on work of Kechris, Martin, Moschovakis and Solovay and on his own prior work, John R. Steel provided in [St83] the first detailed analysis of the scale property at levels of the $L(\mathbb{R})$-hierarchy using the fine structure of that hierarchy.

Starting from this, W. Hugh Woodin discovered in 1991 a method called the *core model induction* where he was able to produce mice with finitely many Woodin cardinals from different hypotheses. The idea is to prove at successor steps that the universe is closed under certain mouse operators, for example $x \mapsto M_n^\#(x)$. It turns out that during such a core model induction at levels in the $L(\mathbb{R})$-hierarchy beyond the projective hierarchy certain *hybrid mice* appear. Here a hybrid mouse is a mouse which contains another mouse and also an iteration strategy for that mouse.

Using this method it is proved in Section 6.2 of [SchSt] that, assuming determinacy for all sets of reals in $L(\mathbb{R})$, certain hybrid mice with finitely many Woodin cardinals exist. But this result is not local: that means to prove the existence of a particular hybrid mouse it uses the hypothesis that *every* set of reals in $L(\mathbb{R})$ is determined instead of a determinacy hypothesis for a level of the $L(\mathbb{R})$-hierarchy.

The second part of this thesis is devoted to a generalization of arguments from the first part to the context of hybrid mice to obtain a more local version of the result mentioned in the previous paragraph. We will show that assuming determinacy for sets of reals at a certain level of the $L(\mathbb{R})$-hierarchy is enough to obtain hybrid mice which capture sets of reals from a lower level of the $L(\mathbb{R})$-hierarchy. More precisely, the hybrid mice we are going to construct will be of the form $M_k^{\Sigma,\#}$, that means they will be ω_1-iterable hybrid mice with finitely many Woodin cardinals.

This motivates further research in this direction to analyze the inner model theory at the individual levels of the $L(\mathbb{R})$-hierarchy in more detail to possibly obtain an exact correspondence between the existence of mice and determinacy for sets of reals at levels of the $L(\mathbb{R})$-hierarchy.

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Contents

Part 1

The Projective Case

Overview

The purpose of this first part of the thesis is to give a proof of the following theorem, which connects inner models with Woodin cardinals and descriptive set theory at projective levels in a direct level-wise way. This theorem is due to W. Hugh Woodin and announced for example in the addendum (§5) of [KW08] and in Theorem 5.3 of [Sch10], but so far a proof of this result has never been published.

Theorem 2.1.1. *Let $n \geq 1$ and assume $\mathbf{\Pi}^1_{n+1}$ determinacy holds. Then $M_n^\#(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$.*

The converse of Theorem 2.1.1 also holds true and is for odd n due to W. Hugh Woodin (unpublished) and for even n due to Itay Neeman (see [Ne95]). From this we can obtain the following corollary, which is Theorem 1.10 in [KW08] for odd n. We will present a proof of the Determinacy Transfer Theorem for the even levels n as a corollary of Theorem 2.1.1 in Section 4.1, using Theorem 2.5 in [Ne95] due to Itay Neeman.

Corollary 4.1.1 (Determinacy Transfer Theorem). *Let $n \geq 1$. Then $\mathbf{\Pi}^1_{n+1}$ determinacy is equivalent to $\Game^{(n)}(< \omega^2 - \mathbf{\Pi}^1_1)$ determinacy.*

In fact we are going to prove the following theorem which will imply Theorem 2.1.1.

Theorem 3.3.2. *Let $n \geq 1$ and assume there is no $\mathbf{\Sigma}^1_{n+2}$-definable ω_1-sequence of pairwise distinct reals. Moreover assume that $\mathbf{\Pi}^1_n$ determinacy and Π^1_{n+1} determinacy hold. Then $M_n^\#$ exists and is ω_1-iterable.*

This is a part of the following theorem.

Theorem 3.3.1. *Let $n \geq 1$ and assume there is no $\mathbf{\Sigma}^1_{n+2}$-definable ω_1-sequence of pairwise distinct reals. Then the following are equivalent.*

(1) $\mathbf{\Pi}^1_n$ determinacy and Π^1_{n+1} determinacy,
(2) for all $x \in {}^\omega\omega$, $M_{n-1}^\#(x)$ exists and is ω_1-iterable, and $M_n^\#$ exists and is ω_1-iterable,
(3) $M_n^\#$ exists and is ω_1-iterable.

Here the direction (3) implies (1) follows from Theorem 2.14 in [Ne02] and is due to W. Hugh Woodin for odd n (unpublished) and due to Itay Neeman for even n. Moreover the direction (2) implies (1) and the equivalence of (2) and (3) as proven in [Ne02] do not need the background hypothesis that there is no $\boldsymbol{\Sigma}^1_{n+2}$-definable ω_1-sequence of pairwise distinct reals.

Furthermore we get the following relativized version of Theorem 3.3.1.

COROLLARY 2.1.2. *Let $n \geq 1$. Then the following are equivalent.*

(1) $\boldsymbol{\Pi}^1_{n+1}$ determinacy, and

(2) for all $x \in {}^\omega\omega$, $M_n^\#(x)$ exists and is ω_1-iterable.

This gives that $\boldsymbol{\Pi}^1_{n+1}$ determinacy is an optimal hypothesis for proving the existence and ω_1-iterability of $M_n^\#(x)$ for all $x \in {}^\omega\omega$.

REMARK. In contrast to the statement of Theorem 1.4 in [Ne02] it is open if $\boldsymbol{\Pi}^1_n$ determinacy and Π^1_{n+1} determinacy alone imply the existence of an ω_1-iterable $M_n^\#$ for $n > 1$ (see also Section 4.2). Whenever we are citing [Ne02] in this thesis we make no use of any consequence of this result stated there.

Outline. This part of the thesis is organized as follows. In Chapter 1 we give a short introduction to determinacy and inner model theory. In particular we state some known results concerning the connection of determinacy for certain sets of reals and the existence of mice with large cardinals.

In Chapter 2 we will construct a proper class inner model with n Woodin cardinals from $\boldsymbol{\Pi}^1_n$ determinacy and Π^1_{n+1} determinacy. For that purpose we will prove in Lemma 2.1.3 from the same determinacy hypothesis that for a cone of reals x the premouse $M_{n-1}(x)|\delta_x$ is a model of OD-determinacy, where δ_x denotes the least Woodin cardinal in $M_{n-1}(x)$. This generalizes a theorem of Kechris and Solovay to the context of mice (see Theorem 3.1 in [KS85]).

Afterwards we will prove in Chapter 3 that, assuming $\boldsymbol{\Pi}^1_{n+1}$ determinacy, there is in fact an ω_1-iterable model which has n Woodin cardinals. More precisely, we will prove under this hypothesis that $M_n^\#(x)$ exists and is ω_1-iterable for all reals x. The proof of this result divides into different steps. In Sections 3.1 and 3.2 we will introduce the concept of n-suitable premice and show that if n is odd, using the results in Chapter 2, such n-suitable premice exist assuming $\boldsymbol{\Pi}^1_n$ determinacy and Π^1_{n+1} determinacy. The rest of Chapter 3 will also be divided into different cases depending if n is even or odd.

Section 3.4 serves as a “warm up” for the rest of the chapter. There we will prove that the premouse $M_{2n-1}^\#$ exists and is ω_1-iterable from a slightly stonger hypothesis than necessary. Namely we will assume that there is no $\boldsymbol{\Sigma}^1_{2n+2}$-definable ω_1-sequence of pairwise distinct reals in addition to $\boldsymbol{\Pi}^1_{2n-1}$ determinacy and Π^1_{2n} determinacy to prove that $M_{2n-1}^\#$ exists and is ω_1-iterable. In Section 3.5 we will show that $\boldsymbol{\Pi}^1_{n+1}$ determinacy already implies that every $\boldsymbol{\Sigma}^1_{n+2}$-definable sequence of pairwise distinct reals is countable.

Then we will show in Section 3.6 that the hypothesis that there is no $\mathbf{\Sigma}^1_{2n+1}$-definable ω_1-sequence of pairwise distinct reals in addition to $\mathbf{\Pi}^1_{2n-1}$ determinacy and Π^1_{2n} determinacy suffices to prove that $M^{\#}_{2n-1}$ exists and is ω_1-iterable. This finishes the proof of Theorem 2.1.1 for odd n.

In Section 3.7 we will finally prove the analogous result for even n, that means we will show that if every $\mathbf{\Sigma}^1_{2n+2}$-definable sequence of pairwise distinct reals is countable and $\mathbf{\Pi}^1_{2n}$ determinacy and Π^1_{2n+1} determinacy hold, then $M^{\#}_{2n}$ exists and is ω_1-iterable. The proof is different for the odd and even levels of the projective hierarchy because of the periodicity in terms of uniformization and correctness, but we will prove the odd and even levels simultaneously by an induction.

We close this part of this thesis with proving the Determinacy Transfer Theorem for even n as an application and mentioning related open questions in Chapter 4.

CHAPTER 1

Introduction

In this chapter we will introduce some relevant notions such as games and mice and their basic properties. In particular we will summarize some known results about the connection between large cardinals and the determinacy of certain games. Then we will have a closer look at mice with finitely many Woodin cardinals and introduce the premouse $M_n^\#$.

1.1. Games and Determinacy

Throughout this thesis we will consider games in the sense of [GS53] if not specified otherwise. We will always identifiy ${}^\omega 2$, ${}^\omega\omega$, and $\mathbb{R}$ with each other, so that we can define Gale-Stewart games as follows.

Definition 1.1.1 (Gale, Stewart). *Let $A \subseteq \mathbb{R}$. By $G(A)$ we denote the following game.*

$$\begin{array}{c|ccccc} \mathrm{I} & i_0 & & i_2 & & \dots \\ \hline \mathrm{II} & & i_1 & & i_3 & & \dots \end{array} \qquad \textit{for } i_n \in \{0,1\} \textit{ and } n \in \omega.$$

We say player I *wins the game $G(A)$ iff $(i_n)_{n<\omega} \in A$. Otherwise we say player* II *wins.*

Definition 1.1.2. *Let $A \subseteq \mathbb{R}$. We say $G(A)$ (or the set A itself) is* determined *iff one of the players has a winning strategy in the game $G(A)$ (in the obvious sense).*

Some famous results concerning the question which sets of reals are determined are the following. The first three theorems can be proven in ZFC.

Theorem 1.1.3 (Gale, Stewart in [GS53]). *Let $A \subset \mathbb{R}$ be open or closed and assume the Axiom of Choice. Then $G(A)$ is determined.*

Theorem 1.1.4 (Gale, Stewart in [GS53]). *Assuming the Axiom of Choice there is a set of reals which is not determined.*

Theorem 1.1.5 (Martin in [Ma75]). *Let $A \subset \mathbb{R}$ be a Borel set and assume the Axiom of Choice. Then $G(A)$ is determined.*

To prove stronger forms of determinacy we need to assume for example large cardinal axioms.

THEOREM 1.1.6 (Martin in [Ma70]). *Assume* ZFC *and that there is a measurable cardinal. Let* $A \subseteq \mathbb{R}$ *be an analytic, i.e.* $\mathbf{\Sigma}^1_1$*-definable, set. Then* $G(A)$ *is determined.*

Determinacy in the projective hierarchy can be obtained from finitely many Woodin cardinals which were introduced by W. Hugh Woodin in 1984 and are defined as follows.

DEFINITION 1.1.7. *Let* $\kappa < \delta$ *be cardinals and* $A \subseteq \delta$*. Then* κ *is called* A-reflecting in δ *iff for every* $\lambda < \delta$ *there exists a transitive model of set theory* M *and an elementary embedding* $\pi : V \rightarrow M$ *with critical point* κ*, such that* $\pi(\kappa) > \lambda$ *and*

$$\pi(A) \cap \lambda = A \cap \lambda.$$

DEFINITION 1.1.8. *A cardinal* δ *is called a* Woodin cardinal *iff for all* $A \subseteq \delta$ *there is a cardinal* $\kappa < \delta$ *which is* A*-reflecting in* δ*.*

THEOREM 1.1.9 (Martin, Steel in [MaSt89]). *Let* $n \geq 1$*. Assume* ZFC *and that there are* n *Woodin cardinals with a measurable cardinal above them all. Then every* $\mathbf{\Sigma}^1_{n+1}$*-definable set of reals is determined.*

See for example Chapter 13 in [Sch14] or Section 5 in [Ne10] for modern write-ups of the proof of Theorem 1.1.9.

The existence of infinitely many Woodin cardinals with a measurable cardinal above them all yields a much stronger form of determinacy.

THEOREM 1.1.10 (Woodin in [KW10]). *Assume* ZFC *and that there are* ω *Woodin cardinals with a measurable cardinal above them all. Then every set of reals in* $L(\mathbb{R})$ *is determined.*

For a definition of the model $L(\mathbb{R})$ see Definition 8.1.1 in the second part of this thesis.

The goal of this thesis is to prove results in the converse direction. That means we want to obtain large cardinal strength from determinacy axioms. This is done using inner model theoretic concepts which we start to introduce in the next section.

1.2. Inner Model Theory

The most important concept in inner model theory is a mouse. Therefore we briefly review the definition of mice in this section and mention some relevant properties without proving them. The reader who is interested in a more detailed introduction to mice is referred to Section 2 of [St10].

We assume that the reader is familiar with some fine structure theory as expounded for example in [MS94] or [SchZe10].

In general the models we are interested in are of the form $L[\vec{E}]$ for some coherent sequence of extenders $\vec{E}$. This notion goes back to Ronald B.

Jensen and William J. Mitchell and is made more precise in the following definition.

DEFINITION 1.2.1. *We say M is a* potential premouse *iff*

$$M = (J_\eta^{\vec{E}}, \in, \vec{E} \restriction \eta, E_\eta)$$

for some fine extender sequence $\vec{E}$ and some ordinal η. We say that such a potential premouse M is active *iff $E_\eta \neq \emptyset$.*
Moreover if $\kappa \leq \eta$, we write

$$M|\kappa = (J_\kappa^{\vec{E}}, \in, \vec{E} \restriction \kappa, E_\kappa).$$

Here *fine extender sequence* is in the sense of Definition 2.4 in [St10]. This definition of a fine extender sequence goes back to Section 1 in [MS94] and [SchStZe02].

DEFINITION 1.2.2. *Let M be a potential premouse. Then we say M is a* premouse *iff every proper initial segment of M is ω-sound.*

We informally say that a *mouse* is an iterable premouse, but since there are several different notions of iterability we try to avoid to use the word "mouse" in formal context, especially if it is not obvious what sort of iterability is meant. Nevertheless whenever it is not specified otherwise "iterable" in this thesis always means "ω_1-iterable" as defined below and therefore a "mouse" will be an ω_1-iterable premouse.

DEFINITION 1.2.3. *We say a premouse M is* ω_1-iterable *iff player* II *has a winning strategy in the iteration game $\mathcal{G}_\omega(M, \omega_1)$ as described in Section* 3.1 *of [St10].*

The iteration trees which are considered in Section 3 in [St10] are called *normal* iteration trees.

Whenever not specified otherwise we will assume throughout this thesis that all iteration trees are normal to simplify the notation. Since normal iteration trees do not suffice to prove for example the Dodd-Jensen Lemma (see Section 4.2 in [St10]) it is necessary to consider stacks of normal trees. See Definition 4.4 in [St10] for a formal definition of iterability for stacks of normal trees.

All arguments to follow easily generalize to countable stacks of normal trees of length $< \omega_1$ instead of just normal trees of length $< \omega_1$. The reason for this is that the iterability we will prove in this thesis for different kinds of premice will in fact always be obtained from the sort of iterability for the model K^c which is proven in Chapter 9 in [St96].

Throughout this thesis we will use the notation from [St10] for iteration trees.

1.3. Mice with Finitely Many Woodin Cardinals

We first fix some notation and give a short background on the mouse $M_n^\#$. Throughout this thesis we always assume $M_n^\#$ to be ω_1-iterable if not specified otherwise.

The premice we are going to consider in this part of this thesis will mostly have the following form.

Definition 1.3.1. *Let $n \geq 1$. A premouse M is called* n-small *iff for every critical point κ of an extender on the M-sequence*

$$M|\kappa \nvDash \text{“there are } n \text{ Woodin cardinals”}.$$

Moreover we say that a premouse M is 0-small *iff M is an initial segment of Gödel's constructible universe L.*

Moreover ω-small premice are defined analogously.

Definition 1.3.2. *Let $n \geq 1$ and $x \in {}^\omega\omega$. Then $M_n^\#(x)$ denotes the unique countable, sound, ω_1-iterable x-premouse which is not n-small, but all of whose proper initial segments are n-small, if it exists and is unique.*

Definition 1.3.3. *Let $n \geq 1$, $x \in {}^\omega\omega$ and assume that $M_n^\#(x)$ exists. Then $M_n(x)$ is the unique x-premouse which is obtained from $M_n^\#(x)$ by iterating its top measure out of the universe.*

Remark. We denote $M_n^\#(0)$ and $M_n(0)$ by $M_n^\#$ and M_n for $n \geq 0$.

Remark. We say that $M_0^\#(x) = x^\#$ for all $x \in {}^\omega\omega$, where $x^\#$ denotes the least active ω_1-iterable premouse if it exists. Moreover we say that $M_0(x) = L[x]$ is Gödel's constructible universe above x.

The two correctness facts due to W. Hugh Woodin about the premouse $M_n^\#(x)$ which are stated in the following lemma are going to be useful later, because they help transferring projective statements from $M_n^\#(x)$ to V or the other way around.

Lemma 1.3.4. *Let $n \geq 0$ and assume that $M_n^\#(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$. Let φ be a Σ^1_{n+2}-formula.*

(1) Assume n is even and let $x \in {}^\omega\omega$. Then we have for every real a in $M_n^\#(x)$,

$$\varphi(a) \;\leftrightarrow\; M_n^\#(x) \vDash \varphi(a).$$

That means $M_n^\#(x)$ is $\boldsymbol{\Sigma}^1_{n+2}$-correct in V.

(2) Assume n is odd, so in particular $n \geq 1$, and let $x \in {}^\omega\omega$. Then we have for every real a in $M_n^\#(x)$,

$$\varphi(a) \;\leftrightarrow\; \Vdash^{M_n^\#(x)}_{\mathrm{Col}(\omega,\delta_0)} \varphi(a),$$

where δ_0 denotes the least Woodin cardinal in $M_n^\#(x)$. Furthermore we have that $M_n^\#(x)$ is $\boldsymbol{\Sigma}^1_{n+1}$-correct in V.

For notational simplicity we sometimes just write a for the standard name $\check{a}$ for a real a in $M_n^\#(x)$.

Proof of Lemma 1.3.4. For $n = 0$ this lemma holds by Shoenfield's Absoluteness Theorem (see for example Theorem 13.15 in [Ka08]) applied to the model we obtain by iterating the top measure of the active premouse $M_0^\#(x)$ and its images until we obtain a model of height $\geq \omega_1^V$, because this model has the same reals as $M_0^\#(x)$.

We simultaneously prove that (1) and (2) hold for all $n \geq 1$ inductively. In fact we are proving a more general statement: We will show inductively that both (1) and (2) hold for all n-iterable x-premice for all reals x in the sense of Definition 1.1 in [Ne95] which have n Woodin cardinals which are countable in V instead of the concrete x-premouse $M_n^\#(x)$ as in the statement of Lemma 1.3.4. Therefore notice that we could replace the z-premouse $M_n^\#(z)$ in the following argument by any z-premouse N which is n-iterable and has n Woodin cardinals which are countable in V.

Proof of (2)**:** We start with proving (2) in the statement of Lemma 1.3.4 for n, assuming inductively that (1) and (2) hold for all $m < n$. For the downward implication assume that n is odd and let φ be a Σ^1_{n+2}-formula such that $\varphi(a)$ holds in V for a parameter $a \in M_n^\#(z) \cap {}^\omega\omega$ for a $z \in {}^\omega\omega$. That means

$$\varphi(a) \equiv \exists x \forall y \, \psi(x, y, a)$$

for a Σ^1_n-formula $\psi(x, y, a)$. Now fix a real $\bar{x}$ in V such that

$$V \vDash \forall y \, \psi(\bar{x}, y, a).$$

We aim to show that

$$\Vdash^{M_n^\#(z)}_{\mathrm{Col}(\omega, \delta_0)} \varphi(a),$$

where δ_0 denotes the least Woodin cardinal in $M_n^\#(z)$.

We first use Corollary 1.8 in [Ne95] to make $\bar{x}$ generic over an iterate M^* of $M_n^\#(z)$ for the collapse of the image of the bottom Woodin cardinal δ_0 in $M_n^\#(z)$. That means there is an iteration tree $\mathcal{T}$ on the n-iterable z-premouse $M_n^\#(z)$ of limit length and a non-dropping branch b through $\mathcal{T}$ such that if

$$i : M_n^\#(z) \rightarrow M^*$$

denotes the corresponding iteration embedding we have that M^* is $(n-1)$-iterable and if g is $\mathrm{Col}(\omega, i(\delta_0))$-generic over M^*, then $\bar{x} \in M^*[g]$.

We have that $M^*[g]$ can be construed as an $(z \oplus \bar{x})$-premouse satisfying the inductive hypothesis and if we construe $M^*[g]$ as an $(z \oplus \bar{x})$-premouse we have that in fact $M^*[g] = M_{n-1}^\#(z \oplus \bar{x})$ (see for example [SchSt09] for the fine structural details). Therefore we have inductively that the premouse $M^*[g]$ is $\boldsymbol{\Sigma}^1_n$-correct in V (in fact it is even $\boldsymbol{\Sigma}^1_{n+1}$-correct in V, but this is not necessary here) and using downwards absoluteness for the Π^1_{n+1}-formula "$\forall y \psi(\bar{x}, y, a)$" it follows that

$$M^*[g] \vDash \forall y\, \psi(\bar{x}, y, a),$$

because $\bar{x}, a \in M^*[g]$. Since the forcing $\mathrm{Col}(\omega, i(\delta_0))$ is homogeneous, we have that

$$\Vdash^{M^*}_{\mathrm{Col}(\omega, i(\delta_0))} \exists x \forall y\, \psi(x, y, a),$$

and by elementarity of the iteration embedding $i : M_n^\#(z) \rightarrow M^*$ it follows that

$$\Vdash^{M_n^\#(z)}_{\mathrm{Col}(\omega, \delta_0)} \exists x \forall y\, \psi(x, y, a),$$

as desired.

For the upward implication of (2) in the statement of Lemma 1.3.4 let n again be odd, let z be a real and assume that

$$\Vdash^{M_n^\#(z)}_{\mathrm{Col}(\omega, \delta_0)} \varphi(a),$$

where as above $\varphi(a) \equiv \exists x \forall y\, \psi(x, y, a)$ is a Σ^1_{n+2}-formula, $\psi(x, y, a)$ is a Σ^1_n-formula and a is a real such that $a \in M_n^\#(z)$. Let g be $\mathrm{Col}(\omega, \delta_0)$-generic over $M_n^\#(z)$ and pick a real $\bar{x}$ such that

$$M_n^\#(z)[g] \vDash \forall y\, \psi(\bar{x}, y, a).$$

Since $M_n^\#(z)|(\delta_0^+)^{M_n^\#(z)}$ is countable in V, we can in fact pick the generic g such that we have $g \in V$. Then we have that $\bar{x} \in V$. Similar as above $M_n^\#(z)[g]$ can be construed as a z^*-premouse for some real z^* which satifies the inductive hypothesis for $n-1$, in fact we again have that $M_n^\#(z)[g] = M_{n-1}^\#(z^*)$ if $M_n^\#(z)[g]$ is construed as a z^*-premouse. Since $n-1$ is even and we inductively assume that (1) in the statement of Lemma 1.3.4 holds for all $m < n$, it follows from (1) applied to the Π^1_{n+1}-formula "$\forall y \psi(\bar{x}, y, a)$" and the premouse $M_n^\#(z)[g]$ that

$$V \vDash \forall y\, \psi(\bar{x}, y, a),$$

and therefore

$$V \vDash \exists x \forall y\, \psi(x, y, a),$$

as desired.

The fact that in this situation $M_n^\#(z)$ is $\mathbf{\Sigma}^1_{n+1}$-correct in V also follows from the inductive hypothesis, because $n-1$ is even and the inductive hypothesis for $n-1$ can be applied to the premouse $M_n^\#(z)$.

Proof of (1)**:** Now we turn to the proof of (1) in the statement of Lemma 1.3.4. Let n be even and assume inductively that (1) and (2) hold for all $m < n$. We again start with the proof of the downward implication, that means we want to prove that

$$M_n^\#(z) \vDash \varphi(a),$$

where as above φ is a Σ^1_{n+2}-formula which holds in V for $a \in M_n^\#(z) \cap {}^\omega\omega$ and $z \in {}^\omega\omega$. That means we again have

$$\varphi(a) \equiv \exists x \forall y\, \psi(x, y, a)$$

for a Σ^1_n-formula $\psi(x, y, a)$. Since n is even, it follows from Moschovakis' Second Periodicity Theorem that the pointclass $\Pi^1_{n+1}(a)$ has the uniformization property (see Theorem 6C.5 in [Mo09]), because Theorem 2.14 in [Ne02] (see also Corollary 2.1.2) yields that $\mathbf{\Pi}^1_n$ determinacy holds from the hypothesis that $M_{n-1}^\#(z)$ exists for all reals z. Consider the $\Pi^1_{n+1}(a)$-definable set

$$\{x \mid \forall y\, \psi(x, y, a)\}.$$

The uniformization property yields the existence of a real $\bar{x}$ such that we have $\{\bar{x}\} \in \Pi^1_{n+1}(a)$ and

$$V \vDash \forall y\, \psi(\bar{x}, y, a).$$

So let ρ be a Π^1_{n+1}-formula such that

$$x = \bar{x} \leftrightarrow \rho(x, a)$$

for all $x \in {}^\omega\omega$. That means we have

$$V \vDash \rho(\bar{x}, a) \wedge \forall y\, \psi(\bar{x}, y, a).$$

Now we use, as in the proof of (2) above, the n-iterability of $M_n^\#(z)$ and Corollary 1.8 in [Ne95] to make $\bar{x}$ generic over an iterate M^* of $M_n^\#(z)$ for the collapse of the image of the bottom Woodin cardinal δ_0 in $M_n^\#(z)$. As in the proof of (2) this means that in fact there is an iteration embedding

$$i : M_n^\#(z) \rightarrow M^*$$

such that M^* is $(n-1)$-iterable and if $g \in V$ is $\mathrm{Col}(\omega, i(\delta_0))$-generic over M^* then $\bar{x} \in M^*[g]$. Since a is a real in $M_n^\#(z)$ we have that

$$\Vdash^{M^*[g]}_{\mathrm{Col}(\omega,\bar{\delta})} \rho(\bar{x}, a) \wedge \forall y\, \psi(\bar{x}, y, a),$$

where $\bar{\delta}$ denotes the least Woodin cardinal inside $M^*[g]$, by the inductive hypothesis applied to the premouse $M^*[g]$, construed as an $(z \oplus \bar{x})$-premouse, and the Π^1_{n+1}-formula "$\rho(\bar{x}, a) \wedge \forall y\, \psi(\bar{x}, y, a)$", because $n-1$ is odd. As above we have that $M^*[g]$, construed as a $(z \oplus \bar{x})$-premouse, satisfies the inductive hypothesis. Moreover we have as above that the inductive hypothesis applied to the model $M^*[g]$ and the Π^1_{n+1}-formula "$\rho(x, a) \wedge \forall y\, \psi(x, y, a)$" yields that in fact for all $x^* \in M^*[g]$

$$\Vdash^{M^*[g]}_{\mathrm{Col}(\omega,\bar{\delta})} \rho(x^*, a) \wedge \forall y\, \psi(x^*, y, a) \text{ iff } V \vDash \rho(x^*, a) \wedge \forall y\, \psi(x^*, y, a).$$

By the homogeneity of the forcing $\mathrm{Col}(\omega, i(\delta_0))$ this implies that the witness $\bar{x}$ for x with $\rho(x, a)$ already exists in the ground model M^*, since $a \in M^*$ and $\bar{x}$ is still the unique witness to the fact that the statement "$\Vdash^{M^*[g]}_{\mathrm{Col}(\omega,\bar{\delta})} \rho(\bar{x}, a) \wedge \forall y\, \psi(\bar{x}, y, a)$" holds true. Therefore it follows by downward absoluteness that

$$M^* \vDash \rho(\bar{x}, a) \wedge \forall y\, \psi(\bar{x}, y, a).$$

This implies that in particular

$$M^* \vDash \exists x \forall y\, \psi(x,y,a).$$

Using the elementarity of the iteration embedding we finally get that

$$M_n^\#(z) \vDash \exists x \forall y\, \psi(x,y,a).$$

For the proof of the upward implication in (1) let n again be even and assume that $M_n^\#(z) \vDash \exists x \forall y\, \psi(x,y,a)$ for $z \in {}^\omega\omega$ and a fixed real $a \in M_n^\#(z)$. Furthermore fix a real $\bar{x} \in M_n^\#(z)$ such that

$$M_n^\#(z) \vDash \forall y\, \psi(\bar{x},y,a).$$

Then we obviously have that $\bar{x} \in V$. We want to show that $V \vDash \forall y\, \psi(\bar{x},y,a)$. Assume not. That means

$$V \vDash \exists y\, \neg\psi(\bar{x},y,a),$$

where "$\exists y\, \neg\psi(\bar{x},y,a)$" is a Σ^1_{n+1}-formula. Therefore the downward implication we already proved applied to the formula "$\exists y\, \neg\psi(\bar{x},y,a)$" and the parameters $\bar{x}, a \in M_n^\#(z)$ yields that

$$M_n^\#(z) \vDash \exists y\, \neg\psi(\bar{x},y,a),$$

which is a contradiction. □

The proof of Lemma 1.3.4 with Shoenfield absoluteness replaced by $\boldsymbol{\Sigma}^1_1$ absoluteness immediately gives the following lemma.

Lemma 1.3.5. *Let $n \geq 0$ and let M be a countable x-premouse with n Woodin cardinals for some $x \in {}^\omega\omega$ such that $M \vDash \mathrm{ZFC}^-$ and M is ω_1-iterable. Let φ be a Σ^1_{n+1}-formula.*

(1) Assume n is even. Then we have for every real a in M,

$$\varphi(a) \;\leftrightarrow\; M \vDash \varphi(a).$$

That means M is $\boldsymbol{\Sigma}^1_{n+1}$-correct in V.

(2) Assume n is odd, so in particular $n \geq 1$. Then we have for every real a in M,

$$\varphi(a) \;\leftrightarrow\; \Vdash^M_{\mathrm{Col}(\omega,\delta_0)} \varphi(a),$$

where δ_0 denotes the least Woodin cardinal in M. Furthermore we have that M is $\boldsymbol{\Sigma}^1_n$-correct in V.

The following lemma shows that Lemma 1.3.4 (1) does not hold if n is odd. Therefore the periodicity in the statement of Lemma 1.3.4 is necessary indeed.

Lemma 1.3.6. *Let $n \geq 1$ be* odd *and assume that $M_n^\#(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$. Then $M_n^\#(x)$ is* not *$\boldsymbol{\Sigma}^1_{n+2}$-correct in V.*

PROOF SKETCH. Consider for example the following Σ^1_{n+2}-formula φ, where Π^1_{n+1}-iterability is defined in Definition 1.6 in [St95] (see also Section 2.2 in this thesis for some results related to Π^1_{n+1}-iterability).

$$\varphi(x) \equiv \exists N \text{ such that } N \text{ is a countable } x\text{-premouse}$$
$$\text{which is } \Pi^1_{n+1}\text{-iterable and not } n\text{-small.}$$

The statement "N is Π^1_{n+1}-iterable" is Π^1_{n+1}-definable uniformly in any code for N (see Lemma 1.7 in [St95]). Therefore φ is a Σ^1_{n+2}-formula.

We have that $\varphi(x)$ holds in V for all reals x as witnessed by the x-premouse $M_n^\#(x)$, because ω_1-iterability implies Π^1_{n+1}-iterability, since we assumed that $M_n^\#(x)$ exists for all $x \in {}^\omega\omega$ (see Lemma 2.2.9 (2) which uses Lemma 2.2 in [St95]).

Assume toward a contradiction that $\varphi(x)$ holds in $M_n^\#(x)$ as witnessed by some x-premouse N in $M_n^\#(x)$ which is Π^1_{n+1}-iterable and not n-small. Since n is odd, Lemma 3.1 in [St95] implies that $\mathbb{R} \cap M_n(x) \subseteq \mathbb{R} \cap N$, which is a contradiction as $N \in M_n^\#(x)$.

Therefore $\varphi(x)$ cannot hold in $M_n^\#(x)$ and thus $M_n^\#(x)$ is not $\boldsymbol{\Sigma}^1_{n+2}$-correct in V. □

See for example [St95] for further information on the premouse $M_n^\#$.

1.4. Mice and Determinacy

Some of the results mentioned in Section 1.1 can be improved using the existence of certain mice instead of large cardinals in V as an hypothesis. In this section we will list results of that form and mention some things known about the converse direction.

In the context of analytic determinacy Harrington was able to prove the converse of Martin's result from [Ma70] and therefore obtained the following theorem.

THEOREM 1.4.1 (Harrington in [Ha78], Martin in [Ma70]). *The following are equivalent over* ZFC.

(*i*) *The mouse* $0^\#$ *exists, and*
(*ii*) *every* Π^1_1*-definable set of reals is determined.*

In the projective hierarchy Neeman improved the result of [MaSt89] as follows. Here "⅁" denotes the game quantifier as also used in Section 3.4 later (see Section 6D in [Mo09] for a definition and some basic facts about the game quantifier "⅁").

THEOREM 1.4.2 (Neeman in [Ne02]). *Let* $n \geq 1$ *and assume that* $M_n^\#$ *exists and is* ω_1*-iterable. Then every* $⅁^{(n)}(< \omega^2 - \Pi^1_1)$*-definable set of reals is determined and thus in particular every* Π^1_{n+1}*-definable set of reals is determined.*

In the first part of this thesis we will present a proof of the boldface version of a converse direction of this theorem due to Woodin which is only assuming that every $\boldsymbol{\Pi}^1_{n+1}$-definable set of reals is determined, see Corollary 2.1.2. The lightface version of the analogous converse direction of Theorem 1.4.2 is still open for $n > 1$ (see also Section 4.2). For $n = 1$ it is due to W. H. Woodin, who proved the following theorem (see Corollary 4.17 in [StW16]).

THEOREM 1.4.3 (Woodin in [StW16] and unpublished). *The following are equivalent over* ZFC*.*

(*i*) *$M_1^\#$ exists and is ω_1-iterable, and*
(*ii*) *every $\boldsymbol{\Pi}^1_1$-definable and every Δ^1_2-definable set of reals is determined.*

Here the implication (*i*) $\Rightarrow$ (*ii*), which also follows from Theorem 2.14 in [Ne02] (see Theorem 1.4.2 above), was first shown by Woodin in unpublished work.

At the level of infinitely many Woodin cardinals we have that the following equivalence, which is also due to W. H. Woodin, holds true (see Theorem 8.4 in [KW10]).

THEOREM 1.4.4 (Woodin in [KW10]). *The following are equivalent over* ZFC*.*

(*i*) *$M_\omega^\#$ exists and is countably iterable, and*
(*ii*) $\mathrm{AD}^{L(\mathbb{R})}$ *holds and $\mathbb{R}^\#$ exists.*

Here we mean by "$\mathrm{AD}^{L(\mathbb{R})}$ holds" that every set of reals in $L(\mathbb{R})$ is determined.

CHAPTER 2

A Model with Woodin Cardinals from Determinacy Hypotheses

In this chapter we are going to construct a proper class model with n Woodin cardinals from $\mathbf{\Pi}^1_n$ determinacy together with Π^1_{n+1} determinacy, but the model constructed in this chapter need not be iterable. We will treat iterability issues for models like the one constructed in this chapter later in Chapter 3.

2.1. Introduction

The main goal of Chapters 2 and 3 is to give a proof of the following theorem due to W. Hugh Woodin.

Theorem 2.1.1. *Let $n \geq 1$ and assume $\mathbf{\Pi}^1_{n+1}$ determinacy holds. Then $M_n^\#(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$.*

The converse of Theorem 2.1.1 also holds true. For odd n it is due to W. Hugh Woodin in never-published work and for even n it is due to Itay Neeman in [Ne95]. This yields the following corollary, where the case $n = 0$ is due to D. A. Martin (see [Ma70]) and L. Harrington (see [Ha78]).

Corollary 2.1.2. *Let $n \geq 0$. Then the following are equivalent.*

(1) $\mathbf{\Pi}^1_{n+1}$ determinacy, and
(2) for all $x \in {}^\omega\omega$, $M_n^\#(x)$ exists and is ω_1-iterable.

The proof of Theorem 2.1.1 is organized inductively. Thereby Harrington's result that analytic determinacy yields the existence of $0^\#$ (see [Ha78]) is the base step of our induction. So we will assume throughout the proof of Theorem 2.1.1 at the n'th level, that Theorem 2.1.1 holds true at the level $n-1$. In fact by Theorem 2.14 in [Ne02] we can assume during the proof at the level n that the existence and ω_1-iterability of $M_{n-1}^\#(x)$ for all $x \in {}^\omega\omega$ is equivalent to $\mathbf{\Pi}^1_n$ determinacy (see Corollary 2.1.2, for odd n this result is due to W. Hugh Woodin). We will use this in what follows without further notice.

We first fix some notation we are going to use for the rest of this part of the thesis.

If M is a premouse let δ_M denote the least Woodin cardinal in M, if it exists. For $n \geq 2$ and $x \in {}^\omega\omega$ let $\delta_x = \delta_{M_{n-1}(x)}$ denote the least Woodin cardinal in $M_{n-1}(x)$, whenever this does not lead to confusion. Moreover in case we are considering $L[x] = M_0(x)$ and a confusion is not possible let δ_x denote the least x-indiscernible in $L[x]$.

Remark. Recall that a real $x \in {}^\omega\omega$ is *Turing reducible* to a real $y \in {}^\omega\omega$ (write "$x \leq_T y$") iff x is recursive in y or equivalently iff there exists an oracle Turing machine that computes x using y as an oracle. Moreover we write $x \equiv_T y$ iff $x \leq_T y$ and $y \leq_T x$ and say in this case that x and y are *Turing equivalent*.

The following lemma generalizes a theorem of Kechris and Solovay (see Theorem 3.1 in [KS85]) to the context of mice with finitely many Woodin cardinals. It is one key ingredient for building inner models with finitely many Woodin cardinals from determinacy hypotheses and therefore in particular for proving Theorem 2.1.1. The following two sections will be devoted to the proof of this lemma.

Lemma 2.1.3. *Let $n \geq 1$. Assume that $M_{n-1}^\#(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$ and that all Σ^1_{n+1}-definable sets of reals are determined. Then there exists a real y_0 such that for all reals $x \geq_T y_0$,*

$$M_{n-1}(x)|\delta_x \vDash \text{OD-}determinacy.$$

The main difficulty in proving Lemma 2.1.3 in our context is the fact that for $n > 1$ the premouse $M_{n-1}(x)|\delta_x$ has lots of total extenders on its sequence. This is the main reason why we cannot generalize the proof of Theorem 3.1 in [KS85] straightforwardly. Therefore we need to prove some preliminary lemmas concerning comparisons and $L[E]$-constructions in our context in the following section. Some of this could have been avoided, if we would only want to prove Lemma 2.1.3 for models like for example lower part models (see Definition 3.7.2), which do not contain total extenders on their sequence.

2.2. Preliminaries

In this section we prove a few general lemmas about $(n-1)$-small premice which we are going to need for the proof of Lemma 2.1.3 and which are also going to be helpful later on.

The following models, called $\mathcal{Q}$-structures, can serve as witnesses for iterability by guiding an iteration strategy as in Definition 2.2.2. See also for example the proof of Lemma 2.2.8 for an application of this iteration strategy.

Informally a $\mathcal{Q}$-structure for a cofinal well-founded branch b through $\mathcal{T}$ is the longest initial segment of $\mathcal{M}_b^\mathcal{T}$ at which $\delta(\mathcal{T})$ is still seen to be a Woodin cardinal. Such $\mathcal{Q}$-structures are also introduced for example in Definition 6.11 in [St10].

Definition 2.2.1. *Let N be an arbitrary premouse and let $\mathcal{T}$ be an iteration tree on N of limit length.*

*(1) We say $\mathcal{Q} = \mathcal{Q}(\mathcal{T})$ is a $\mathcal{Q}$-*structure *for $\mathcal{T}$ iff $\mathcal{M}(\mathcal{T}) \trianglelefteq \mathcal{Q}$, $\delta(\mathcal{T})$ is a cutpoint of $\mathcal{Q}$, $\mathcal{Q}$ is ω_1-iterable above $\delta(\mathcal{T})$,*

$$\mathcal{Q} \vDash \text{“}\delta(\mathcal{T}) \text{ is a Woodin cardinal”},$$

if $\mathcal{Q} \neq \mathcal{M}(\mathcal{T})$ and either

(i) over $\mathcal{Q}$ there exists an $r\Sigma_n$-definable set $A \subset \delta(\mathcal{T})$ such that there is no $\kappa < \delta(\mathcal{T})$ such that κ is strong up to $\delta(\mathcal{T})$ with respect to A as being witnessed by extenders on the sequence of $\mathcal{Q}$ for some $n < \omega$, or

(ii) $\rho_n(\mathcal{Q}) < \delta(\mathcal{T})$ for some $n < \omega$.

*(2) Let b be a cofinal well-founded branch through $\mathcal{T}$. Then we say $\mathcal{Q} = \mathcal{Q}(b, \mathcal{T})$ is a $\mathcal{Q}$-*structure *for b* in *$\mathcal{T}$ iff $\mathcal{Q} = \mathcal{M}_b^{\mathcal{T}} | \gamma$, where $\gamma \leq \mathcal{M}_b^{\mathcal{T}} \cap \mathrm{Ord}$ is the least ordinal such that either*

$$\gamma < \mathcal{M}_b^{\mathcal{T}} \cap \mathrm{Ord} \text{ and } \mathcal{M}_b^{\mathcal{T}} | (\gamma + 1) \vDash \text{“}\delta(\mathcal{T}) \text{ is not Woodin”},$$

or

$$\gamma = \mathcal{M}_b^{\mathcal{T}} \cap \mathrm{Ord} \text{ and } \rho_n(\mathcal{M}_b^{\mathcal{T}}) < \delta(\mathcal{T})$$

for some $n < \omega$ or over $\mathcal{M}_b^{\mathcal{T}}$ there exists an $r\Sigma_n$-definable set $A \subset \delta(\mathcal{T})$ such that there is no $\kappa < \delta(\mathcal{T})$ such that κ is strong up to $\delta(\mathcal{T})$ with respect to A as being witnessed by extenders on the sequence of $\mathcal{M}_b^{\mathcal{T}}$ for some $n < \omega$.

If no such ordinal $\gamma \leq \mathcal{M}_b^{\mathcal{T}} \cap \mathrm{Ord}$ exists, we let $\mathcal{Q}(b, \mathcal{T})$ be undefined.

For the notion of an $r\Sigma_n$-definable set see for example §2 in [MS94].

Remark. We are also going to use the notion of a Π^1_n-*iterable $\mathcal{Q}$-structure* $\mathcal{Q}(\mathcal{T})$, meaning that $\mathcal{Q}(\mathcal{T})$ is Π^1_n-iterable above $\delta(\mathcal{T})$, $\delta(\mathcal{T})$-solid and that $\mathcal{Q}(\mathcal{T})$ satisfies all properties in (1) except for the ω_1-iterability above $\delta(\mathcal{T})$. It will be clear from the context if we include ω_1-iterability in the definition of $\mathcal{Q}$-structure or not. Here Π^1_n-iterability is defined as in Definitions 1.4 and 1.6 in [St95] (see also the explanations before Lemma 2.2.9).

Remark. In Case (1)(*i*) in Definition 2.2.1 we have that in particular $\rho_n(\mathcal{Q}) \leq \delta$ and if $\mathcal{Q} \lhd M$ for a premouse M and if we let $\gamma = \mathcal{Q} \cap \mathrm{Ord}$, then we have that $\delta(\mathcal{T})$ is not Woodin in $J^M_{\gamma+1}$. The same thing for M as above holds true in Case (1)(*ii*) in Definition 2.2.1, because in this case $\delta(\mathcal{T})$ is not even a cardinal in $J^M_{\gamma+1}$.

Remark. Let $n \geq 1$ and assume that $M_n^\#(x)$ exists and is ω_1-iterable for all reals x. Then any $\mathcal{Q}$-structure $\mathcal{Q}$ for an iteration tree $\mathcal{T}$ on an n-small premouse is unique by a comparison argument as the one we will see in the proof of Lemma 2.2.8.

Definition 2.2.2. *Let N be a premouse. Then a possibly partial iteration strategy Σ for N is the $\mathcal{Q}$-*structure iteration strategy *for N or we say that*

Σ is guided by $\mathcal{Q}$-structures*, iff Σ is defined as follows. For a tree $\mathcal{U}$ on N of limit length and a branch b through $\mathcal{U}$ we let*

$$\Sigma(\mathcal{U}) = b \textit{ iff } \mathcal{Q}(\mathcal{U}) \textit{ exists and } \mathcal{Q}(b, \mathcal{U}) = \mathcal{Q}(\mathcal{U}),$$

if such a branch b exists and is unique. If no such unique branch b through $\mathcal{U}$ exists, we let $\Sigma(\mathcal{U})$ be undefined.

Lemma 2.2.3. *Let $n \geq 1$ and assume that $M_n^\#(x)$ exists and is ω_1-iterable for all reals x. Let N be an n-small premouse and let $\mathcal{T}$ be an iteration tree on N. Then the branch b through $\mathcal{T}$ which satisfies $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$ as in the definition of the $\mathcal{Q}$-structure iteration strategy Σ is in fact unique.*

This lemma holds true because for two branches b and c through an iteration tree $\mathcal{T}$ as above, we have that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(c, \mathcal{T})$ implies that $b = c$ by the Branch Uniqueness Theorem (see Theorem 6.10 in [St10]). This is proven in Corollary 6 in §6 of [Je97] (written by Martin Zeman).

Remark. Let $n \geq 1$ and assume that $M_n^\#(x)$ exists and is ω_1-iterable for all reals x. Let N be an n-small premouse and let $\mathcal{T}$ be an iteration tree on N. Then the branch $b = \Sigma(\mathcal{T})$ given by the $\mathcal{Q}$-structure iteration strategy is in fact cofinal.

The following notion will be important in what follows to ensure that $\mathcal{Q}$-structures exist.

Definition 2.2.4. *Let M be a premouse and let δ be a cardinal in M or $\delta = M \cap \mathrm{Ord}$. We say that δ* is not definably Woodin over M *iff there exists an ordinal $\gamma \leq M \cap \mathrm{Ord}$ such that $\gamma \geq \delta$ and either*

(i) over J_γ^M there exists an $r\Sigma_n$-definable set $A \subset \delta$ such that there is no $\kappa < \delta$ such that κ is strong up to δ with respect to A as witnessed by extenders on the sequence of M for some $n < \omega$, or

(ii) $\rho_n(J_\gamma^M) < \delta$ for some $n < \omega$.

For several iterability arguments to follow we need our premice to satisfy the following property. By a fine structural argument this property is preserved during an iteration and will therefore ensure that $\mathcal{Q}$-structures exist in an iteration of a premouse M satisfying this property.

Definition 2.2.5. *Let M be a premouse. We say M has* no definable Woodin cardinals *iff for all $\delta \leq M \cap \mathrm{Ord}$ we have that δ is not definably Woodin over M.*

Remark. Let M be a premouse which has no definable Woodin cardinals. Note that M might still have Woodin cardinals. Consider for example the premouse $M_1^\#$, which by definition has a Woodin cardinal, but no definable Woodin cardinals since $\rho_\omega(M_1^\#) = \omega$.

In what follows we sometimes want to consider premice which are obtained from $M_n^\#$ "constructed on top" of a premouse N. The following definition makes precise what we mean by that.

DEFINITION 2.2.6. *Let $n \geq 1$ and assume that $M_n^\#(x)$ exists for all reals x. Let N be a countable x-premouse for some $x \in {}^\omega\omega$. Then we say $M_n^\#(N)$ is the smallest x-premouse $M \trianglerighteq N$ with*

$$\rho_\omega(M) \leq N \cap \mathrm{Ord}$$

which is ω_1-iterable above $N \cap \mathrm{Ord}$, sound above $N \cap \mathrm{Ord}$ and such that either M is not fully sound, or M is not n-small above $N \cap \mathrm{Ord}$.

In the first case, i.e. if M is not fully sound, we sometimes say that the construction of $M_n^\#(N)$ breaks down.

REMARK. We can define a premouse $M_n(N)$ in a similar fashion, by iterating the top extender of $M = M_n^\#(N)$ out of the universe in the case that M is not n-small above $N \cap \mathrm{Ord}$. In the case that M is not fully sound, we just let $M_n(N) = M$. So in particular $M_n(N)$ is a proper class model in the first case and a set in the latter case.

In what follows we will point out if $M_n^\#(N)$ denotes the premouse constructed in the sense of Definition 2.2.6 or if it denotes the premouse $M_n^\#(x)$ in the usual sense, where x is for example a real coding the countable premouse N. Note that these two notions are different since extenders on the N-sequence are included in the $M_n^\#(N)$-sequence if $M_n^\#(N)$ is constructed in the sense of Definition 2.2.6.

We will need the following notation.

DEFINITION 2.2.7. *(1) Let $x, y \in {}^\omega\omega$ be such that $x = (x_n \mid n < \omega)$ for $x_n \in \omega$ and $y = (y_n \mid n < \omega)$ for $y_n \in \omega$. Then we let $x \oplus y = (x_0, y_0, x_1, y_1, \dots) \in {}^\omega\omega$.*
(2) Let M and N be countable premice. We say a real x codes $M \oplus N$ iff $x \geq_T x_M \oplus x_N$ for a real x_M coding M and a real x_N coding N.

The following lemma proves that under the right hypothesis comparison works for certain ω_1-iterable premice instead of $(\omega_1 + 1)$-iterable premice as in the usual Comparison Lemma (see Theorem 3.11 in [St10]). Moreover the proof of this lemma will use arguments that are explained here in full detail and will show up in several different proofs throughout this thesis again with possibly less details given.

Recall that δ_x denotes the least Woodin cardinal in $M_{n-1}^\#(x)$ for $n \geq 2$.

LEMMA 2.2.8. *Let $n \geq 1$ and assume that $M_{n-1}^\#(x)$ exists and is ω_1-iterable for all reals x. Let M and N be countable premice, such that M and N have a common cutpoint δ. Assume that M and N both do not have definable Woodin cardinals above δ and that every proper initial segment of M and N is $(n-1)$-small above δ.*

(1) Let x be an arbitrary real and let $n \geq 2$. Then $H_{\delta_x}^{M_{n-1}^{\#}(x)}$ is closed under the operation

$$a \mapsto M_{n-2}^{\#}(a)$$

and moreover this operation $a \mapsto M_{n-2}^{\#}(a)$ for $a \in H_{\delta_x}^{M_{n-1}^{\#}(x)}$ is contained in $M_{n-1}^{\#}(x)|\delta_x$.

(2) Let x be a real coding M and assume that the premouse M is ω_1-iterable above δ. If Σ denotes the ω_1-iteration strategy for M above δ, then

$$\Sigma \upharpoonright H_{\delta_x}^{M_{n-1}^{\#}(x)} \in M_{n-1}^{\#}(x)|\delta_x.$$

(3) Let x be a real coding $M \oplus N$ and assume that the premice M and N are both ω_1-iterable above δ. Moreover assume that

$$M|\delta = N|\delta.$$

Then we can successfully coiterate M and N above δ inside the model $M_{n-1}^{\#}(x)$. That means there are iterates M^* of M and N^* of N above δ such that we have

(a) $M^* \trianglelefteq N^*$ and the iteration from M to M^* does not drop, or

(b) $N^* \trianglelefteq M^*$ and the iteration from N to N^* does not drop.

In particular the coiteration is successful in V in the same sense.

(4) Let x be a real coding $M \oplus N$ and assume that the premice M and N are both ω_1-iterable above δ. Moreover assume that

$$M|\delta = N|\delta,$$

M and N are δ-sound, $\rho_\omega(M) \leq \delta$ and $\rho_\omega(N) \leq \delta$. Then we have

$$M \trianglelefteq N \text{ or } N \trianglelefteq M.$$

Remark. This lemma holds in particular for $\delta = \omega$. That means if we assume that $M_{n-1}^{\#}(x)$ exists and is ω_1-iterable for all reals x and M and N are ω_1-iterable countable premice such that both do not have definable Woodin cardinals and such that every proper initial segment of M and N is $(n-1)$-small, then we can successfully compare M and N as in Lemma 2.2.8 (3).

Proof of Lemma 2.2.8. **Proof of** (1)**:** Let $a \in H_{\delta_x}^{M_{n-1}^{\#}(x)}$ be arbitrary and perform a fully backgrounded extender construction $L[E](a)^{M_{n-1}^{\#}(x)|\kappa}$ in the sense of [MS94] (with the smallness hypothesis weakened to allow ω-small premice in the construction) above a inside the model $M_{n-1}^{\#}(x)|\kappa$, where κ denotes the critical point of the top measure of the active premouse $M_{n-1}^{\#}(x)$. Then we have that the premouse $L[E](a)^{M_{n-1}^{\#}(x)|\kappa}$ has $n-1$ Woodin cardinals by a generalization of Theorem 11.3 in [MS94]. So in particular it follows that $L[E](a)^{M_{n-1}^{\#}(x)|\kappa}$ is not $(n-2)$-small. Moreover the premouse $L[E](a)^{M_{n-1}^{\#}(x)|\kappa}$ inherits the iterability from $M_{n-1}^{\#}(x)$ and therefore we have that

$$M_{n-2}^{\#}(a) \lhd L[E](a)^{M_{n-1}^{\#}(x)|\kappa}.$$

In fact the operation $a \mapsto M_{n-2}^{\#}(a)$ for $a \in H_{\delta_x}^{M_{n-1}^{\#}(x)}$ is contained in $M_{n-1}^{\#}(x)|\delta_x$, because $M_{n-2}^{\#}(a)$ can be obtained from an $L[E]$-construction above a.

Proof of (2) + (3) + (4)**:** We prove (2),(3) and (4) simultaneously by an inductive argument. For $n = 1$ there is nothing to show, because we defined M to be 0-small iff M is an initial segment of Gödel's constructible universe L. That means if M and N are such that every proper initial segment of M or N is 0-small above some common cutpoint δ as in the statement of Lemma 2.2.8, then every iteration tree on M or N above δ is linear and there is nothing to show for (2). Moreover we easily get that one is an initial segment of the other since every proper initial segment of M or N is above δ as an initial segment of L. Therefore (3) and (4) hold.

So let $n \geq 2$ and assume that (2),(3) and (4) hold for $n-2$. We first want to show that (2) holds for $n-1$. Let us assume for notational simplicity that $\delta = \omega$ and let M be an ω_1-iterable premouse, such that every proper initial segment of M is $(n-1)$-small and such that M has no definable Woodin cardinals. Let x be a real coding the premouse M and assume that $M_{n-1}^{\#}(x)$ exists and is ω_1-iterable. Further let Σ be the ω_1-iteration strategy for M and let $\mathcal{T}$ be an iteration tree on M in V of length $\lambda + 1$ for some limit ordinal $\lambda < \omega_1^V$ such that $\mathcal{T}$ is according to Σ and

$$\mathcal{T} \upharpoonright \lambda \in H_{\delta_x}^{M_{n-1}^{\#}(x)}.$$

By assumption the premouse M is ω_1-iterable in V and has no definable Woodin cardinals. Therefore Σ is the $\mathcal{Q}$-structure iteration strategy (see Definition 2.2.2) and there exists a $\mathcal{Q}$-structure $\mathcal{Q} \trianglelefteq \mathcal{M}_\lambda^{\mathcal{T}}$ for $\mathcal{T} \upharpoonright \lambda$ which is ω_1-iterable in V. We first want to show that such an ω_1-iterable $\mathcal{Q}$-structure already exists in the model $M_{n-1}^{\#}(x)|\delta_x$.

First consider the case

$$\mathcal{Q} = \mathcal{M}(\mathcal{T} \upharpoonright \lambda),$$

where the latter denotes the common part model of $\mathcal{T} \upharpoonright \lambda$. In this case $\mathcal{Q}$ is also a $\mathcal{Q}$-structure for $\mathcal{T} \upharpoonright \lambda$ inside the model $M_{n-1}^{\#}(x)|\delta_x$ for trivial reasons, because the condition that $\mathcal{Q}$ needs to be ω_1-iterable above $\delta(\mathcal{T} \upharpoonright \lambda) = \mathcal{M}(\mathcal{T} \upharpoonright \lambda) \cap \mathrm{Ord}$ is empty here and therefore $\mathcal{Q}$ can be isolated as the $\mathcal{Q}$-structure for $\mathcal{T} \upharpoonright \lambda$ in $M_{n-1}^{\#}(x)|\delta_x$.

So we can assume now that

$$\mathcal{M}(\mathcal{T} \upharpoonright \lambda) \lhd Q.$$

That means $\delta(\mathcal{T} \upharpoonright \lambda) \in \mathcal{Q}$ and therefore $\mathcal{Q}$ is by definition the longest initial segment of $\mathcal{M}_\lambda^\mathcal{T}$ in V such that

$$\mathcal{Q} \vDash \text{“}\delta(\mathcal{T} \upharpoonright \lambda) \text{ is Woodin”}.$$

Every proper initial segment of $\mathcal{M}_\lambda^\mathcal{T}$ is $(n-1)$-small since the same holds for M. Thus every proper initial segment of $\mathcal{Q}$ is $(n-1)$-small. Together with the fact that

$$\mathcal{Q} \vDash \text{“}\delta(\mathcal{T} \upharpoonright \lambda) \text{ is Woodin”},$$

this implies that every proper initial segment of $\mathcal{Q}$ in fact has to be $(n-2)$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$.

Now consider the premouse $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ in the sense of Definition 2.2.6, which exists inside $M_{n-1}^\#(x)|\delta_x$ because of (1), since we have that $\mathcal{T} \upharpoonright \lambda \in H_{\delta_x}^{M_{n-1}^\#(x)}$. Note that all proper initial segments of $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ are $(n-2)$-small above

$$\delta(\mathcal{T} \upharpoonright \lambda) = \mathcal{M}(\mathcal{T} \upharpoonright \lambda) \cap \mathrm{Ord},$$

regardless of whether the case that $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is not fully sound or the case that $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is not $(n-2)$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$ in the definition of $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ (see Definition 2.2.6) holds. Moreover we have by definition that

$$\mathcal{Q} \,|\, \delta(\mathcal{T} \upharpoonright \lambda) = \mathcal{M}(\mathcal{T} \upharpoonright \lambda) = M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda)) \,|\, \delta(\mathcal{T} \upharpoonright \lambda).$$

Thus a coiteration of $\mathcal{Q}$ and $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ would take place above $\delta(\mathcal{T} \upharpoonright \lambda)$. Moreover $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ and $\mathcal{Q}$ are both $\delta(\mathcal{T} \upharpoonright \lambda)$-sound and project to $\delta(\mathcal{T} \upharpoonright \lambda)$, so in particular they both do not have definable Woodin cardinals above $\delta(\mathcal{T} \upharpoonright \lambda)$. This implies by the inductive hypothesis (4) that the comparison of $\mathcal{Q}$ and $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is successful in V, because all proper initial segments of both $\mathcal{Q}$ and $M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ are $(n-2)$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$. Moreover both sides do not move in the comparison as in (4) and therefore we can distinguish two cases as follows.

Case 1. Assume that

$$M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda)) \lhd \mathcal{Q}.$$

Then we have that

$$M_{n-2}^\#(\mathcal{M}(\mathcal{T} \upharpoonright \lambda)) \vDash \text{“}\delta(\mathcal{T} \upharpoonright \lambda) \text{ is Woodin”},$$

because we have by definition of the $\mathcal{Q}$-structure $\mathcal{Q}$ (see Definition 2.2.1) that

$$\mathcal{Q} \vDash \text{“}\delta(\mathcal{T} \upharpoonright \lambda) \text{ is Woodin”}.$$

Case 1.1. Assume that the premouse $M^{\#}_{n-2}(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is not $(n-2)$-small above $\mathcal{M}(\mathcal{T} \upharpoonright \lambda) \cap \mathrm{Ord}$.

That means we were able to construct the full premouse $M^{\#}_{n-2}$ on top of $\mathcal{M}(\mathcal{T} \upharpoonright \lambda)$ as in the sense of Definition 2.2.6. In this case $M^{\#}_{n-2}(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is not $(n-1)$-small, because $\delta(\mathcal{T} \upharpoonright \lambda)$ is a Woodin cardinal in $M^{\#}_{n-2}(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$. But we argued earlier that every proper initial segment of $\mathcal{Q}$ is $(n-1)$-small, which contradicts $M^{\#}_{n-2}(\mathcal{M}(\mathcal{T} \upharpoonright \lambda)) \lhd \mathcal{Q}$.

Case 1.2. Assume now that we are in the other case of Definition 2.2.6. That means we have that $M^{\#}_{n-2}(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is not fully sound.

So $\rho_m(M^{\#}_{n-2}(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))) < \delta(\mathcal{T} \upharpoonright \lambda)$ for some $m < \omega$. This implies that $M^{\#}_{n-2}(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ is already a $\mathcal{Q}$-structure for $\mathcal{T} \upharpoonright \lambda$ inside the model $M^{\#}_{n-1}(x)|\delta_x$, because $M^{\#}_{n-2}(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ exists inside $M^{\#}_{n-1}(x)|\delta_x$ by part (1) of this lemma as argued earlier and is by definition ω_1-iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$, which finishes the argument for this case.

Case 2. Assume that

$$\mathcal{Q} \unlhd M^{\#}_{n-2}(\mathcal{M}(\mathcal{T} \upharpoonright \lambda)).$$

This implies that $\mathcal{Q}$ is in $M^{\#}_{n-1}(x)|\delta_x$ and furthermore is ω_1-iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$ in $M^{\#}_{n-1}(x)|\delta_x$ since the same holds for $M^{\#}_{n-2}(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$ by part (1) of this lemma.

So we showed that in both cases there exists an ω_1-iterable $\mathcal{Q}$-structure $\mathcal{Q}$ for $\mathcal{T} \upharpoonright \lambda$ in $M^{\#}_{n-1}(x)|\delta_x$. We now aim to show that the cofinal well-founded branch through $\mathcal{T}$ in V which is given by the $\mathcal{Q}$-structure iteration strategy Σ, that means the branch b for which we have

$$\mathcal{Q}(b, \mathcal{T} \upharpoonright \lambda) = \mathcal{Q},$$

is also contained in $M^{\#}_{n-1}(x)|\delta_x$.

Consider the statement

$$\phi(\mathcal{T} \upharpoonright \lambda, \mathcal{Q}) \equiv \text{“there is a cofinal branch } b \text{ through } \mathcal{T} \upharpoonright \lambda \text{ such that there is a } \mathcal{Q}^* \unlhd \mathcal{M}_b^{\mathcal{T}} \text{ with } \mathcal{Q}^* \cong \mathcal{Q}\text{”}.$$

This statement is Σ^1_1-definable uniformly in codes for $\mathcal{T} \upharpoonright \lambda$ and $\mathcal{Q}$ and obviously true in V as witnessed by the branch b given by the ω_1-iterability of the premouse M. Since the iteration tree $\mathcal{T} \upharpoonright \lambda \in H_{\delta_x}^{M^{\#}_{n-1}(x)}$ need not be countable in the model $M^{\#}_{n-1}(x)|\delta_x$, we consider the model $(M^{\#}_{n-1}(x)|\delta_x)^{\mathrm{Col}(\omega,\gamma)}$ instead, where $\gamma < \delta_x$ is an ordinal such that

$$\mathcal{T} \upharpoonright \lambda, \mathcal{Q} \in (M^{\#}_{n-1}(x)|\delta_x)^{\mathrm{Col}(\omega,\gamma)}$$

are countable inside the model $(M_{n-1}^{\#}(x)|\delta_x)^{\mathrm{Col}(\omega,\gamma)}$, where with the model $(M_{n-1}^{\#}(x)|\delta_x)^{\mathrm{Col}(\omega,\gamma)}$ we denote an arbitrary $\mathrm{Col}(\omega,\gamma)$-generic extension of the model $M_{n-1}^{\#}(x)|\delta_x$.

Then it follows by $\mathbf{\Sigma}^1_1$-absoluteness that the statement $\phi(\mathcal{T} \upharpoonright \lambda, \mathcal{Q})$ holds in $(M_{n-1}^{\#}(x)|\delta_x)^{\mathrm{Col}(\omega,\gamma)}$ as witnessed by some branch $\bar{b}$ and some model $\bar{\mathcal{Q}}$.

Since by the argument above $\mathcal{Q}$ is a $\mathcal{Q}$-structure for $\mathcal{T}$ in $M_{n-1}^{\#}(x)|\delta_x$, we have that $\mathcal{Q} = \bar{\mathcal{Q}}$ and it follows that $\bar{b}$ is the unique cofinal branch through $\mathcal{T}$ with $\mathcal{Q}(\bar{b}, \mathcal{T} \upharpoonright \lambda) = \mathcal{Q}$ by Lemma 2.2.3.

Since the branch $\bar{b}$ is uniquely definable from $\mathcal{T} \upharpoonright \lambda$ and $\mathcal{Q}$, and we have that $\mathcal{T} \upharpoonright \lambda, \mathcal{Q} \in M_{n-1}^{\#}(x)|\delta_x$, it follows by homogeneity of the forcing $\mathrm{Col}(\omega,\gamma)$ that actually $\bar{b} \in M_{n-1}^{\#}(x)|\delta_x$.

Thus we have that $\Sigma(\mathcal{T}) = \bar{b} \in M_{n-1}^{\#}(x)|\delta_x$ and our argument shows that in fact the operation $\mathcal{T} \mapsto \Sigma(\mathcal{T})$ for iteration trees $\mathcal{T} \in H_{\delta_x}^{M_{n-1}^{\#}(x)}$ on M of limit length is in the model $M_{n-1}^{\#}(x)|\delta_x$ for the following reason. Let $\mathcal{T}$ be an iteration tree on M of limit length such that $\mathcal{T} \in H_{\delta_x}^{M_{n-1}^{\#}(x)}$. Then we showed in the first part of this proof that $M_{n-1}^{\#}(x)|\delta_x$ can find a $\mathcal{Q}$-structure $\mathcal{Q}$ for $\mathcal{T}$. Now $M_{n-1}^{\#}(x)|\delta_x$ can compute $\Sigma(\mathcal{T})$ from $\mathcal{T}$, because $\Sigma(\mathcal{T})$ is the unique cofinal branch b through $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}$ and we showed that this branch b exists inside $M_{n-1}^{\#}(x)|\delta_x$. Therefore we proved that (2) holds.

To show (3) assume now in addition that N is an ω_1-iterable premouse such that every proper initial segment of N is $(n-1)$-small and N has no definable Woodin cardinals. Moreover let x be a real coding $M \oplus N$. We again assume that $\delta = \omega$ for notational simplicity.

By our hypothesis we have that $M_{n-1}^{\#}(x)$ exists, so we work inside the model $M_{n-1}^{\#}(x)$. Moreover it follows from (2) that M and N are iterable inside $M_{n-1}^{\#}(x)$, particularly with respect to trees in $H_{\omega_2}^{M_{n-1}^{\#}(x)}$, since $M_{n-1}^{\#}(x)$ is countable in V and thus

$$\omega_2^{M_{n-1}^{\#}(x)} < \omega_1^V.$$

In particular M and N are $(\omega_1 + 1)$-iterable in $M_{n-1}^{\#}(x)$ and the coiteration of M and N terminates successfully inside the model $M_{n-1}^{\#}(x)$ by the usual Comparison Lemma (see Theorem 3.11 in [St10]) applied inside $M_{n-1}^{\#}(x)$. This shows that (3) holds.

To prove (4) we assume that we moreover have that M and N are ω-sound and that $\rho_\omega(M) = \omega$ and $\rho_\omega(N) = \omega$. Then it follows as in Corollary 3.12 in [St10] that we have $M \trianglelefteq N$ or $N \trianglelefteq M$. $\square$

Remark. As in the previous lemma let $n \geq 1$ and assume that $M_{n-1}^{\#}(x)$ exists and is ω_1-iterable for all reals x. Then we can also successfully coiterate countable ω_1-iterable premice M and N which agree up to a common cutpoint δ and are $(n-1)$-small above δ in the sense of Lemma 2.2.8 (3), if we only assume that M and N both do not have Woodin cardinals, for the following reason.

Let x be a real coding $M \oplus N$. If $M \cap \mathrm{Ord}$ and $N \cap \mathrm{Ord}$ are both not definably Woodin over M and N respectively, then this implies together with the assumption that M and N both do not have Woodin cardinals that M and N both do not have definable Woodin cardinals and we can apply Lemma 2.2.8. So assume now for example that $M \cap \mathrm{Ord}$ is definably Woodin over M. By the proof of Lemma 2.2.8 the coiteration of M and N can only fail on the M-side because of the lack of a $\mathcal{Q}$-structure for an iteration tree $\mathcal{T}$ of limit length on M inside $M_{n-1}^{\#}(x)|\delta_x$. But in this case we must have that

$$\mathcal{M}(\mathcal{T}) = \mathcal{M}_\lambda^{\mathcal{T}},$$

where $\mathcal{M}_\lambda^{\mathcal{T}}$ is the limit model for $\mathcal{T}$ which exists in V. This implies that the coiteration on the M-side already finished because the M-side can no longer be iterated and by the same argument for N we have that the coiteration is successful, even if $M \cap \mathrm{Ord}$ and $N \cap \mathrm{Ord}$ are both definably Woodin over M and N respectively.

In what follows we also want to consider premice which are not fully ω_1-iterable but only Π^1_n-iterable for some $n \in \omega$. This notion was defined by John Steel in [St95] and he proved that for a premouse M the statement "M is Π^1_n-iterable" is Π^1_n-definable uniformly in any code for M. See Definitions 1.4 and 1.6 in [St95] for a precise definition of Π^1_n-iterability. He proves in Lemma 2.2 in [St95] that for an $(n-1)$-small premouse N which is ω-sound and such that $\rho_\omega(N) = \omega$, Π^1_n-iterability is enough to perform the standard comparison arguments with an (ω_1+1)-iterable premouse which has the same properties.

This implies that using Lemma 1.3.4 and Lemma 2.2.8 (2) the following version of Lemma 2.2 proven in [St95] holds true for ω_1-iterability (instead of (ω_1+1)-iterability as in [St95]).

Lemma 2.2.9. *Let $n \geq 2$ and assume that $M_{n-1}^{\#}(x)$ exists and is ω_1-iterable for all reals x. Let M and N be countable premice which have a common cutpoint δ such that M and N are $(n-1)$-small above δ. Assume that M and N are δ-sound and that $\rho_\omega(M) \leq \delta$ and $\rho_\omega(N) \leq \delta$.*

(1) Assume that M is ω_1-iterable above δ. Let $\mathcal{T}$ be a normal iteration tree on M above δ of length λ for some limit ordinal $\lambda < \omega_1$ and let b be the unique cofinal well-founded branch through $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T})$ is ω_1-iterable above $\delta(\mathcal{T})$. Then b is the unique cofinal branch c through $\mathcal{T}$ such that $\mathcal{M}_c^{\mathcal{T}}$ is well-founded and, if $n \geq 3$, $\mathcal{Q}(c, \mathcal{T})$ is Π^1_{n-1}-iterable above $\delta(\mathcal{T})$.

(2) *Assume M is ω_1-iterable above δ. Then M is Π^1_n-iterable above δ.*
(3) *Assume M is ω_1-iterable above δ and N is Π^1_n-iterable above δ. Moreover assume that*
$$M|\delta = N|\delta.$$
Then we can successfully coiterate M and N above δ, that means we have that
$$M \trianglelefteq N \text{ or } N \trianglelefteq M.$$

Proof. Apply Lemma 2.2 from [St95] inside the model $M^\#_{n-1}(x)$, where x is a real coding $M \oplus N$. This immediately gives Lemma 2.2.9 using Lemma 2.2.8 (2), because we have that
$$\omega_2^{M^\#_{n-1}(x)} < \omega_1^V$$
and therefore M is $(\omega_1 + 1)$-iterable inside $M^\#_{n-1}(x)$ and by Lemma 1.3.4 we have that N is Π^1_n-iterable inside the model $M^\#_{n-1}(x)$ since $M^\#_{n-1}(x)$ is $\boldsymbol{\Sigma}^1_n$-correct in V. □

Remark. Similarly as for Lemma 2.2.8 this lemma also holds true in the special case that $\delta = \omega$. More precisely in this case Lemma 2.2.9 (3) holds true in the following sense. If M and N are ω-sound, $\rho_\omega(M) = \omega$ and $\rho_\omega(N) = \omega$, and if we assume ω_1-iterability and Π^1_n-iterability for M and N respectively, then we have that
$$M \trianglelefteq N \text{ or } N \trianglelefteq M.$$

In Chapter 3 we in fact need the following strengthening of Lemma 2.2.9 for odd n, which is proved in Lemma 3.3 in [St95]. That it holds for ω_1-iterability instead of $(\omega_1 + 1)$-iterability follows by the same argument as the one we gave for Lemma 2.2.9 above. This lemma only holds for odd n because of the periodicity in the projective hierarchy. For more details see [St95].

Lemma 2.2.10. *Let $n \geq 1$ be* odd *and assume that $M^\#_{n-1}(x)$ exists and is ω_1-iterable for all reals x. Let M and N be countable premice which agree up to a common cutpoint δ such that M and N are δ-sound and such that $\rho_\omega(M) \leq \delta$ and $\rho_\omega(N) \leq \delta$. Assume that M is $(n-1)$-small above δ and N is not $(n-1)$-small above δ. Moreover assume that M is Π^1_n-iterable above δ and that N is ω_1-iterable above δ. Then we have that*
$$M \trianglelefteq N.$$

For the proof of Lemma 2.1.3 we need the following variant of Lemma 2.2.9 which is a straightforward consequence of the proof of Lemma 2.2 in [St95], because the assumption that the premice M and N both do not have definable Woodin cardinals yields that $\mathcal{Q}$-structures exist in a coiteration of M and N.

We say that an iteration tree $\mathcal{U}$ is a *putative iteration tree* if $\mathcal{U}$ satisfies all properties of an iteration tree, but we allow the last model of $\mathcal{U}$ to be ill-founded, in case $\mathcal{U}$ has a last model.

LEMMA 2.2.11. *Let $n \geq 2$ and assume that $M_{n-1}^{\#}(x)$ exists and is ω_1-iterable for all reals x. Let M and N be countable premice which have a common cutpoint δ such that M and N are $(n-1)$-small above δ and solid above δ. Assume that M and N both do not have definable Woodin cardinals and assume in addition that M is ω_1-iterable above δ and that N is Π^1_n-iterable above δ. Moreover assume that*

$$M|\delta = N|\delta.$$

Then we can successfully coiterate M and N above δ in the analogous to Lemma 2.2.8 (3)*, that means here that there is an iteration tree $\mathcal{T}$ on M and a putative iteration tree $\mathcal{U}$ on N of length $\lambda+1$ for some ordinal λ such that we have*

$$\mathcal{M}_\lambda^\mathcal{T} \trianglelefteq \mathcal{M}_\lambda^\mathcal{U} \text{ or } \mathcal{M}_\lambda^\mathcal{U} \trianglelefteq \mathcal{M}_\lambda^\mathcal{T}.$$

So in the first case $\mathcal{M}_\lambda^\mathcal{U}$ need not be fully well-founded, but it is well-founded up to $\mathcal{M}_\lambda^\mathcal{T} \cap \mathrm{Ord}$. In the second case we have that $\mathcal{M}_\lambda^\mathcal{U}$ is fully well-founded and $\mathcal{U}$ is in fact an iteration tree.

Analogous to the remark after the proof of Lemma 2.2.8 we get that the following strengthening of Lemma 2.2.11 holds true, where we replace "no definable Woodin cardinals" by "no Woodin cardinals".

COROLLARY 2.2.12. *Let $n \geq 2$ and assume that $M_{n-1}^{\#}(x)$ exists and is ω_1-iterable for all reals x. Let M and N be countable premice which have a common cutpoint δ such that M and N are $(n-1)$-small above δ and solid above δ. Assume that M and N both do not have Woodin cardinals and assume in addition that M is ω_1-iterable above δ and that N is Π^1_n-iterable above δ. Moreover assume that*

$$M|\delta = N|\delta.$$

Then we can successfully coiterate M and N above δ in the analogous to Lemma 2.2.8 (3)*, that means here that there is an iteration tree $\mathcal{T}$ on M and a putative iteration tree $\mathcal{U}$ on N of length $\lambda+1$ for some ordinal λ such that we have*

$$\mathcal{M}_\lambda^\mathcal{T} \trianglelefteq \mathcal{M}_\lambda^\mathcal{U} \text{ or } \mathcal{M}_\lambda^\mathcal{U} \trianglelefteq \mathcal{M}_\lambda^\mathcal{T}.$$

So we again have that in the first case $\mathcal{M}_\lambda^\mathcal{U}$ need not be fully well-founded, but it is well-founded up to $\mathcal{M}_\lambda^\mathcal{T} \cap \mathrm{Ord}$, and in the second case we have that $\mathcal{M}_\lambda^\mathcal{U}$ is fully well-founded and $\mathcal{U}$ is in fact an iteration tree.

PROOF. To simplify the notation we again assume that $\delta = \omega$. Analogous to the remark after the proof of Lemma 2.2.8 we can just use Lemma 2.2.11 in the case that $M \cap \mathrm{Ord}$ and $N \cap \mathrm{Ord}$ are both not definably Woodin over M and N respectively, because in this case M and N in fact do not have definable Woodin cardinals.

So assume for example that $N \cap \mathrm{Ord}$ is definably Woodin over N. Let x be a real coding $M \oplus N$ and consider the coiteration of M and N inside $M_{n-1}^{\#}(x)$. Let $\mathcal{T}$ and $\mathcal{U}$ be the resulting trees on M and N respectively. Assume that the coiteration breaks down on the N-side, that means $\mathcal{U}$ is an iteration tree of limit length λ such that there is no $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{U})$ for $\mathcal{U}$ in $M_{n-1}^{\#}(x)$. Since N has no Woodin cardinals, this can only be the case if

$$\mathcal{M}(\mathcal{U}) = \mathcal{M}_{\lambda}^{\mathcal{U}},$$

where $\mathcal{M}_{\lambda}^{\mathcal{U}}$ is the limit model for $\mathcal{U}$ which exists in V since N is Π_n^1-iterable in V. Therefore we have as in the remark after the proof of Lemma 2.2.8 that the coiteration on the N-side already finished.

If we assume that $M \cap \mathrm{Ord}$ is definably Woodin over M it follows by the same argument that the coiteration on the M-side already finished if it breaks down, because M is ω_1-iterable in V.

Therefore the coiteration of M and N is successful in the sense of Corollary 2.2.12 even if $M \cap \mathrm{Ord}$ or $N \cap \mathrm{Ord}$ or both of them are definably Woodin over M or N respectively. □

We now aim to fix an order on OD-sets. Therefore we first introduce the following notion.

Definition 2.2.13. *Let $x \in \mathrm{OD}$. Then we say $(n, (\alpha_0, \dots, \alpha_n), \varphi)$ is the* minimal triple defining x *iff $(n, (\alpha_0, \dots, \alpha_n), \varphi)$ is the minimal triple according to the lexicographical order on triples (using the lexicographical order on tuples of ordinals of length $n+1$ and the order on Gödel numbers for formulae) such that*

$$x = \{z \mid V_{\alpha_0} \vDash \varphi(z, \alpha_1, \dots, \alpha_n)\}.$$

Definition 2.2.14. *Let $x, y \in \mathrm{OD}$. Moreover let $(n, (\alpha_0, \dots, \alpha_n), \varphi)$ and $(m, (\beta_0, \dots, \beta_m), \psi)$ be the minimal triples defining x and y respectively as in Definition 2.2.13. Then we say x* is less than y in the order on the OD-sets *and write*

$$x <_{\mathrm{OD}} y,$$

iff $(n, (\alpha_0, \dots, \alpha_n), \varphi)$ is smaller than $(m, (\beta_0, \dots, \beta_m), \psi)$ in the lexicographical order on triples, that means iff either

(a) $n < m$, or
(b) $n = m$ and $(\alpha_0, \dots, \alpha_n) <_{lex} (\beta_0, \dots, \beta_m)$, where $<_{lex}$ denotes the lexicographical order on tuples of ordinals of length $n+1$, or
(c) $(\alpha_0, \dots, \alpha_n) = (\beta_0, \dots, \beta_m)$ and $\ulcorner\varphi\urcorner < \ulcorner\psi\urcorner$, where $\ulcorner\varphi\urcorner$ and $\ulcorner\psi\urcorner$ denote the Gödel numbers of the formulae φ and ψ respectively.

Using this order on OD-sets we can prove the following lemma, which will also be used in the proof of Lemma 2.1.3.

Lemma 2.2.15. *Let $n \geq 2$ and assume that $M_{n-1}^{\#}(z)$ exists and is ω_1-iterable for all reals z. Moreover let x be a real. Then we have for all reals $y \geq_T x$*

such that $y \in M_{n-1}(x)$, that the premice $M_{n-1}(x)$ and $L[E](y)^{M_{n-1}(x)}$ have the same sets of reals and the same OD*-sets of reals in the same order.*

REMARK. With $L[E](y)^{M_{n-1}(x)}$ we denote the resulting model of a fully backgrounded extender construction above the real y as in [MS94] performed inside the model $M_{n-1}(x)$, but with the smallness hypothesis weakened to allow ω-small premice in the construction. For more details about such a construction in a more general setting see also [St93].

PROOF OF LEMMA 2.2.15. We start with some general remarks about the premice we consider. The premouse $L[E](y)^{M_{n-1}(x)}$ still has $n-1$ Woodin cardinals and is ω_1-iterable via an iteration strategy which is induced by the ω_1-iteration strategy for $M_{n-1}(x)$ by §11 and §12 in [MS94]. Therefore the premouse

$$L[E](x)^{L[E](y)^{M_{n-1}(x)}},$$

which again denotes the fully backgrounded extender construction in the sense of [MS94] with the smallness hypothesis weakened to allow not only 1-small premice, but now performed inside the model $L[E](y)^{M_{n-1}(x)}$, also has $n-1$ Woodin cardinals and is ω_1-iterable.

CLAIM 1. *The premice $L[E](y)^{M_{n-1}(x)}$ and $L[E](x)^{L[E](y)^{M_{n-1}(x)}}$ as defined above are $(n-1)$-small.*

PROOF. We first show that the y-premouse $L[E](y)^{M_{n-1}(x)}$ is $(n-1)$-small, using the $(n-1)$-smallness of $M_{n-1}(x)$. Then it follows by the same argument that the x-premouse $L[E](x)^{L[E](y)^{M_{n-1}(x)}}$ is $(n-1)$-small.
So assume toward a contradiction that the premouse $L[E](y)^{M_{n-1}(x)}$ is not $(n-1)$-small and let $N_y^\#$ be the shortest initial segment of $L[E](y)^{M_{n-1}(x)}$ which is not $(n-1)$-small. That means we choose $N_y^\# \trianglelefteq L[E](y)^{M_{n-1}(x)}$ such that it is not $(n-1)$-small, but every proper initial segment of $N_y^\#$ is $(n-1)$-small. In particular $N_y^\#$ is an active y-premouse, so let F be the top extender of $N_y^\#$. Moreover let N_y be the model obtained from $N_y^\#$ by iterating the top extender F out of the universe inside the model $M_{n-1}(x)$. Now consider $L[E](x)^{N_y}$ and let N be the active x-premouse obtained by adding $F \cap L[E](x)^{N_y}$ as a top extender to an initial segment of $L[E](x)^{N_y}$, analogous to Section 2 in [FNS10] to ensure that N is a premouse. The main result in [FNS10] yields that N is ω_1-iterable in V, because $L[E](y)^{M_{n-1}(x)}$, $N_y^\#$ and $L[E](x)^{N_y}$ inherit the iterability from $M_{n-1}(x)$ as is §11 and §12 in [MS94]. Moreover N is not $(n-1)$-small.
Let $N_x^\#$ be the shortest initial segment of N which is not $(n-1)$-small. By Lemma 2.2.8 we can successfully compare the x-premice $N_x^\#$ and $M_{n-1}^\#(x)$, because both premice are ω_1-iterable in V and every proper initial segment of one of them is $(n-1)$-small. Therefore we have that in fact $N_x^\# = M_{n-1}^\#(x)$

and thus

$$\mathbb{R} \cap N_x^{\#} = \mathbb{R} \cap M_{n-1}^{\#}(x) = \mathbb{R} \cap M_{n-1}(x).$$

But $N_x^{\#}$ is by construction a countable premouse in $M_{n-1}(x)$, so this is a contradiction. □

Now fix a real z which codes the x-premouse $M_{n-1}^{\#}(x)$ and work inside the model $M_{n-1}(z)$ for a while.

We have that every proper initial segment of $M_{n-1}^{\#}(x)$ is $(n-1)$-small, $\rho_\omega(M_{n-1}^{\#}(x)) = \omega$, and $M_{n-1}^{\#}(x)$ is ω-sound. Since

$$\omega_2^{M_{n-1}^{\#}(z)} < \omega_1^V,$$

this yields that $M_{n-1}^{\#}(x)$ is (ω_1+1)-iterable inside $M_{n-1}^{\#}(z)$ by Lemma 2.2.8 (2). So in particular, working inside $M_{n-1}(z)$, the x-premouse $M_{n-1}(x)$, obtained from $M_{n-1}^{\#}(x)$ by iterating the top measure out of the universe, is (ω_1+1)-iterable.

Claim 2. *The x-premice $M_{n-1}(x)$ and $L[E](x)^{L[E](y)^{M_{n-1}(x)}}$ agree below the least measurable cardinal in $M_{n-1}(x)$.*

Proof. The x-premouse $L[E](x)^{L[E](y)^{M_{n-1}(x)}}$ is (ω_1+1)-iterable inside $M_{n-1}(z)$ via an iteration strategy which is induced by the iteration strategy for $M_{n-1}(x)$. In particular we can successfully compare the x-premice $M_{n-1}(x)$ and $L[E](x)^{L[E](y)^{M_{n-1}(x)}}$ inside the model $M_{n-1}(z)$ using this iteration strategy and they coiterate to the same premouse. This yields the claim. □

Now we can finally show the following claim, which will finish the proof of Lemma 2.2.15.

Claim 3. *$M_{n-1}(x)$ and $L[E](y)^{M_{n-1}(x)}$ have the same sets of reals and the same* OD*-sets of reals in the same order.*

Proof. Claim 2 implies that if κ denotes the least measurable cardinal in $M_{n-1}(x)$, then we have that

$$V_\kappa^{M_{n-1}(x)} \supseteq V_\kappa^{L[E](y)^{M_{n-1}(x)}} \supseteq V_\kappa^{L[E](x)^{L[E](y)^{M_{n-1}(x)}}} = V_\kappa^{M_{n-1}(x)},$$

and therefore

$$V_\kappa^{M_{n-1}(x)} = V_\kappa^{L[E](y)^{M_{n-1}(x)}}.$$

Thus we can consider $M_{n-1}(x)$ and $L[E](y)^{M_{n-1}(x)}$ as $V_\kappa^{M_{n-1}(x)}$-premice and, still working in $M_{n-1}(z)$, we can successfully compare them by the same argument as in the proof of Claim 2, using the iteration strategy for $L[E](y)^{M_{n-1}(x)}$ which is induced by the iteration strategy for $M_{n-1}(x)$. As $V_\kappa^{M_{n-1}(x)}$-premice $M_{n-1}(x)$ and $L[E](y)^{M_{n-1}(x)}$ coiterate to the same model and hence they have the same sets of reals and the same OD-sets of reals in the same order. □

This proves Lemma 2.2.15. □

Motivated by this lemma we introduce the following notation.

DEFINITION 2.2.16. *For premice M and N we write $M \sim N$ iff M and N have the same sets of reals and the same* OD*-sets of reals in the same order.*

In the proof of Lemma 2.1.3 we will also need the following lemma.

LEMMA 2.2.17. *Let M be an ω_1-iterable premouse with a Woodin cardinal and let $\delta = \delta_M$ denote the least Woodin cardinal in M. Let*

$$(\mathcal{M}_\xi, \mathcal{N}_\xi \mid \xi \in \mathrm{Ord})$$

be the sequence of models obtained from a fully backgrounded extender construction inside M in the sense of [MS94] but with the smallness hypothesis weakened, such that

$$\mathcal{M}_{\xi+1} = \mathcal{C}_\omega(\mathcal{N}_{\xi+1})$$

and let $L[E]$ be the resulting model. Then we have for all $\xi \geq \delta$ that

$$\rho_\omega(\mathcal{M}_\xi) \geq \delta,$$

and therefore

$$\mathcal{M}_\delta = L[E]|\delta.$$

REMARK. This lemma also holds true if we perform the fully backgrounded extender construction to obtain the model $L[E]$ relativized to a real $x \in M$, which we then denote by $L[E](x)$.

PROOF OF LEMMA 2.2.17. Work in M. Assume not and let $\mathcal{M}_\xi$ be the least model with $\xi \geq \delta$ such that $\rho = \rho_\omega(\mathcal{M}_\xi) < \delta$.

The background universe $M|\delta$ is generic over $\mathcal{M}_\xi$ via the δ-version of the Extender Algebra $\mathbb{Q}_\delta$ (see Lemma 1.3 in [SchSt09]), because all extenders that are part of the fully backgrounded extender construction satisfy the axioms of the Extender Algebra. We have that $\mathbb{Q}_\delta$ satisfies the δ-chain condition by the same argument as in the proof that the "classical" version of the Extender Algebra satisfies the δ-chain condition (see Theorem 7.14 in [St10] for the "classical" version and Lemma 1.3 in [SchSt09] for the δ-version of the Extender Algebra).

By definition of ρ there exists an $r\Sigma_{n+1}$-formula ϕ for some $n < \omega$ and a parameter $b \in \mathcal{M}_\xi$ such that

$$a \overset{def}{=} \{x < \rho \mid \mathcal{M}_\xi \vDash \phi(x, b)\} \notin \mathcal{M}_\xi.$$

But we have that $a \in \mathcal{M}_\xi[M|\delta]$ because $a \subseteq \rho < \delta$ and we are working inside the model M. Moreover the set a is bounded in δ, so there exists a name $\tau \in \mathcal{M}_\delta$ such that $\tau^{M|\delta} = a$. Then there exists a condition $p \in \mathbb{Q}_\delta$ such that

$$p \Vdash^{\mathcal{M}_\xi}_{\mathbb{Q}_\delta} \text{“}\tau = \{x < \check{\rho} \mid \mathcal{M}_\xi \vDash \phi(x, \check{b})\}\text{”}.$$

Therefore

$$a = \{x < \rho \mid p \Vdash^{\mathcal{M}_\xi}_{\mathbb{Q}_\delta} \text{“}\check{x} \in \tau\text{”}\}.$$

Note that $\mathbb{Q}_\delta$ is definable over $\mathcal{M}_\delta$.

Claim 1. *We have that*

$$p \Vdash^{\mathcal{M}_\xi}_{\mathbb{Q}_\delta} \text{“}\check{x} \in \tau\text{”} \text{ iff } p \Vdash^{\mathcal{M}_\delta}_{\mathbb{Q}_\delta} \text{“}\check{x} \in \tau\text{”}.$$

Proof. Suppose we have that $p \Vdash^{\mathcal{M}_\xi}_{\mathbb{Q}_\delta}$ “$\check{x} \in \tau$” and $p \nVdash^{\mathcal{M}_\delta}_{\mathbb{Q}_\delta}$ “$\check{x} \in \tau$”. Fix a condition $q \leq p$ such that

$$q \Vdash^{\mathcal{M}_\delta}_{\mathbb{Q}_\delta} \text{“}\neg\check{x} \in \tau\text{”}.$$

Pick an $\mathcal{M}_\xi$ generic filter g such that $q \in g$. This contradicts $p \Vdash^{\mathcal{M}_\xi}_{\mathbb{Q}_\delta}$ “$\check{x} \in \tau$”, because g is also generic over $\mathcal{M}_\delta$.

For the converse suppose that we have $p \Vdash^{\mathcal{M}_\delta}_{\mathbb{Q}_\delta}$ “$\check{x} \in \tau$” and $p \nVdash^{\mathcal{M}_\xi}_{\mathbb{Q}_\delta}$ “$\check{x} \in \tau$”. Fix a $q \leq p$ such that

$$q \Vdash^{\mathcal{M}_\xi}_{\mathbb{Q}_\delta} \text{“}\neg\check{x} \in \tau\text{”}.$$

As above pick an $\mathcal{M}_\xi$ generic filter g such that $q \in g$. Then g is also generic over $\mathcal{M}_\delta$ again and so this contradicts $p \Vdash^{\mathcal{M}_\delta}_{\mathbb{Q}_\delta}$ “$\check{x} \in \tau$”. □

Therefore we have that

$$a = \{x < \rho \mid p \Vdash^{\mathcal{M}_\delta}_{\mathbb{Q}_\delta} \text{“}\check{x} \in \tau\text{”}\}.$$

That means the set a is definable over $\mathcal{M}_\delta$ and thus $a \in \mathcal{M}_\delta$. Therefore we have in particular that $a \in \mathcal{M}_\xi$, a contradiction. □

2.3. OD-Determinacy for an Initial Segment of M_{n-1}

Now we can turn to the proof of Lemma 2.1.3. Recall

Lemma 2.1.3. *Let $n \geq 1$. Assume that $M^\#_{n-1}(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$ and that all Σ^1_{n+1}-definable sets of reals are determined. Then there exists a real y_0 such that for all reals $x \geq_T y_0$,*

$$M_{n-1}(x)|\delta_x \vDash \text{OD-}\textit{determinacy},$$

where δ_x denotes the least Woodin cardinal in $M_{n-1}(x)$ if $n > 1$ and δ_x denotes the least x-indiscernible in $M_0(x) = L[x]$ if $n = 1$.

Proof of Lemma 2.1.3. For $n = 1$ we have that Lemma 2.1.3 immediately follows from Theorem 3.1 in [KS85]. So assume that $n > 1$ and assume further toward a contradiction that there is no such real y_0 as in the statement of Lemma 2.1.3.

Then there are cofinally (in the Turing degrees) many $y \in {}^\omega\omega$ such that there exists an $M \trianglelefteq M_{n-1}(y)|\delta_y$ with

$$M \vDash \neg\text{OD-determinacy}.$$

We want to consider x-premice M for $x \in {}^\omega\omega$ which have the following properties:

(1) $M \vDash$ ZFC, M is countable, and we have that the following formula holds true:

$$\text{for all } k < \omega \text{ and for all } (x_n \mid n < k) \text{ and } (M^n \mid n \leq k) \text{ such that}$$
$$x_n \in M^n \cap {}^\omega\omega,\ x_{n+1} \geq_T x_n \geq_T x,$$
$$M^0 = M,\ M^{n+1} = L[E](x_n)^{M^n}, \text{ and we have}$$
$$\text{for all } n < k \text{ that } M^{n+1} \sim M^n \text{ and}$$
$$M^n \text{ does not have a Woodin cardinal,}$$

where the relation "$\sim$" is as defined in Definition 2.2.16,

(2) M is $(n-1)$-small, and

$$M \vDash \neg\text{OD-determinacy},$$

(3) for all $\xi < M \cap \mathrm{Ord}$ such that $M|\xi$ satisfies property (1),

$$M|\xi \vDash \text{OD-determinacy},$$

and

(4) M is Π^1_n-iterable.

If M is an x-premouse, then we have for every real y such that $y \geq_T x$ and $y \in M$, that

$$M \vDash (1) \quad \Rightarrow \quad L[E](y)^M \vDash (1),$$

where $L[E](y)^M$ as above denotes the resulting model of a fully backgrounded extender construction above the real y inside M as in [MS94] with the smallness hypothesis weakened as for example in the remark after Lemma 2.2.15.

We first show that there exists a Turing cone of reals x such that there exists an x-premouse M satisfying properties (1) - (4) as above.

Claim 1. *For cofinally many $x \in {}^\omega\omega$, $M_{n-1}(x)|\delta_x$ satisfies properties* (1) *and* (2).

Proof. By assumption there are cofinally many $x \in {}^\omega\omega$ such that

$$M_{n-1}(x)|\delta_x \vDash \neg\text{OD-determinacy}.$$

Pick such an $x \in {}^\omega\omega$. Then $M_{n-1}(x)|\delta_x$ is obviously a countable ZFC model without a Woodin cardinal. Moreover it is a proper initial segment of $M_{n-1}(x)$ and therefore $(n-1)$-small. This already implies that property (2) holds true for $M_{n-1}(x)|\delta_x$. Lemma 2.2.15 yields that

$$M_{n-1}(x) \sim L[E](y)^{M_{n-1}(x)},$$

for all reals $y \geq_T x$ such that $y \in M_{n-1}(x)$. Then we have for such a real y that

$$M_{n-1}(x)|\delta_x \sim L[E](y)^{M_{n-1}(x)}|\delta_x = L[E](y)^{M_{n-1}(x)|\delta_x},$$

where the $\sim$-equivalence follows from Lemma 2.2.15 and the equality is given by Lemma 2.2.17. Therefore the formula in property (1) holds true for $M_{n-1}(x)|\delta_x$ as we also have that $L[E](y)^{M_{n-1}(x)|\delta_x}$ does not have a Woodin cardinal. □

By Claim 1 there are cofinally many $x \in {}^\omega\omega$ such that there exists an x-premouse M which satisfies properties $(1) - (4)$ defined above. Such x-premice M can be obtained by taking the smallest initial segment of $M_{n-1}(x)|\delta_x$ which satisfies properties $(1) - (4)$ defined above. Moreover the set

$$A = \{x \in {}^\omega\omega \mid \text{there is an } x\text{-premouse } M \text{ with } (1) - (4)\}$$

is Σ^1_{n+1}-definable and Turing invariant. So by Σ^1_{n+1} (Turing-)determinacy there exists a cone of such reals $x \in A$, because the set A cannot be completely disjoint from a cone of reals since there are cofinally many reals $x \in A$ as argued above. Let v be a base of this cone and consider a real $x \geq_T v$ in the cone.

Claim 2. *If there is an ω_1-iterable x-premouse M with properties* (1) *and* (2)*, then every x-premouse N satisfying properties* $(1)-(4)$ *is in fact already ω_1-iterable.*

Proof. Assume there is such an x-premouse M and let N be an arbitrary x-premouse satisfying properties $(1) - (4)$. By Corollary 2.2.12 we can successfully coiterate the x-premice M and N because they are both $(n-1)$-small, solid and without Woodin cardinals and moreover M is ω_1-iterable and N is Π^1_n-iterable. Let $\mathcal{T}$ and $\mathcal{U}$ be the resulting trees of length $\lambda+1$ for some ordinal λ on M and N respectively, in particular $\mathcal{U}$ might be a putative iteration tree.

We consider three possible cases as follows.

Case 1. We have $\mathcal{M}^{\mathcal{U}}_\lambda \trianglelefteq \mathcal{M}^{\mathcal{T}}_\lambda$ and the iteration from N to $\mathcal{M}^{\mathcal{U}}_\lambda$ is without drops on the main branch.

That means we have that $[0,\lambda]_U \cap \mathcal{D}^{\mathcal{U}} = \emptyset$, where the set $\mathcal{D}^{\mathcal{U}}$ denotes the set of drops in model or degree in the iteration tree $\mathcal{U}$ (see Section 3.1 in [St10] for a formal definition of $\mathcal{D}^{\mathcal{U}}$). In this case we have that the model $\mathcal{M}^{\mathcal{U}}_\lambda$ is fully well-founded and N is elementary embedded into an ω_1-iterable model and thus ω_1-iterable itself.

Case 2. We have $\mathcal{M}^{\mathcal{T}}_\lambda \triangleleft \mathcal{M}^{\mathcal{U}}_\lambda$ and the iteration from M to $\mathcal{M}^{\mathcal{T}}_\lambda$ is without drops on the main branch.

In this case $\mathcal{M}^{\mathcal{U}}_\lambda$ need not be fully well-founded, but this will not affect our argument to follow, because we have that $\mathcal{M}^{\mathcal{U}}_\lambda$ is well-founded up to $\mathcal{M}^{\mathcal{T}}_\lambda \cap \mathrm{Ord}$. So there exists an ordinal $\alpha < \mathcal{M}^{\mathcal{U}}_\lambda \cap \mathrm{Ord}$ such that $\mathcal{M}^{\mathcal{U}}_\lambda|\alpha = \mathcal{M}^{\mathcal{T}}_\lambda$ and we have by elementarity that

$$\mathcal{M}^{\mathcal{T}}_\lambda \vDash \neg\text{OD-determinacy},$$

since property (2) holds for M. Since $\alpha = \mathcal{M}^{\mathcal{T}}_\lambda \cap \mathrm{Ord}$ it follows that

$$\mathcal{M}^{\mathcal{U}}_\lambda|\alpha \vDash \neg\text{OD-determinacy}.$$

Moreover we have that $\mathcal{M}_\lambda^{\mathcal{U}}|\alpha \vDash (1)$, because we have by elementarity that $\mathcal{M}_\lambda^{\mathcal{T}} \vDash (1)$. If there is no drop in the iteration from N to $\mathcal{M}_\lambda^{\mathcal{U}}$, this contradicts the minimality property (3) for N by elementarity. But even if there is a drop on the main branch in $\mathcal{U}$ this statement is transferred along the branch to N by the following argument.

Assume there is a drop at stage $\beta + 1$ on the main branch through $\mathcal{U}$, that means $\mathcal{M}_{\beta+1}^*$ is a proper initial segment of $\mathcal{M}_\gamma$ for $\gamma = \text{pred}_U(\beta+1)$, where $\mathcal{M}_{\beta+1}^*$ is the model to which the next extender F_β from the $\mathcal{M}_\beta$ sequence is applied in the iteration as introduced in Section 3.1 in [St10]. Since there can only be finitely many drops along the main branch through the iteration tree $\mathcal{U}$, we can assume further without loss of generality that this is the only drop along the main branch through $\mathcal{U}$. (If there is more than one drop on the main branch through $\mathcal{U}$, we repeat the argument to follow for the remaining drops.) Then by elementarity there is an ordinal $\alpha' < \mathcal{M}_{\beta+1}^* \cap \text{Ord} < \mathcal{M}_\gamma \cap \text{Ord}$ such that

$$\mathcal{M}_{\beta+1}^*|\alpha' \vDash (1) + \neg\text{OD-determinacy}.$$

But then also

$$\mathcal{M}_\gamma|\alpha' \vDash (1) + \neg\text{OD-determinacy},$$

and therefore by elementarity there is an ordinal $\alpha'' < N \cap \text{Ord}$ such that

$$N|\alpha'' \vDash (1) + \neg\text{OD-determinacy}.$$

This now again contradicts the minimality property (3) for N.

Case 3. We have $\mathcal{M}_\lambda^{\mathcal{T}} = \mathcal{M}_\lambda^{\mathcal{U}}$, there is no drop on the main branch in the iteration from M to $\mathcal{M}_\lambda^{\mathcal{T}}$, but there is a drop on the main branch in the iteration from N to $\mathcal{M}_\lambda^{\mathcal{U}}$.

This immediately is a contradiction because it implies that we have $\mathcal{M}_\lambda^{\mathcal{T}} \vDash$ ZFC, but at the same time $\rho_\omega(\mathcal{M}_\lambda^{\mathcal{U}}) < \mathcal{M}_\lambda^{\mathcal{U}} \cap \text{Ord}$. □

Since by Claim 1 there are cofinally many reals x such that the ω_1-iterable x-premouse $M_{n-1}(x)|\delta_x$ satisfies properties (1) and (2), Claim 2 yields that the following claim holds true.

Claim 3. *There are cofinally many reals x such that every x-premouse satisfying properties* $(1) - (4)$ *is in fact ω_1-iterable.*

Consider the game G which is defined as follows.

$$\begin{array}{c|cc} \text{I} & x \oplus a & \\ \hline \text{II} & & y \oplus b \end{array} \qquad \text{for } x, a, y, b \in {}^\omega\omega.$$

The players I and II alternate playing natural numbers and the game lasts ω steps. Say player I produces a real $x \oplus a$ and player II produces a real $y \oplus b$. Then player I wins G iff there exists an $(x \oplus y)$-premouse M which satisfies properties $(1) - (4)$ and if A_M denotes the least OD-set of reals in M which is not determined, then $a \oplus b \in A_M$.

This game is Σ^1_{n+1}-definable and therefore determined. So say first player I has a winning strategy τ for G. Recall that v denotes a base for a cone of reals x such that there exists an x-premouse which satisfies properties $(1) - (4)$ and pick a real $z^* \geq_T x_\tau \oplus v$, where x_τ is a real coding the winning strategy τ for player I, such that every z^*-premouse which satisfies properties $(1) - (4)$ is in fact ω_1-iterable (using Claim 3).

We now aim to construct a real $z \geq_T z^*$ such that there is a z-premouse N which satisfies properties $(1) - (4)$, is ω_1-iterable, and satisfies the following additional property (3^*).

(3^*) For all reals $y \in N$ such that $y \geq_T z$ there exists no ordinal $\xi < N \cap \mathrm{Ord}$ such that

$$L[E](y)^N | \xi \vDash (1) + \neg\text{OD-determinacy}.$$

Let M^0 be an arbitrary z^*-premouse satisfying properties $(1) - (4)$, which is therefore ω_1-iterable. Assume that if we let $N = M^0$ and $z = z^*$, then property (3^*) is not satisfied. So there is a real $y \in M^0$ with $y \geq_T z^*$ witnessing the failure of property (3^*) in M^0. Let $y_0 \in M^0$ with $y_0 \geq_T z^*$ be a witness for that fact such that the ordinal $\xi^0 < L[E](y_0)^{M^0} \cap \mathrm{Ord}$ with

$$L[E](y_0)^{M^0} | \xi^0 \vDash (1) + \neg\text{OD-determinacy}$$

is minimal. Let $M^1 = L[E](y_0)^{M^0} | \xi^0$ and assume that

$$M^1 \nvDash (3^*),$$

because otherwise we could pick $N = M^1$ and $z = y_0$ and the construction would be finished. Then as before there is a real $y_1 \in M^1$ with $y_1 \geq_T y_0$ and a minimal ordinal $\xi^1 < L[E](y_1)^{M^1} \cap \mathrm{Ord} = M^1 \cap \mathrm{Ord} = \xi^0$ such that

$$L[E](y_1)^{M^1} | \xi^1 \vDash (1) + \neg\text{OD-determinacy}.$$

This construction has to stop at a finite stage, because otherwise we have that $\xi^0 > \xi^1 > \dots$ is an infinite descending chain of ordinals. Therefore there is a natural number $n < \omega$ such that

$$M^n = L[E](y_{n-1})^{M^{n-1}} | \xi^{n-1},$$

and

$$M^n \vDash (3^*).$$

Let $z = y_{n-1}$ and $N = M^n$. Then we have that $z \geq_T z^*$ and N is a z-premouse which satisfies properties (1) and (3^*). Moreover by minimality of the ordinal ξ^{n-1} we have that N satisfies property (3). From the construction we also get that

$$N \vDash \neg\text{OD-determinacy}.$$

Furthermore N inherits properties (2) and (4) and the ω_1-iterability from M^0 since it is obtained by performing multiple fully backgrounded extender constructions inside the ω_1-iterable premouse M^0. Here the fact that N is $(n-1)$-small follows from the $(n-1)$-smallness of M^0 by an argument we already gave earlier in the proof of Lemma 2.2.15. Thus N and z are as desired.

Let A_N denote the least non-determined OD-set of reals in N. We define a strategy τ^* for player I in the usual Gale-Stewart game $G(A_N)$ with payoff set A_N played inside the model N as follows. Assume player II produces the real $b \in N$. Then we consider the following run of the original game G defined above:

$$\begin{array}{c|ll} \text{I} & x \oplus a = \tau((z \oplus b) \oplus b) & \\ \hline \text{II} & & (z \oplus b) \oplus b \end{array}$$

Player II plays the real $(z \oplus b) \oplus b$ and player I responds with the real $x \oplus a$ according to his winning strategy τ in G. Note that this run of the game G is in the model N. We define the strategy τ^* such that in a run of the game $G(A_N)$ inside N according to τ^* player I has to respond to the real b with producing the real a.

$$\begin{array}{c|ll} \text{I} & a = \tau^*(b) & \\ \hline \text{II} & & b \end{array}$$

Claim 4. *τ^* is a winning strategy for player* I *in the Gale-Stewart game with payoff set A_N played in N.*

This claim implies that A_N is determined in N, contradicting that A_N was assumed to be the least non-determined OD-set of reals in N.

Proof of Claim 4. Since τ is a winning strategy for player I in the original game G, there exists an $(x \oplus (z \oplus b))$-premouse N' which satisfies properties $(1) - (4)$ such that

$$a \oplus b \in A_{N'},$$

where $A_{N'}$ denotes the least non-determined OD-set of reals in N'.

We want to show that

$$A_{N'} = A_N$$

in order to conclude that τ^* is a winning strategy for player I in the Gale-Stewart game with payoff set A_N played in N.

Property (1) yields that

$$L[E](x \oplus (z \oplus b))^N \sim N,$$

because $x, z, b \in N$ and $x \oplus (z \oplus b) \geq_T z$. Therefore $L[E](x \oplus (z \oplus b))^N$ is an $(x \oplus (z \oplus b))$-premouse which satisfies property (2), because it has the same sets of reals and the same OD-sets of reals as N and hence

$$L[E](x \oplus (z \oplus b))^N \vDash \neg\text{OD-determinacy}.$$

The $(n-1)$-smallness of $L[E](x\oplus(z\oplus b))^N$ follows from the $(n-1)$-smallness of N by an argument we already gave earlier in the proof of Lemma 2.2.15. Moreover $L[E](x \oplus (z \oplus b))^N$ inherits property (1) from N.

Since $L[E](x \oplus (z \oplus b))^N$ is the result of a fully backgrounded extender construction inside the ω_1-iterable premouse N, it is ω_1-iterable itself. Therefore Claim 2 yields that in particular the $(x \oplus (z \oplus b))$-premouse N' is also ω_1-iterable, because it was choosen such that it satisfies properties $(1)-(4)$. So we can coiterate $L[E](x \oplus (z \oplus b))^N$ and N' by the remark after Lemma 2.2.8 since they are both $(n-1)$-small ω_1-iterable $(x\oplus(z\oplus b))$-premice which do not have Woodin cardinals. Thus by the minimality of N' from property (3) and an argument analogous to the one we already gave in the proof of Claim 2, we have that

$$L[E](x \oplus (z \oplus b))^N \geq^* N',$$

where $\leq^*$ denotes the usual mouse order[1]. Moreover we have minimality for the premouse $L[E](x \oplus (z \oplus b))^N$ in the sense of property (3^*) for N. This yields again by an argument analogous to the one we already gave in the proof of Claim 2 that

$$L[E](x \oplus (z \oplus b))^N \leq^* N'.$$

Therefore we have that in fact

$$L[E](x \oplus (z \oplus b))^N =^* N',$$

and hence

$$L[E](x \oplus (z \oplus b))^N \sim N'.$$

Using $L[E](x \oplus (z \oplus b))^N \sim N$ it follows that

$$N \sim N',$$

and thus $A_N = A_{N'}$. □

Now suppose player II has a winning strategy σ in the game G introduced above and recall that v is a base of a cone of reals x such that there exists an x-premouse which satisfies properties $(1)-(4)$. Analogous to the situation when player I has a winning strategy, we pick a real $z^* \geq_T x_\sigma \oplus v$, where x_σ is a real coding the winning strategy σ for player II, such that every z^*-premouse which satisfies properties $(1)-(4)$ is already ω_1-iterable (see Claims 1 and 2).

As in the argument for player I we can construct a real $z \geq_T z^*$ such that there exists a z-premouse N which satisfies properties $(1)-(4)$, is ω_1-iterable, and satisfies the additional property (3^*). As before we let A_N denote the

[1]We say an ω_1-iterable premouse M is smaller or equal in the *mouse order* than an ω_1-iterable premouse N and write "$M \leq^* N$" iff M and N successfully coiterate to premice M^* and N^* such that $M^* \trianglelefteq N^*$. Moreover we say that N and M are equal in the mouse order and write "$M =^* N$" iff $M \leq^* N$ and $N \leq^* M$.

least non-determined OD-set in N, which exists because the z-premouse N satisfies property (2).

Now we define a strategy σ^* for player II in the usual Gale-Stewart game $G(A_N)$ with payoff set A_N played inside N as follows. Assume that player I produces a real $a \in N$ in a run of the game $G(A_N)$ inside the model N. Then we consider the following run of the game G:

I	$(z \oplus a) \oplus a$
II	$y \oplus b = \sigma((z \oplus a) \oplus a)$

Player I plays a real $(z \oplus a) \oplus a$ and player II responds with a real $y \oplus b$ according to his winning strategy σ in G. We define the strategy σ^* such that in a run of the game $G(A_N)$ inside the model N according to σ^* player II has to respond to the real a with producing the real b.

I	a
II	$b = \sigma^*(a)$

Claim 5. *σ^* is a winning strategy for player* II *in the Gale-Stewart game with payoff set A_N played in N.*

This claim implies that A_N is determined in N, again contradicting that A_N was assumed to be the least non-determined OD-set of reals in N.

Proof of Claim 5. We first want to show that the $((z \oplus a) \oplus y)$-premouse $L[E]((z \oplus a) \oplus y)^N$ satifies properties $(1)-(4)$.

First property (1) for N yields that

$$L[E]((z \oplus a) \oplus y)^N \sim N,$$

because we have $z, a, y \in N$. Therefore $L[E]((z \oplus a) \oplus y)^N$ is a $((z \oplus a) \oplus y)$-premouse which satisfies property (2), because it has the same sets of reals and the same OD-sets of reals as N and hence as before

$$L[E]((z \oplus a) \oplus y)^N \vDash \neg\text{OD-determinacy}.$$

The $(n-1)$-smallness of $L[E]((z \oplus a) \oplus y)^N$ again follows by an argument we already gave in the proof of Lemma 2.2.15. Moreover $L[E]((z \oplus a) \oplus y)^N$ inherits condition (1) from N.

Since $L[E]((z \oplus a) \oplus y)^N$ is a fully backgrounded extender construction inside the ω_1-iterable mouse N it is ω_1-iterable itself by §12 in [MS94]. Therefore $L[E]((z \oplus a) \oplus y)^N$ satisfies properties $(1), (2)$ and (4). By property (3^*) for the z-premouse N we additionally have that $L[E]((z \oplus a) \oplus y)^N$ also satisfies property (3).

Since σ is a winning strategy for player II in the original game G, we have that if a, b, z and y are as above, then

$$a \oplus b \notin A_{N'},$$

for all $((z \oplus a) \oplus y)$-premice N' satisfying properties $(1) - (4)$, where $A_{N'}$ denotes the least non-determined OD-set of reals in N'. Now let

$$N' = L[E]((z \oplus a) \oplus y)^N,$$

so we have that in particular N' is a $((z \oplus a) \oplus y)$-premouse and satisfies properties $(1) - (4)$ as above.
As in the previous case where we assumed that player I has a winning strategy in G, we want to show that

$$A_{N'} = A_N$$

in order to conclude that σ^* is a winning strategy for player II in the Gale-Stewart game with payoff set A_N played inside N, using that N' satisfies properties $(1) - (4)$ as argued above.
Using $L[E]((z \oplus a) \oplus y)^N \sim N$ it follows that

$$N \sim N',$$

and thus $A_N = A_{N'}$, as desired. □

This finishes the proof of Lemma 2.1.3. □

2.4. Applications

This section is devoted to two important corollaries of Lemma 2.1.3 which are going to be used in Sections 2.5 and 3.7.

Corollary 2.4.1. *Let $n \geq 1$. Assume that $M^\#_{n-1}(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$ and that all Σ^1_{n+1}-definable sets of reals are determined. Then*

$$\omega_1^{M_{n-1}(x)} \textit{ is measurable in } \mathrm{HOD}^{M_{n-1}(x)|\delta_x},$$

for a cone of reals x.

Proof. This follows from Lemma 2.1.3 with a generalized version of Solovay's theorem that, under the Axiom of Determinacy AD, ω_1 is measurable. For the readers convenience, we will present a proof of this result, following the proof of the classical result as in Theorem 12.18 (b) in [Sch14] or Lemma 6.2.2 in [SchSt].
Lemma 2.1.3 yields that

$$M_{n-1}(x)|\delta_x \vDash \text{OD-determinacy},$$

for a cone of reals x. Let $x \in {}^\omega\omega$ be an arbitrary element of this cone and let us work inside $M_{n-1}(x)|\delta_x$ for the rest of the proof. We aim to define a $< \omega_1$-complete ultrafilter $\mathcal{U}$ inside $\mathrm{HOD}^{M_{n-1}(x)|\delta_x}$ on $\omega_1 \overset{def}{=} \omega_1^{M_{n-1}(x)|\delta_x} = \omega_1^{M_{n-1}(x)}$, witnessing that ω_1 is measurable in $\mathrm{HOD}^{M_{n-1}(x)|\delta_x}$.

Let $n, m \mapsto \langle n, m\rangle$ be the Gödel pairing function for $n, m < \omega$ and recall that

$$\mathsf{WO} \stackrel{def}{=} \{x \in {}^\omega\omega \mid R_x \text{ is a well-ordering}\},$$

where we let $(n, m) \in R_x$ iff $x(\langle n, m\rangle) = 1$ for $x \in {}^\omega\omega$. For $y \in \mathsf{WO}$ we write $||y||$ for the order type of R_y and for $x \in {}^\omega\omega$ we let

$$|x| = \sup\{||y|| \mid y \in \mathsf{WO} \wedge y \equiv_T x\}.$$

Consider the set $S = \{|x| \mid x \in {}^\omega\omega\}$ and let $\pi : \omega_1 \to S$ be an order isomorphism. Now we define the filter $\mathcal{U}$ on ω_1 as follows. For $A \subset \omega_1$ such that $A \in \mathrm{OD}$ we let

$$A \in \mathcal{U} \ \text{ iff } \ \{x \in {}^\omega\omega \mid |x| \in \pi\text{"}A\} \text{ contains a cone of reals.}$$

Claim 1. *$\mathcal{U} \cap \mathrm{HOD}$ is a $< \omega_1^{M_{n-1}(x)}$-complete ultrafilter in* HOD*.*

Proof. The set $\{x \in {}^\omega\omega \mid |x| \in \pi\text{"}A\}$ is Turing invariant. Therefore we have that OD-Turing-determinacy implies that the set $\{x \in {}^\omega\omega \mid |x| \in \pi\text{"}A\}$ for $A \in \mathrm{OD}$ either contains a cone of reals or is completely disjoint from a cone of reals. Hence we have that $\mathcal{U} \cap \mathrm{HOD}$ is an ultrafilter on $\omega_1 = \omega_1^{M_{n-1}(x)}$ in HOD. Moreover the following argument shows that this ultrafilter $\mathcal{U} \cap \mathrm{HOD}$ is $< \omega_1^{M_{n-1}(x)}$-complete in HOD.
Let $\{A_\alpha \mid \alpha < \gamma\} \subset \mathcal{U} \cap \mathrm{HOD}$ be such that $\{A_\alpha \mid \alpha < \gamma\} \in \mathrm{HOD}$ for an ordinal $\gamma < \omega_1^{M_{n-1}(x)}$. Then there is a sequence $(a_\alpha \mid \alpha < \gamma)$ of reals such that for each $\alpha < \gamma$, the real a_α is a base for a cone of reals contained in $\{x \in {}^\omega\omega \mid |x| \in \pi\text{"}A_\alpha\}$. Since $\gamma < \omega_1^{M_{n-1}(x)}$, we can fix a bijection $f : \omega \to \gamma$ in $M_{n-1}(x)$. But then $\bigoplus_{n<\omega} a_{f(n)}$ is a base for a cone of reals contained in

$$\bigcap_{\alpha<\gamma} \{x \in {}^\omega\omega \mid |x| \in \pi\text{"}A_\alpha\} = \{x \in {}^\omega\omega \mid |x| \in \pi\text{"} \bigcap_{\alpha<\gamma} A_\alpha\}.$$

So we have that $\bigcap_{\alpha<\gamma} A_\alpha \in \mathcal{U} \cap \mathrm{HOD}$ and thus the filter $\mathcal{U} \cap \mathrm{HOD}$ is $< \omega_1^{M_{n-1}(x)}$-complete. □

Therefore $\mathcal{U} \cap \mathrm{HOD}$ witnesses that $\omega_1^{M_{n-1}(x)}$ is measurable in $\mathrm{HOD}^{M_{n-1}(x)|\delta_x}$. □

In what follows we will prove that in the same situation as above $\omega_2^{M_{n-1}(x)}$ is strongly inaccessible in $\mathrm{HOD}^{M_{n-1}(x)|\delta_x}$, which is another consequence of Lemma 2.1.3. This is going to be used later in Section 3.7.
In fact the following theorem holds true. It is due to W. Hugh Woodin and a consequence of the "Generation Theorems" in [KW10] (see Theorem 5.4 in [KW10]).

Theorem 2.4.2. *Let $n \geq 1$. Assume that $M_{n-1}^\#(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$ and that all Σ^1_{n+1}-definable sets of reals are determined. Then* *for a cone of reals x,*

$$\omega_2^{M_{n-1}(x)} \text{ is a Woodin cardinal in } \mathrm{HOD}^{M_{n-1}(x)|\delta_x}.$$

In order to make this thesis more self-contained, we shall not use Theorem 2.4.2 here, though, and we will give a proof of the following theorem, which is essentially due to Moschovakis, and will be used in Section 3.7. A version of it can also be found in [KW10] (see Theorem 3.9 in [KW10]).

THEOREM 2.4.3. *Let $n \geq 1$. Assume that $M_{n-1}^{\#}(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$ and that all Σ^1_{n+1}-definable sets of reals are determined. Then for a cone of reals x,*

$$\omega_2^{M_{n-1}(x)} \text{ is strongly inaccessible in } \mathrm{HOD}^{M_{n-1}(x)|\delta_x}.$$

PROOF. Using Lemma 2.1.3 we have as above that there is a cone of reals x such that

$$M_{n-1}(x)|\delta_x \vDash \text{OD-determinacy}.$$

Let x be an element of that cone.

CLAIM 1. *We have that $\omega_2^{M_{n-1}(x)} = (\Theta_0)^{M_{n-1}(x)}$, where*

$$\Theta_0 = \sup\{\alpha \mid \text{ there exists an } \mathrm{OD}\text{-surjection } f : {}^\omega\omega \to \alpha\}.$$

PROOF. Work inside the model $M_{n-1}(x)$. Since CH holds in $M_{n-1}(x)$, it follows that $\Theta_0 \leq \omega_2$. For the other inequality let $\alpha < \omega_2$ be arbitrary. Then there exists an OD_x-surjection $g : {}^\omega\omega \to \alpha$ because by definability (using the definability results from [St95]) we have that

$$M_{n-1}(x)|\omega_2^{M_{n-1}(x)} \subseteq \mathrm{HOD}_x^{M_{n-1}(x)}.$$

This implies that there is an OD-surjection $f : {}^\omega\omega \times {}^\omega\omega \to \alpha$ by varying x and thus it follows that $\alpha \leq \Theta_0$. □

Work inside the model $M_{n-1}(x)|\delta_x$ from now on and note that it is trivial that $\Theta_0 = \omega_2$ is a regular cardinal in HOD. So we focus on proving that ω_2 is a strong limit. For this purpose we fix an arbitrary ordinal $\alpha < \omega_2$ and prove that $|\mathcal{P}(\alpha)^{\mathrm{HOD}}| < \omega_2$.

Since $\alpha < \omega_2 = \Theta_0$, we can fix a surjection $f : {}^\omega\omega \to \alpha$ such that $f \in \mathrm{OD}$. This surjection f induces a prewellordering $\leq_f \in \mathrm{OD}$ on ${}^\omega\omega$ if we let

$$x \leq_f y \text{ iff } f(x) \leq f(y)$$

for $x, y \in {}^\omega\omega$. Now consider the pointclass $\boldsymbol{\Sigma}^1_1(\leq_f)$ which is defined as follows. For a set of reals A (or analogously for a set $A \subset ({}^\omega\omega)^k$ for some $k < \omega$) we say $A \in \boldsymbol{\Sigma}^1_1(\leq_f)$ iff there is a Σ_0-formula φ and a real $z \in {}^\omega\omega$ such that

$$A = \{y \in {}^\omega\omega \mid \exists x \in {}^\omega\omega \; \varphi(y, x, \leq_f, {}^\omega\omega \setminus \leq_f, z)\}.$$

The pointclass $\Sigma^1_1(\leq_f)$ is defined analogous without the parameter z. We have that there exists a universal $\Sigma^1_1(\leq_f)$-definable set $U \subseteq {}^\omega\omega \times {}^\omega\omega$ for the pointclass $\mathbf{\Sigma}^1_1(\leq_f)$. So we have that for every $\mathbf{\Sigma}^1_1(\leq_f)$-definable set A there exists a $z \in {}^\omega\omega$ such that $A = U_z = \{x \in {}^\omega\omega \mid (z, x) \in U\}$. Now it suffices to prove the following claim.

Claim 2. *Let $X \subset \alpha$ with $X \in \mathrm{OD}$ be arbitrary. Then there exists a $\mathbf{\Sigma}^1_1(\leq_f)$-definable set $A \subset {}^\omega\omega$ such that $X = f\,"A$.*

Using Claim 2 we can define a surjection

$$g : {}^\omega\omega \to \mathcal{P}(\alpha) \cap \mathrm{OD}$$

such that $g \in \mathrm{OD}$ by letting $g(z) = f\,"U_z$ for $z \in {}^\omega\omega$. This yields that we have $|\mathcal{P}(\alpha)^{\mathrm{HOD}}| < \omega_2$ as desired.

Therefore we are left with proving Claim 2 to finish the proof of Theorem 2.4.3. The proof of Claim 2 is mainly a special case of Moschovakis' Coding Lemma as in Theorem 3.2 in [KW10], so we will outline the proof in this special case.

Proof of Claim 2. Let $X \in \mathcal{P}(\alpha) \cap \mathrm{OD}$ be arbitrary. We aim to show that there is a real $z \in {}^\omega\omega$ such that $X = f\,"U_z$. Let

$$B = \{z \in {}^\omega\omega \mid f\,"U_z \subseteq X\}.$$

Moreover let α_z for $z \in B$ be the minimal ordinal β such that $\beta \in X \setminus f\,"U_z$, if it exists. We aim to show that there exists a real $z \in B$ such that α_z does not exist. So assume toward a contradiction that the ordinal α_z exists for all $z \in B$.

Now consider the following game G of length ω, where player I and player II alternate playing natural numbers such that in the end player I plays a real x and player II plays a real y.

$$\begin{array}{c|cc} \mathrm{I} & x & \\ \hline \mathrm{II} & & y \end{array} \quad \text{for } x, y \in {}^\omega\omega.$$

We define that player I wins the game G iff

$$x \in B \wedge (y \in B \rightarrow \alpha_x \geq \alpha_y).$$

Note that we have $B \in \mathrm{OD}$ since $f, U, X \in \mathrm{OD}$. Therefore the game G is OD and thus determined by our hypothesis.

Assume first that player I has a winning strategy σ in G. For a real y let $(\sigma * y)_{\mathrm{I}}$ denote player I's moves in a run of the game G, where player II plays y and player I responds according to his winning strategy σ. Then there exists a real z_0 such that

$$U_{z_0} = \bigcup \{U_{(\sigma * y)_{\mathrm{I}}} \mid y \in {}^\omega\omega\}$$

because the right hand side of this equation is $\mathbf{\Sigma}^1_1(\leq_f)$-definable by choice of U. Since σ is a winning strategy for player I, we have that $(\sigma * y)_{\mathrm{I}} \in B$

for all $y \in {}^\omega\omega$ and thus it follows that $z_0 \in B$. Moreover we have that $\alpha_{(\sigma * y)_{\mathrm{I}}} \leq \alpha_{z_0}$ for all $y \in {}^\omega\omega$ by definition of z_0.

Now we aim to construct a play z^* for player II defeating the strategy σ. Since $f : {}^\omega\omega \to \alpha$ is a surjection we can choose $a \in {}^\omega\omega$ such that $f(a) = \alpha_{z_0}$. Moreover we let $z^* \in {}^\omega\omega$ be such that $U_{z^*} = U_{z_0} \cup \{a\}$. Then we have that

$$f\,"U_{z^*} = f\,"U_{z_0} \cup \{f(a)\} = f\,"U_{z_0} \cup \{\alpha_{z_0}\} \subset X,$$

since $z_0 \in B$. Hence $z^* \in B$. Moreover we have that

$$\alpha_{z^*} > \alpha_{z_0} \geq \alpha_{(\sigma * y)_{\mathrm{I}}}$$

for all $y \in {}^\omega\omega$. Therefore player II can defeat σ by playing the real z^*, contradicting the fact that σ is a winning strategy for player I.

Assume now that player II has a winning strategy τ in the game G. Let

$$h_0 : {}^\omega\omega \times {}^\omega\omega \to {}^\omega\omega$$

be a $\Sigma^1_1(\leq_f)$-definable function such that for all $y, z \in {}^\omega\omega$,

$$U_{h_0(z,y)} = U_z \cap \{x \in {}^\omega\omega \mid f(x) < f(y)\}.$$

Choose $h_1 : {}^\omega\omega \to {}^\omega\omega$ such that h_1 is $\boldsymbol{\Sigma}^1_1(\leq_f)$-definable and

$$U_{h_1(z)} = \bigcup \{U_{(h_0(z,y) * \tau)_{\mathrm{II}}} \cap \{x \in {}^\omega\omega \mid f(x) = f(y)\} \mid y \in {}^\omega\omega\},$$

where the notion $(h_0(z,y) * \tau)_{\mathrm{II}}$ is defined analogous to the corresponding notion for player I introduced above. By Kleene's Recursion Theorem (see for example Theorem 3.1 in [KW10]) there exists a fixed point for h_1 with respect to the set U, that means there exists a real $z^* \in {}^\omega\omega$ such that we have

$$U_{z^*} = U_{h_1(z^*)}.$$

Now our first step is to prove that $z^* \in B$. Assume toward a contradiction that $(f\,"U_{z^*}) \setminus X \neq \emptyset$ and let $\gamma_0 \in (f\,"U_{z^*}) \setminus X$ be minimal. Moreover let $y_0 \in U_{z^*}$ be such that $f(y_0) = \gamma_0$. Then

$$\gamma_0 \in f\,"U_{z^*} = f\,"U_{h_1(z^*)}$$

and by definition of the function h_1 it follows that $\gamma_0 \in f\,"U_{(h_0(z^*,y_0) * \tau)_{\mathrm{II}}}$. Since γ_0 was picked to be minimal in $(f\,"U_{z^*}) \setminus X$, we have $h_0(z^*, y_0) \in B$ because we have by definition that

$$U_{h_0(z^*,y_0)} = U_{z^*} \cap \{x \in {}^\omega\omega \mid f(x) < f(y_0)\} = U_{z^*} \cap \{x \in {}^\omega\omega \mid f(x) < \gamma_0\}$$

and thus $f\,"U_{h_0(z^*,y_0)} \subseteq X$. Since τ is a winning strategy for player II, we have that $(h_0(z^*, y_0) * \tau)_{\mathrm{II}} \in B$. Taken all together it follows that

$$\gamma_0 \in f\,"U_{(h_0(z^*,y_0) * \tau)_{\mathrm{II}}} \subseteq X.$$

This contradicts the fact that $\gamma_0 \in (f\,"U_{z^*}) \setminus X$.

Recall that we assumed toward a contradiction that the ordinal α_{z^*} exists. Let $a^* \in {}^\omega\omega$ be such that

$$f(a^*) = \alpha_{z^*}$$

and note that such an a^* exists since $f : {}^\omega\omega \rightarrow \alpha$ is a surjection and $\alpha_{z^*} < \alpha$. Then we have by definition of the function h_0 that $h_0(z^*, a^*) \in B$ because $z^* \in B$. Moreover we have that $\alpha_{z^*} = \alpha_{h_0(z^*,a^*)}$ holds by definition of α_{z^*} since $f(a^*) = \alpha_{z^*}$. As τ is a winning strategy for player II in the game G, we finally have that

$$\alpha_{(h_0(z^*,a^*)*\tau)_{\mathrm{II}}} > \alpha_{h_0(z^*,a^*)} = \alpha_{z^*},$$

because $h_0(z^*, a^*) \in B$. This contradicts the fact that

$$U_{(h_0(z^*,a^*)*\tau)_{\mathrm{II}}} \subset U_{h_1(z^*)} = U_{z^*},$$

by definition of α_{z^*} and $\alpha_{(h_0(z^*,a^*)*\tau)_{\mathrm{II}}}$. Therefore the ordinal α_{z^*} does not exist and thus we finally have that $f\,{}''U_{z^*} = X$, as desired. □

This finishes the proof of Theorem 2.4.3. □

2.5. A Proper Class Inner Model with n Woodin Cardinals

In this section we are now able to apply the results from the previous sections to show the existence of a proper class inner model with n Woodin cardinals from determinacy for $\boldsymbol{\Pi}^1_n$- and Π^1_{n+1}-definable sets (if we assume inductively that $\boldsymbol{\Pi}^1_n$ determinacy implies that $M^{\#}_{n-1}(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$). This is done in the following theorem, which is a generalization of Theorem 7.7 in [St96] using Lemma 2.1.3 and Corollary 2.4.1.

THEOREM 2.5.1. *Let $n \geq 1$. If $M^{\#}_{n-1}(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$ and all Σ^1_{n+1}-definable sets of reals are determined, then there exists a proper class inner model with n Woodin cardinals.*

We are not claiming here that the model obtained in Theorem 2.5.1 is iterable in any sense. We will show how to construct an ω_1-iterable premouse with n Woodin cardinals using this model in the next chapter, but for that we need to assume slightly more determinacy (namely a consequence of determinacy for all $\boldsymbol{\Sigma}^1_{n+1}$-definable sets of reals).

PROOF. As before let δ_x denote the least Woodin cardinal in $M_{n-1}(x)$ if $n > 1$ and let δ_x denote the least x-indiscernible in $L[x] = M_0(x)$ if $n = 1$. Then we have that according to Lemma 2.1.3, there is a real x such that for all reals $y \geq_T x$,

$$M_{n-1}(y)|\delta_y \vDash \text{OD-determinacy}.$$

Fix such a real x.

In the case $n = 1$ we have that Theorem 2.5.1 immediately follows from Theorem 7.7 in [St96], so assume $n > 1$.

Let $(K^c)^{M_{n-1}(x)|\delta_x}$ denote the result of a K^c-construction in the sense of Chapter 1 in [St96] performed inside the model $M_{n-1}(x)|\delta_x$. Then we distinguish three cases as follows.

Case 1. Assume that $(K^c)^{M_{n-1}(x)|\delta_x}$ has no Woodin cardinals and is fully iterable inside $M_{n-1}(x)|\delta_x$ via the iteration strategy Σ which is guided by $\mathcal{Q}$-structures as in Definition 2.2.2.

In this case we can isolate the core model $K^{M_{n-1}(x)|\delta_x}$ below δ_x as in Theorem 1.1 in [JS13]. Then the core model $K^{M_{n-1}(x)|\delta_x}$ is absolute for all forcings of size less than δ_x over $M_{n-1}(x)|\delta_x$ and moreover $K^{M_{n-1}(x)|\delta_x}$ satisfies weak covering using [MSch95]. That means we have that $M_{n-1}(x)|\delta_x \vDash$ "$(\alpha^+)^K = \alpha^+$" for all singular cardinals α.
Let $\alpha = \aleph_\omega^{M_{n-1}(x)}$. Then α is singular in $M_{n-1}(x)$, so we have in particular that

$$M_{n-1}(x)|\delta_x \vDash \text{“}(\alpha^+)^K = \alpha^+\text{”}.$$

Moreover we have that α is a cutpoint of $M_{n-1}(x)$. So let $z \in {}^\omega\omega$ be generic over $M_{n-1}(x)$ for $\mathrm{Col}(\omega, \alpha)$. Then for $y = x \oplus z$ we have that

$$M_{n-1}(x)[z] = M_{n-1}(y),$$

where we construe $M_{n-1}^\#(x)[z]$ as a y-mouse and as a y-mouse $M_{n-1}^\#(x)[z]$ is sound and $\rho_\omega(M_{n-1}^\#(x)[z]) = y$ (see [SchSt09] for the fine structural details). Moreover we have that

$$M_{n-1}(y)|\delta_y \vDash \text{OD-determinacy},$$

since $y \geq_T x$. This implies that

$$M_{n-1}(x)[z]|\delta_y \vDash \text{OD-determinacy}.$$

Now work in the model $M_{n-1}(x)[z]|\delta_y$. Then we have that OD-determinacy implies that ω_1 is measurable in HOD as in Corollary 2.4.1.
Since $K \subseteq$ HOD and we have that $\omega_1 = (\alpha^+)^K$, it follows that $\omega_1 = (\alpha^+)^{\mathrm{HOD}}$. But HOD $\vDash$ AC, so in particular in HOD all measurable cardinals are inaccessible. This is a contradiction.

Case 2. Assume that there is a Woodin cardinal in $(K^c)^{M_{n-1}(x)|\delta_x}$.

In this case we aim to show that there exists a proper class inner model with n Woodin cardinals, which is obtained by performing a fully backgrounded extender construction inside $M_{n-1}(x)$ on top of the model

$$(K^c)^{M_{n-1}(x)|\delta_x} \mid \delta,$$

where δ denotes the largest Woodin cardinal in $(K^c)^{M_{n-1}(x)|\delta_x}$.
We can assume without loss of generality that there is a largest Woodin cardinal in the model $(K^c)^{M_{n-1}(x)|\delta_x}$ if it has a Woodin cardinal, because if there is no largest one, then $(K^c)^{M_{n-1}(x)|\delta_x}$ already yields a proper class inner model with n Woodin cardinals by iterating some large enough extender out of the universe. By the same argument we can in fact assume that $(K^c)^{M_{n-1}(x)|\delta_x}$ is $(n-1)$-small above δ.

Let

$$(\mathcal{M}_\xi, \mathcal{N}_\xi \mid \xi \in \mathrm{Ord})$$

be the sequence of models obtained from a fully backgrounded extender construction above $(K^c)^{M_{n-1}(x)|\delta_x} \mid \delta$ inside $M_{n-1}(x)$ in the sense of [MS94] but with the smallness hypothesis weakened where

$$\mathcal{M}_{\xi+1} = \mathcal{C}_\omega(\mathcal{N}_{\xi+1})$$

and let

$$L[E]((K^c)^{M_{n-1}(x)|\delta_x} \mid \delta)^{M_{n-1}(x)}$$

denote the resulting model.

Case 2.1. There is no $\xi \in \mathrm{Ord}$ such that δ is not definably Woodin over the model $\mathcal{M}_{\xi+1}$.

In this case δ is a Woodin cardinal inside $L[E]((K^c)^{M_{n-1}(x)|\delta_x} \mid \delta)^{M_{n-1}(x)}$ and it follows by a generalization of Theorem 11.3 in [MS94] that we have that $L[E]((K^c)^{M_{n-1}(x)|\delta_x} \mid \delta)^{M_{n-1}(x)}$ is a proper class inner model with n Woodin cardinals, as desired.

Case 2.2. There exists a $\xi \in \mathrm{Ord}$ such that δ is not definably Woodin over the model $\mathcal{M}_{\xi+1}$.

Let ξ be the minimal such ordinal. In this case the premouse $\mathcal{M}_{\xi+1}$ is $(n-1)$-small above δ (see the proof of Claim 1 in the proof of Lemma 2.2.15) and we have that

$$\mathcal{M}_{\xi+1} \in M_{n-1}(x)|\delta_x.$$

Consider the coiteration of $\mathcal{M}_{\xi+1}$ and $(K^c)^{M_{n-1}(x)|\delta_x}$ inside $M_{n-1}(x)|\delta_x$.

Claim 1. *The coiteration of $\mathcal{M}_{\xi+1}$ and $(K^c)^{M_{n-1}(x)|\delta_x}$ inside $M_{n-1}(x)|\delta_x$ is successful.*

Proof. First of all we have that the coiteration takes place above δ and the premouse $\mathcal{M}_{\xi+1}$ is ω_1-iterable above δ in V by construction (see [MS94]). Therefore the proof of Lemma 2.2.8 (2) yields that in the model $M_{n-1}(x)|\delta_x$ we have that $\mathcal{M}_{\xi+1}$ is iterable for iteration trees in $H_{\delta_x}^{M_{n-1}(x)}$ which are above δ, since $\mathcal{M}_{\xi+1} \in M_{n-1}(x)|\delta_x$ is $(n-1)$-small above δ and $\rho_\omega(\mathcal{M}_{\xi+1}) \leq \delta$.

Moreover we have that $(K^c)^{M_{n-1}(x)|\delta_x}$ is countably iterable above δ inside $M_{n-1}(x)|\delta_x$ by the iterability proof in Chapter 9 in [St96].

Assume now toward a contradiction that the coiteration of $(K^c)^{M_{n-1}(x)|\delta_x}$ with $\mathcal{M}_{\xi+1}$ inside $M_{n-1}(x)|\delta_x$ is not successful. Since as argued above $\mathcal{M}_{\xi+1}$ is iterable above δ inside $M_{n-1}(x)|\delta_x$ and the coiteration takes place above δ this means that the coiteration has to fail on the $(K^c)^{M_{n-1}(x)|\delta_x}$-side.

The premouse $(K^c)^{M_{n-1}(x)|\delta_x}$ is assumed to be $(n-1)$-small above δ and therefore the fact that the coiteration of $(K^c)^{M_{n-1}(x)|\delta_x}$ and $\mathcal{M}_{\xi+1}$ fails on the $(K^c)^{M_{n-1}(x)|\delta_x}$-side, implies that there exists an iteration tree $\mathcal{T}$ on $(K^c)^{M_{n-1}(x)|\delta_x}$ of limit length such that there is no $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{T})$ for $\mathcal{T}$ such that $\mathcal{Q}(\mathcal{T}) \trianglelefteq M_{n-2}^{\#}(\mathcal{M}(\mathcal{T}))$ and hence

$$M_{n-2}^{\#}(\mathcal{M}(\mathcal{T})) \vDash \text{“}\delta(\mathcal{T}) \text{ is Woodin”}.$$

In particular we have that the premouse $M_{n-2}^{\#}(\mathcal{M}(\mathcal{T}))$ constructed in the sense of Definition 2.2.6 is not $(n-2)$-small above $\delta(\mathcal{T})$ since otherwise it would already provide a $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{T})$ for $\mathcal{T}$ which is $(n-2)$-small above $\delta(\mathcal{T})$.

Let $\bar{M}$ be the Mostowski collapse of a countable substructure of $M_{n-1}(x)|\delta_x$ containing the iteration tree $\mathcal{T}$. That means for a large enough natural number m we let $\bar{M}, X$ and σ be such that

$$\bar{M} \overset{\sigma}{\cong} X \prec_{\Sigma_m} M_{n-1}(x)|\delta_x,$$

where

$$\sigma : \bar{M} \rightarrow M_{n-1}(x)|\delta_x$$

denotes the uncollapse map such that we have a model $\bar{K}$ in $\bar{M}$ with $\sigma(\bar{K}|\gamma) = (K^c)^{M_{n-1}(x)|\delta_x}|\sigma(\gamma)$ for every ordinal $\gamma < \bar{M} \cap \mathrm{Ord}$, and we have an iteration tree $\bar{\mathcal{T}}$ on $\bar{K}$ in $\bar{M}$ with $\sigma(\bar{\mathcal{T}}) = \mathcal{T}$. Moreover we let $\bar{\delta} \in \bar{M}$ be such that $\sigma(\bar{\delta}) = \delta$.

By the iterability proof of Chapter 9 in [St96] applied inside the model $M_{n-1}(x)|\delta_x$, there exists a cofinal well-founded branch b through the iteration tree $\bar{\mathcal{T}}$ on $\bar{K}$ above $\bar{\delta}$. Moreover we have that

$$M_{n-2}^{\#}(\mathcal{M}(\bar{\mathcal{T}})) \vDash \text{“}\delta(\bar{\mathcal{T}}) \text{ is Woodin”}$$

and $M_{n-2}^{\#}(\mathcal{M}(\bar{\mathcal{T}}))$ is not $(n-2)$-small above $\delta(\bar{\mathcal{T}})$.

Consider the coiteration of $\mathcal{M}_b^{\bar{\mathcal{T}}}$ with $M_{n-2}^{\#}(\mathcal{M}(\bar{\mathcal{T}}))$ and note that it takes place above $\delta(\bar{\mathcal{T}})$. Since $M_{n-2}^{\#}(\mathcal{M}(\bar{\mathcal{T}}))$ is ω_1-iterable above $\delta(\bar{\mathcal{T}})$ and $\mathcal{M}_b^{\bar{\mathcal{T}}}$ is iterable above $\bar{\delta} < \delta(\bar{\mathcal{T}})$ by the iterability proof of Chapter 9 in [St96] applied inside $M_{n-1}(x)|\delta_x$, the coiteration is successful using Lemma 2.2.8 (2). We have that $\mathcal{M}_b^{\bar{\mathcal{T}}}$ cannot loose the coiteration by the following argument. If there is no drop along the branch b, then $\mathcal{M}_b^{\bar{\mathcal{T}}}$ cannot loose the coiteration, because then there is no definable Woodin cardinal in $\mathcal{M}_b^{\bar{\mathcal{T}}}$ above $\bar{\delta}$ by elementarity, but at the same time we have that

$$M_{n-2}^{\#}(\mathcal{M}(\bar{\mathcal{T}})) \vDash \text{“}\delta(\bar{\mathcal{T}}) > \bar{\delta} \text{ is Woodin”}.$$

If there is a drop along b, then $\mathcal{M}_b^{\bar{\mathcal{T}}}$ also has to win the coiteration, because we have that $\rho_\omega(\mathcal{M}_b^{\bar{\mathcal{T}}}) < \delta(\bar{\mathcal{T}})$ and $\rho_\omega(M_{n-2}^{\#}(\mathcal{M}(\bar{\mathcal{T}}))) = \delta(\bar{\mathcal{T}})$.

That means there is an iterate $\mathcal{R}^*$ of $\mathcal{M}_b^{\bar{\mathcal{T}}}$ and a non-dropping iterate $\mathcal{M}^*$ of $M_{n-2}^{\#}(\mathcal{M}(\bar{\mathcal{T}}))$ such that $\mathcal{M}^* \trianglelefteq \mathcal{R}^*$. We have that $\mathcal{M}^*$ is not $(n-1)$-small above $\bar{\delta}$, because $M_{n-2}^{\#}(\mathcal{M}(\bar{\mathcal{T}}))$ is not $(n-1)$-small above $\bar{\delta}$ as argued above and the iteration from $M^{\#}_{n-2}(\mathcal{M}(\bar{\mathcal{T}}))$ to $\mathcal{M}^*$ is non-dropping. Therefore it follows that $\mathcal{R}^*$ is not $(n-1)$-small above $\bar{\delta}$ and thus $\mathcal{M}_b^{\bar{\mathcal{T}}}$ is not $(n-1)$-small above $\bar{\delta}$. By the iterability proof of Chapter 9 in [St96] we can re-embed the model $\mathcal{M}_b^{\bar{\mathcal{T}}}$ into a model of the $(K^c)^{M_{n-1}(x)|\delta_x}$-construction above $(K^c)^{M_{n-1}(x)|\delta_x} \mid \delta$. This yields that $(K^c)^{M_{n-1}(x)|\delta_x}$ is not n-small, contradicting our assumption that it is $(n-1)$-small above δ. □

From Claim 1 it now follows by universality of $(K^c)^{M_{n-1}(x)|\delta_x}$ above δ (see Corollary 3.6 in [St96]) that the $(K^c)^{M_{n-1}(x)|\delta_x}$-side has to win the comparison. That means there is an iterate K^* of $(K^c)^{M_{n-1}(x)|\delta_x}$ and an iterate N^* of $\mathcal{M}_{\xi+1}$ which is non-dropping on the main branch such that

$$N^* \trianglelefteq K^*.$$

But this is a contradiction, because we assumed that δ is not definably Woodin over $\mathcal{M}_{\xi+1}$ and at the same time we have that

$$(K^c)^{M_{n-1}(x)|\delta_x} \vDash \text{“}\delta \text{ is a Woodin cardinal”}.$$

This finishes the case that there is a Woodin cardinal in $(K^c)^{M_{n-1}(x)|\delta_x}$.

Case 3. Assume that there is no Woodin cardinal in $(K^c)^{M_{n-1}(x)|\delta_x}$ and that the premouse $(K^c)^{M_{n-1}(x)|\delta_x}$ is not fully iterable inside $M_{n-1}(x)|\delta_x$ via the iteration strategy Σ.

The failure of the attempt to iterate $(K^c)^{M_{n-1}(x)|\delta_x}$ via the $\mathcal{Q}$-structure iteration strategy Σ implies that there exists an iteration tree $\mathcal{T}$ of limit length on $(K^c)^{M_{n-1}(x)|\delta_x}$ in $M_{n-1}(x)|\delta_x$ such that there exists no $\mathcal{Q}$-structure for $\mathcal{T}$ inside the model $M_{n-1}(x)|\delta_x$.
Let

$$(\mathcal{M}_\xi, \mathcal{N}_\xi \mid \xi \in \mathrm{Ord})$$

be the sequence of models obtained from a fully backgrounded extender construction above $\mathcal{M}(\mathcal{T})$ inside $M_{n-1}(x)$ in the sense of [MS94] but with the smallness hypothesis weakened where

$$\mathcal{M}_{\xi+1} = \mathcal{C}_\omega(\mathcal{N}_{\xi+1})$$

and let

$$L[E](\mathcal{M}(\mathcal{T}))^{M_{n-1}(x)}$$

denote the resulting model.

Case 3.1. There is no $\xi \in \mathrm{Ord}$ such that $\delta(\mathcal{T})$ is not definably Woodin over the model $\mathcal{M}_{\xi+1}$.

In this case $\delta(\mathcal{T})$ is a Woodin cardinal inside $L[E](\mathcal{M}(\mathcal{T}))^{M_{n-1}(x)}$ and it follows as in Case 2.1 by a generalization of Theorem 11.3 in [MS94] that $L[E](\mathcal{M}(\mathcal{T}))^{M_{n-1}(x)}$ is a proper class inner model with n Woodin cardinals, as desired.

Case 3.2. There exists a $\xi \in \mathrm{Ord}$ such that $\delta(\mathcal{T})$ is not definably Woodin over the model $\mathcal{M}_{\xi+1}$.

Let ξ be the minimal such ordinal. In this case the premouse $\mathcal{M}_{\xi+1}$ is $(n-1)$-small above $\delta(\mathcal{T})$ and we have that

$$\mathcal{M}_{\xi+1} \in M_{n-1}(x)|\delta_x.$$

But then $\mathcal{M}_{\xi+1} \rhd \mathcal{M}(\mathcal{T})$ already provides a $\mathcal{Q}$-structure for $\mathcal{T}$ inside the model $M_{n-1}(x)|\delta_x$ because $\delta(\mathcal{T})$ is not definably Woodin over $\mathcal{M}_{\xi+1}$. This is a contradiction. $\square$

Note that all results we proved in this chapter under a lightface determinacy hypothesis relativize to all $x \in {}^\omega\omega$ if we assume the analogous boldface determinacy hypothesis. We just decided to present the results without additional parameters to simplify the notation.

CHAPTER 3

Proving Iterability

With Theorem 2.5.1 we found a candidate for M_n in the previous chapter, but we still have to show its iterability. We will in fact not prove that this candidate is iterable, but we will use it to construct an ω_1-iterable premouse $M_n^\#$ in the case that n is odd. (Note here already that we will give a different argument if n is even.)

We do parts of this in a slightly more general context and therefore introduce the concept of an n-suitable premouse in Section 3.1, which will be a natural candidate for the premouse $M_n|(\delta_0^+)^{M_n}$, where δ_0 denotes the least Woodin cardinal in M_n. Using n-suitable premice we will show under a determinacy hypothesis that $M_n^\#$ exists and is ω_1-iterable if n is odd. Before we prove this as Theorem 2.1.1 in Sections 3.5 and 3.6, we start with a "warm up" in Section 3.4 to introduce some concepts of the proof which are going to be reused later. We will show that $M_n^\#$ exists and is ω_1-iterable for even n in Section 3.7.

In this chapter again all results we are going to prove under a lightface determinacy hypothesis relativize to all $x \in {}^\omega\omega$ under the analogous boldface determinacy hypothesis.

3.1. Existence of n-suitable Premice

After introducing pre-n-suitable premice and proving their existence from the results in the previous chapter, we aim to show in this section that pre-$(2n-1)$-suitable premice, which are premice with one Woodin cardinal which satisfy certain fullness conditions, also satisfy a weak form of iterability, namely short tree iterability. In fact we are going to show a slightly stronger form of iterability which includes that fullness properties are preserved during non-dropping iterations. This will in particular enable us to perform certain comparison arguments for $(2n-1)$-small premice and will therefore help us to conclude ω_1-iterability for some candidate for $M_{2n-1}^\#$.

Recall that in what follows by "$M_n^\#$ exists" we always mean that "$M_n^\#$ exists and is ω_1-iterable".

A lot of the results in this section only hold true for premice at the odd levels of our argument, namely $(2n-1)$-suitable premice. This results from the periodicity in the projective hierarchy in terms of the uniformization property (see Lemma 3.4.2) and the periodicity in the correctness of $M_n^\#$

(see Lemmas 1.3.4 and 1.3.6). This behaviour forces us to give a different proof for the even levels of our argument in Section 3.7.

We start by introducing pre-n-suitable and n-suitable premice. Our definition will generalize the notion of suitability from Definition 3.4 in [StW16] to $n > 1$. For technical reasons our notion slightly differs from n-suitability as defined in Definition 5.2 in [Sa13].

Definition 3.1.1. *Let $n \geq 1$ and assume that $M_{n-1}^{\#}(x)$ exists for all $x \in {}^{\omega}\omega$. Then we say a countable premouse N is* pre-n-suitable *iff there is an ordinal $\delta < \omega_1^V$ such that*

(1) $N \vDash$ "ZFC$^-$ + δ is the largest cardinal",

$$N = M_{n-1}(N|\delta) \,|\, (\delta^+)^{M_{n-1}(N|\delta)},$$

and for every $\gamma < \delta$,

$$M_{n-1}(N|\gamma) \,|\, (\gamma^+)^{M_{n-1}(N|\gamma)} \lhd N,$$

(2) $M_{n-1}(N|\delta)$ is a proper class model and

$$M_{n-1}(N|\delta) \vDash \text{“}\delta \text{ is Woodin”},$$

(3) for every $\gamma < \delta$, $M_{n-1}(N|\gamma)$ is a set, or

$$M_{n-1}(N|\gamma) \nvDash \text{“}\gamma \text{ is Woodin”},$$

and

(4) for every $\eta < \delta$, $M_{n-1}(N|\delta) \vDash$ "$N|\delta$ is η-iterable".

Recall the definition of the premouse $M_{n-1}(N|\delta)$ from Definition 2.2.6. If N is a pre-n-suitable premouse, we denote the unique ordinal δ from Definition 3.1.1 by δ_N, analogous to the notation fixed in Section 2.1.

Whenever we assume that some premouse N is pre-n-suitable for some $n \geq 1$, we in fact tacitly assume in addition that the premouse $M_{n-1}^{\#}(x)$ exists for all $x \in {}^{\omega}\omega$ (or at least that the premouse $M_{n-1}^{\#}(N|\delta)$ exists).

Remark. Clearly, if it exists, $M_n|(\delta^+)^{M_n}$ is a pre-n-suitable premouse for $n \geq 1$, whenever δ denotes the least Woodin cardinal in M_n.

Remark. We have that for $n \geq 1$, if N is a pre-n-suitable premouse, then N is n-small.

We first show that the proper class inner model with n Woodin cardinals we constructed in the proof of Theorem 2.5.1 yields a pre-n-suitable premouse, if we cut it off at the successor of its least Woodin cardinal and minimize it.

Lemma 3.1.2. *Let $n \geq 1$. Assume that $M_{n-1}^{\#}(x)$ exists for all $x \in {}^{\omega}\omega$ and that all Σ_{n+1}^1-definable sets of reals are determined. Then there exists a pre-n-suitable premouse.*

Proof. Let W be the model constructed in Cases 2 and 3 in the proof of Theorem 2.5.1. Cutting off at the successor of the bottom Woodin cardinal δ yields a premouse $N = W|(\delta^+)^W$ which satisfies conditions (1) and (2) in the definition of pre-n-suitability, in the case that the premouse $(K^c)^{M_{n-1}(x)|\delta_x}$ from the proof of Theorem 2.5.1 is $(n-1)$-small above δ. Otherwise we can easily consider an initial segment N of $(K^c)^{M_{n-1}(x)|\delta_x}$ which satisfies conditions (1) and (2). Let N' be the minimal initial segment of N which satisfies conditions (1) and (2) and note that we then have that N' satisfies condition (3).

Now the iterability proof from Chapter 9 in [St96] shows that this premouse $N'|\delta'$ is countably iterable inside $M_{n-1}(x)$, where x is a real as in the proof of Theorem 2.5.1 such that the model W as above is constructed inside the model $M_{n-1}(x)$ and δ' denotes the largest cardinal in N'. The $\mathcal{Q}$-structures for iteration trees $\mathcal{T}$ on $N'|\delta'$ are $(n-1)$-small above the common part model and are therefore contained in the model $M_{n-1}(N'|\delta')$ by arguments we already gave several times before. Thus it follows that $N'|\delta'$ is η-iterable inside $M_{n-1}(N'|\delta')$ for all $\eta < \delta'$. Therefore we have that condition (4) holds as well for N'. □

We can show that the following weak form of condensation holds for pre-n-suitable premice.

Lemma 3.1.3 (Weak Condensation Lemma). *Let N be a pre-n-suitable premouse for some $n \geq 1$ and let δ_N denote the largest cardinal in N. Let γ be a large enough countable ordinal in V and let H be the Mostowski collapse of $Hull_m^{M_{n-1}(N|\delta_N)|\gamma}(\{\delta_N\})$ for some large enough natural number m. Then*

$$H \lhd M_{n-1}(N|\delta_N).$$

Proof. Consider the coiteration of the premice $N|\delta_N$ and H inside the model $M_{n-1}(N|\delta_N)$. By condition (4) in the definition of pre-n-suitability we have that $N|\delta_N$ is iterable enough for the comparison with H inside the model $M_{n-1}(N|\delta_N)$. Moreover H is also iterable enough for the comparison via the realization strategy (see Section 4 in [MaSt94]). So the comparison is successful and there are iterates N^* of $N|\delta_N$ and H^* of H such that $H^* \unlhd N^*$ or $N^* \unlhd H^*$. The Weak Dodd-Jensen Lemma (see Theorem 4.10 in [St10]) yields that in fact $H^* \unlhd N^*$ and that the iteration from H to H^* does not drop.

Assume toward a contradiction that $H \neq H^*$, that means there is at least one extender used in the iteration from H to H^*. Since $\rho_\omega(H) = \omega$, this implies that H^* is not sound and yields a contradiction to the fact that we have $H^* \unlhd N^*$. So it follows that H is not moved in the coiteration and therefore $H = H^*$.

Assume now that the premouse $N|\delta_N$ moves in the coiteration, that means we have that $N^* \neq N|\delta_N$. Let E_α for some ordinal α be the first extender used in the iteration from $N|\delta_N$ to N^*. Then the index α of the extender

E_α is a cardinal in N^*. Since we have $\rho_\omega(H) = \omega$ and $H \trianglelefteq N^*$, it follows that $\alpha > H \cap \mathrm{Ord}$. Therefore there was no need to iterate $N|\delta_N$ at all and we have that

$$H \trianglelefteq N|\delta_N \triangleleft M_{n-1}(N|\delta_N),$$

as desired. □

Analogous to Definitions 3.6 and 3.9 in [StW16] we define a notion of short tree iterability for pre-n-suitable premice. Informally a pre-n-suitable premouse is short tree iterable if it is iterable with respect to iteration trees for which there are $\mathcal{Q}$-structures (see Definition 2.2.1) which are not too complicated. For this definition we again tacitly assume that $M_{n-1}^{\#}(x)$ exists for all reals x.

Definition 3.1.4. *Let $\mathcal{T}$ be a normal iteration tree of length $< \omega_1^V$ on a pre-n-suitable premouse N for some $n \geq 1$. We say $\mathcal{T}$ is* short *iff for all limit ordinals $\lambda < \mathrm{lh}(\mathcal{T})$ the $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ exists, is $(n-1)$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$ and we have that*

$$\mathcal{Q}(\mathcal{T} \upharpoonright \lambda) \trianglelefteq \mathcal{M}_\lambda^{\mathcal{T}},$$

and if $\mathcal{T}$ has limit length we in addition have that $\mathcal{Q}(\mathcal{T})$ exists and

$$\mathcal{Q}(\mathcal{T}) \trianglelefteq M_{n-1}(\mathcal{M}(\mathcal{T})).$$

Moreover we say $\mathcal{T}$ is maximal *iff $\mathcal{T}$ is not short.*

The premouse $M_{n-1}(\mathcal{M}(\mathcal{T}))$ in Definition 3.1.4 is defined as in Definition 2.2.6.

Definition 3.1.5. *Let N be a pre-n-suitable premouse for some $n \geq 1$. We say N is* short tree iterable *iff whenever $\mathcal{T}$ is a short tree on N,*

(i) if $\mathcal{T}$ has a last model, then every putative[1] iteration tree $\mathcal{U}$ extending $\mathcal{T}$ such that $\mathrm{lh}(\mathcal{U}) = \mathrm{lh}(\mathcal{T}) + 1$ has a well-founded last model, and
(ii) if $\mathcal{T}$ has limit length, then there exists a unique cofinal well-founded branch b through $\mathcal{T}$ such that

$$\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T}).$$

Remark. At this point in contrast to the notion of short tree iterability for 1-suitable premice in [StW16] we do not require the iterate of a pre-n-suitable premouse via a short tree to be pre-n-suitable again. The reason for this is that in the general case for $n > 1$ it is not obvious that this property holds assuming only our notion of short tree iterability as defined above. We will be able to prove later in Lemma 3.1.9 that this property in fact does hold true.

[1]Recall that we say an iteration tree $\mathcal{U}$ is a *putative iteration tree* if $\mathcal{U}$ satisfies all properties of an iteration tree, but we allow the last model of $\mathcal{U}$ to be ill-founded, in case $\mathcal{U}$ has a last model.

Because of the periodicity in the projective hierarchy (see also [St95] for the periodicity in the definition of Π^1_n-iterability) the proof of the following lemma only works for odd levels of suitability.

LEMMA 3.1.6. *Let $n \geq 0$ and assume that $M^{\#}_{2n}(x)$ exists for all $x \in {}^\omega\omega$. Let N be a pre-$(2n+1)$-suitable premouse. Then the statement "N is short tree iterable" as in Definition 3.1.5 is Π^1_{2n+2}-definable uniformly in any code for the countable premouse N.*

PROOF. The statement "N is short tree iterable" can be phrased as follows. We first consider trees of limit length.

$$\begin{aligned}
&\forall \mathcal{T} \text{ tree on } N \text{ of limit length } \forall\, (\mathcal{Q}_\lambda \mid \lambda \leq \mathrm{lh}(\mathcal{T}) \text{ limit ordinal}),\\
&\quad \text{if for all limit ordinals } \lambda \leq \mathrm{lh}(\mathcal{T}),\\
&\qquad \mathcal{Q}_\lambda \text{ is } \Pi^1_{2n+1}\text{-iterable above } \delta(\mathcal{T} \upharpoonright \lambda), 2n\text{-small above } \delta(\mathcal{T} \upharpoonright \lambda),\\
&\qquad \text{solid above } \delta(\mathcal{T} \upharpoonright \lambda) \text{ and a } \mathcal{Q}\text{-structure for } \mathcal{T} \upharpoonright \lambda, \text{ and}\\
&\quad \text{if for all limit ordinals } \lambda < \mathrm{lh}(\mathcal{T}) \text{ we have } \mathcal{Q}_\lambda \trianglelefteq \mathcal{M}^{\mathcal{T}}_\lambda, \text{ then}\\
&\quad \exists b \text{ cofinal branch through } \mathcal{T} \text{ such that } \mathcal{Q}_{\mathrm{lh}(\mathcal{T})} \trianglelefteq \mathcal{M}^{\mathcal{T}}_b.
\end{aligned}$$

This statement is Π^1_{2n+2}-definable uniformly in any code for N since Π^1_{2n+1}-iterability above $\delta(\mathcal{T} \upharpoonright \lambda)$ for $\mathcal{Q}_\lambda$ is Π^1_{2n+1}-definable uniformly in any code for $\mathcal{Q}_\lambda$. For trees of successor length we get a similar statement as follows.

$$\begin{aligned}
&\forall \mathcal{T} \text{ putative tree on } N \text{ of successor length } \forall\, (\mathcal{Q}_\lambda \mid \lambda < \mathrm{lh}(\mathcal{T}) \text{ limit ordinal}),\\
&\quad \text{if for all limit ordinals } \lambda < \mathrm{lh}(\mathcal{T}),\\
&\qquad \mathcal{Q}_\lambda \text{ is } \Pi^1_{2n+1}\text{-iterable above } \delta(\mathcal{T} \upharpoonright \lambda), 2n\text{-small above } \delta(\mathcal{T} \upharpoonright \lambda),\\
&\qquad \text{solid above } \delta(\mathcal{T} \upharpoonright \lambda) \text{ and a } \mathcal{Q}\text{-structure for } \mathcal{T} \upharpoonright \lambda, \text{ and}\\
&\quad \text{if for all limit ordinals } \lambda < \mathrm{lh}(\mathcal{T}) \text{ we have } \mathcal{Q}_\lambda \trianglelefteq \mathcal{M}^{\mathcal{T}}_\lambda, \text{ then}\\
&\quad \text{the last model of } \mathcal{T} \text{ is well-founded.}
\end{aligned}$$

As above this statement is also Π^1_{2n+2}-definable uniformly in any code for N. Moreover the conjunction of these two statements is equivalent to the statement "N is short tree iterable", because the relevant $\mathcal{Q}$-structures $\mathcal{Q}_\lambda$ for limit ordinals $\lambda \leq \mathrm{lh}(\mathcal{T})$ are $2n$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$ and thus Lemma 2.2.10 implies that for them it is enough to demand Π^1_{2n+1}-iterability above $\delta(\mathcal{T} \upharpoonright \lambda)$ to identify them as a $\mathcal{Q}$-structure for $\mathcal{T} \upharpoonright \lambda$ since we assumed that $M^{\#}_{2n}(x)$ exists for all $x \in {}^\omega\omega$. □

From this we can obtain the following corollary using Lemma 1.3.4.

COROLLARY 3.1.7. *Let $n \geq 0$ and assume that $M^{\#}_{2n}(x)$ exists for all $x \in {}^\omega\omega$. If N is a pre-$(2n+1)$-suitable premouse, then N is short tree iterable iff N is short tree iterable inside the model $M_{2n}(N|\delta_N)^{\mathrm{Col}(\omega,\delta_N)}$, where δ_N again denotes the largest cardinal in N.*

PROOF. Let N be an arbitrary pre-$(2n+1)$-suitable premouse. By Lemma 3.1.6 we have that short tree iterability for N is a Π^1_{2n+2}-definable statement uniformly in any code for N. Therefore we have by Lemma 1.3.4 that N is short tree iterable inside the model $M_{2n}(N|\delta_N)^{\mathrm{Col}(\omega,\delta_N)}$ iff N is short tree iterable in V, because the model $M_{2n}(N|\delta_N)^{\mathrm{Col}(\omega,\delta_N)}$ is $\mathbf{\Sigma}^1_{2n+2}$-correct in V. □

In what follows we aim to show that every pre-$(2n+1)$-suitable premouse N is short tree iterable. In fact we are going to show a stronger form of iterability for pre-$(2n+1)$-suitable premice, including for example fullness-preservation for short trees. This means that for non-dropping short trees $\mathcal{T}$ on N of length $\lambda+1$ for some ordinal $\lambda < \omega_1^V$ the resulting model of the iteration $\mathcal{M}^{\mathcal{T}}_\lambda$ is again pre-$(2n+1)$-suitable. Here we mean by "non-dropping" that the tree $\mathcal{T}$ does not drop on the main branch $[0,\lambda]_T$. If this property holds for a pre-$(2n+1)$-suitable premouse N we say that N *has a fullness preserving iteration strategy for short trees.* Moreover we also want to show some form of iterability including fullness-preservation for maximal trees on N. Premice which satify all these kinds of iterability we will call $(2n+1)$-suitable.

The exact form of iterability we are aiming for is introduced in the following definition.

DEFINITION 3.1.8. *Assume that $M^{\#}_{n-1}(x)$ exists for all $x \in {}^\omega\omega$ and let N be a pre-n-suitable premouse for some $n \geq 1$. Then we say that the premouse N is n-*suitable *iff*

(i) N is short tree iterable and whenever $\mathcal{T}$ is a short tree on N of length $\lambda+1$ for some ordinal $\lambda < \omega_1^V$ which is non-dropping on the main branch $[0,\lambda]_T$, then the final model $\mathcal{M}^{\mathcal{T}}_\lambda$ is pre-n-suitable, and

(ii) whenever $\mathcal{T}$ is a maximal iteration tree on N of length λ for some limit ordinal $\lambda < \omega_1^V$ according to the $\mathcal{Q}$-structure iteration strategy, then there exists a cofinal well-founded branch b through $\mathcal{T}$ such that b is non-dropping and the model $\mathcal{M}^{\mathcal{T}}_b$ is pre-n-suitable. In fact we have in this case that

$$\mathcal{M}^{\mathcal{T}}_b = M_{n-1}(\mathcal{M}(\mathcal{T}))|(\delta(\mathcal{T})^+)^{M_{n-1}(\mathcal{M}(\mathcal{T}))}.$$

Now we are ready to prove that every pre-$(2n+1)$-suitable premouse is in fact already $(2n+1)$-suitable, using the iterability we build into condition (4) of Definition 3.1.1 in form of the Weak Condensation Lemma (see Lemma 3.1.3).

LEMMA 3.1.9. *Let $n \geq 0$ and assume that $M^{\#}_{2n}(x)$ exists for all $x \in {}^\omega\omega$. Let N be an arbitrary pre-$(2n+1)$-suitable premouse. Then N is $(2n+1)$-suitable.*

PROOF. Let N be an arbitrary pre-$(2n+1)$-suitable premouse and let $W = M_{2n}(N|\delta_N)$ be a premouse in the sense of Definition 2.2.6, where δ_N

as usual denotes the largest cardinal in N. That means in particular that $N = W|(\delta_N^+)^W$.

We want to show that N is $(2n+1)$-suitable. So we assume toward a contradiction that this is not the case. Say this is witnessed by an iteration tree $\mathcal{T}$ on N.

We want to reflect this statement down to a countable hull. Therefore let m be a large enough natural number, let θ be a large enough ordinal such that in particular

$$W|\theta \prec_{\Sigma_m} W$$

and

$$W|\theta \vDash \mathrm{ZFC}^-,$$

and let

$$\bar{W} \overset{\sigma}{\cong} Hull_m^{W|\theta}(\{\delta_N\}) \prec W|\theta,$$

where $\bar{W}$ is the Mostowski collapse of $Hull_m^{W|\theta}(\{\delta_N\})$ and

$$\sigma : \bar{W} \to Hull_m^{W|\theta}(\{\delta_N\})$$

denotes the uncollapse map such that $\delta_N \in \mathrm{ran}(\sigma)$ and $\sigma(\bar{\delta}) = \delta_N$ for some ordinal $\bar{\delta}$ in $\bar{W}$. Then we have that $\bar{W}$ is sound, $\rho_{m+1}(\bar{W}) = \omega$, and the Weak Condensation Lemma 3.1.3 yields that

$$\bar{W} \lhd W.$$

Case 1. $\mathcal{T}$ is short and witnesses that N is not short tree iterable.

For simplicity assume in this case that $\mathcal{T}$ has limit length since the other case is easier. Then $\mathcal{T}$ witnesses that the following statement $\phi_1(N)$ holds in V.

$$\begin{aligned}
\phi_1(N) \equiv \exists\, & \mathcal{T} \text{ tree on } N \text{ of length } \lambda \text{ for some limit ordinal } \lambda < \omega_1^V \\
& \exists\, (\mathcal{Q}_\gamma \mid \gamma \leq \lambda \text{ limit ordinal}),\ \text{such that for all limit ordinals } \gamma \leq \lambda, \\
& \quad \mathcal{Q}_\gamma \text{ is } \Pi^1_{2n+1}\text{-iterable above } \delta(\mathcal{T} \restriction \gamma), 2n\text{-small above } \delta(\mathcal{T} \restriction \gamma), \\
& \quad \text{solid above } \delta(\mathcal{T} \restriction \gamma) \text{ and a } \mathcal{Q}\text{-structure for } \mathcal{T} \restriction \gamma,\ \text{and} \\
& \text{for all limit ordinals } \gamma < \lambda \text{ we have } \mathcal{Q}_\gamma \trianglelefteq \mathcal{M}_\gamma^{\mathcal{T}},\ \text{but} \\
& \text{there exists no cofinal branch } b \text{ through } \mathcal{T} \text{ such that } \mathcal{Q}_\lambda \trianglelefteq \mathcal{M}_b^{\mathcal{T}}.
\end{aligned}$$

We have that $\phi_1(N)$ is Σ^1_{2n+2}-definable uniformly in any code for N as in the proof of Lemma 3.1.6. See also the proof of Lemma 3.1.6 for the case that $\mathcal{T}$ has successor length.

Case 2. $\mathcal{T}$ is a short tree on N of length $\lambda + 1$ for some ordinal $\lambda < \omega_1^V$ which is non-dropping on the main branch such that the final model $\mathcal{M}_\lambda^{\mathcal{T}}$ is not pre-$(2n+1)$-suitable.

Assume that

$$M_{2n}(\mathcal{M}_\lambda^\mathcal{T} | \delta_{\mathcal{M}_\lambda^\mathcal{T}}) \nvDash \text{“}\delta_{\mathcal{M}_\lambda^\mathcal{T}} \text{ is Woodin”},$$

where $\delta_{\mathcal{M}_\lambda^\mathcal{T}}$ denotes the largest cardinal in $\mathcal{M}_\lambda^\mathcal{T}$. This means that we have $\delta_{\mathcal{M}_\lambda^\mathcal{T}} = i_{0\lambda}^\mathcal{T}(\delta_N)$, where $i_{0\lambda}^\mathcal{T} : N \rightarrow \mathcal{M}_\lambda^\mathcal{T}$ denotes the iteration embedding, which exists since the iteration tree $\mathcal{T}$ is assumed to be non-dropping on the main branch.

Then $\mathcal{T}$ witnesses that the following statement $\phi_2(N)$ holds true in V.

$$\begin{aligned} \phi_2(N) \equiv \exists \mathcal{T} \text{ tree on } N &\text{ of length } \lambda + 1 \text{ for some } \lambda < \omega_1^V \text{ such that} \\ &\mathcal{T} \text{ is non-dropping along } [0, \lambda]_T \text{ and} \\ &\forall \gamma < \text{lh}(\mathcal{T}) \text{ limit } \exists \mathcal{Q} \trianglelefteq \mathcal{M}_\gamma^\mathcal{T} \text{ such that} \\ &\quad \mathcal{Q} \text{ is } \Pi^1_{2n+1}\text{-iterable above } \delta(\mathcal{T} \upharpoonright \gamma), 2n\text{-small above } \delta(\mathcal{T} \upharpoonright \gamma), \\ &\quad \text{solid above } \delta(\mathcal{T} \upharpoonright \gamma), \text{ and a } \mathcal{Q}\text{-structure for } \mathcal{T} \upharpoonright \gamma, \text{ and} \\ &\exists \mathcal{P} \rhd \mathcal{M}_\lambda^\mathcal{T} | \delta_{\mathcal{M}_\lambda^\mathcal{T}} \text{ such that } \mathcal{P} \text{ is } \Pi^1_{2n+1}\text{-iterable above } i_{0\lambda}^\mathcal{T}(\delta_N), \\ &\quad 2n\text{-small above } i_{0\lambda}^\mathcal{T}(\delta_N), i_{0\lambda}^\mathcal{T}(\delta_N)\text{-sound}, \rho_\omega(\mathcal{P}) \leq i_{0\lambda}^\mathcal{T}(\delta_N), \\ &\quad\ \text{and } i_{0\lambda}^\mathcal{T}(\delta_N) \text{ is not definably Woodin over } \mathcal{P}, \end{aligned}$$

where δ_N as above denotes the largest cardinal in N. Recall Definition 2.2.4 for the notion of a definable Woodin cardinal.

We have that $\phi_2(N)$ is Σ^1_{2n+2}-definable uniformly in any code for N.

Case 3. $\mathcal{T}$ is a maximal tree on N of length λ for some limit ordinal $\lambda < \omega_1^V$ such that there is no cofinal well-founded branch b through $\mathcal{T}$ or for every such branch b the premouse $\mathcal{M}_b^\mathcal{T}$ is not pre-$(2n+1)$-suitable.

As $\mathcal{T}$ is maximal, we have that every such branch b is non-dropping and in the case that for every such branch b the premouse $\mathcal{M}_b^\mathcal{T}$ is not pre-$(2n+1)$-suitable, assume that we have

$$M_{2n}(\mathcal{M}_b^\mathcal{T} | \delta_{\mathcal{M}_b^\mathcal{T}}) \nvDash \text{“}\delta_{\mathcal{M}_b^\mathcal{T}} \text{ is Woodin”},$$

where $\delta_{\mathcal{M}_b^\mathcal{T}}$ denotes the largest cardinal in $\mathcal{M}_b^\mathcal{T}$. Then the iteration tree $\mathcal{T}$ witnesses that the following statement $\phi_3(N)$ holds true in V.

$$\begin{aligned} \phi_3(N) \equiv \exists \mathcal{T} \text{ tree on } N &\text{ of length } \lambda \text{ for some limit ordinal } \lambda < \omega_1^V \text{ such that} \\ &\forall \gamma < \lambda \text{ limit } \ \exists \mathcal{Q} \trianglelefteq \mathcal{M}_\gamma^\mathcal{T} \text{ such that} \\ &\quad \mathcal{Q} \text{ is } \Pi^1_{2n+1}\text{-iterable above } \delta(\mathcal{T} \upharpoonright \gamma), 2n\text{-small above } \delta(\mathcal{T} \upharpoonright \gamma), \\ &\quad \text{solid above } \delta(\mathcal{T} \upharpoonright \gamma), \text{ and a } \mathcal{Q}\text{-structure for } \mathcal{T} \upharpoonright \gamma, \text{ and} \end{aligned}$$

$$\exists \mathcal{P} \rhd \mathcal{M}(\mathcal{T}) \text{ such that } \mathcal{P} \text{ is } \Pi^1_{2n+1}\text{-iterable above } \delta(\mathcal{T}),$$
$$\rho_\omega(\mathcal{P}) \leq \delta(\mathcal{T}), \mathcal{P} \text{ is } 2n\text{-small above } \delta(\mathcal{T}), \delta(\mathcal{T})\text{-sound, and}$$
$$\mathcal{P} \vDash \text{“ZFC}^- + \delta(\mathcal{T}) \text{ is the largest cardinal } + \delta(\mathcal{T}) \text{ is Woodin”},$$
$$\text{and there is no branch } b \text{ through } \mathcal{T} \text{ such that } \mathcal{P} \trianglelefteq \mathcal{M}^{\mathcal{T}}_b.$$

We have that $\phi_3(N)$ is Σ^1_{2n+2}-definable uniformly in any code for N.
Now we consider all three cases together again and let

$$\phi(N) = \phi_1(N) \vee \phi_2(N) \vee \phi_3(N).$$

As argued in the individual cases the iteration tree $\mathcal{T}$ witnesses that $\phi(N)$ holds in V (still assuming for simplicity that $\mathcal{T}$ has limit length if $\mathcal{T}$ is as in Case 1).

Since ϕ is a $\boldsymbol{\Sigma}^1_{2n+2}$-definable statement and $W = M_{2n}(N|\delta_N)$, we have by Lemma 1.3.4 that

$$\Vdash^W_{\mathrm{Col}(\omega,\delta_N)} \phi(N).$$

So since we picked m and θ large enough we have that

$$\bar{W}[g] \vDash \phi(\bar{N}),$$

if we let $\bar{N} \in \bar{W}$ be such that $\sigma(\bar{N}) = N$ and if g is $\mathrm{Col}(\omega, \bar{\delta})$-generic over $\bar{W}$. Let $\bar{\mathcal{T}}$ be a tree on $\bar{N}$ in $\bar{W}[g]$ witnessing that $\phi(\bar{N})$ holds. Since $\bar{N}$ is countable in W, we can pick $g \in W$ and then have that $\bar{\mathcal{T}} \in W$.

We have that $\bar{W}[g]$ is $\boldsymbol{\Sigma}^1_{2n+1}$-correct in V using Lemma 1.3.5, because it is a countable model with $2n$ Woodin cardinals. Since $\bar{\mathcal{T}}$ witnesses the statement $\phi(\bar{N})$ in $\bar{W}[g]$, it follows that $\bar{\mathcal{T}}$ also witnesses $\phi(\bar{N})$ in V, because $\phi(\bar{N})$ is Σ^1_{2n+2}-definable in any code for $\bar{N}$. The $\mathcal{Q}$-structures for $\bar{\mathcal{T}}$ in $\bar{W}[g]$ in the statement $\phi(\bar{N})$ are Π^1_{2n+1}-iterable above $\delta(\bar{\mathcal{T}} \restriction \gamma)$ and $2n$-small above $\delta(\bar{\mathcal{T}} \restriction \gamma)$ for limit ordinals $\gamma < \mathrm{lh}(\bar{\mathcal{T}})$ (and also for $\gamma = \mathrm{lh}(\bar{\mathcal{T}})$ if $\bar{\mathcal{T}}$ witnesses that $\phi_1(\bar{N})$ holds in $\bar{W}[g]$).

Since this amount of iterability suffices to witness $\mathcal{Q}$-structures using Lemma 2.2.10 and since as mentioned above $\bar{W}[g]$ is $\boldsymbol{\Sigma}^1_{2n+1}$-correct in V, the $\mathcal{Q}$-structures for $\bar{\mathcal{T}}$ in $\bar{W}[g]$ are also $\mathcal{Q}$-structures for $\bar{\mathcal{T}}$ inside V. Since $W = M_{2n}(N|\delta_N)$ is also $\boldsymbol{\Sigma}^1_{2n+1}$-correct in V using Lemma 1.3.4 and $\bar{N}$ and $\bar{\mathcal{T}}$ are countable in W, it follows that the $\mathcal{Q}$-structures for $\bar{\mathcal{T}}$ in $\bar{W}[g]$ (which are Π^1_{2n+1}-iterable above $\delta(\bar{\mathcal{T}} \restriction \gamma)$ for γ as above) are also $\mathcal{Q}$-structures for $\bar{\mathcal{T}}$ inside W. Therefore the branches choosen in the tree $\bar{\mathcal{T}}$ on $\bar{N}$ inside $\bar{W}[g]$ are the same branches as the $\mathcal{Q}$-structure iteration strategy Σ as in Definition 2.2.2 would choose inside the model W when iterating the premouse $\bar{W}$. That means if $\mathcal{T}^*$ is the tree on $\bar{W}$ obtained by considering $\bar{\mathcal{T}}$ as a tree on $\bar{W} \rhd \bar{N}$, then the iteration strategy Σ picks the same branches for the tree $\mathcal{T}^*$ as it does for the tree $\bar{\mathcal{T}}$.

Now we again distinguish three cases as before.

Case 1. $\bar{\mathcal{T}}$ witnesses that $\phi_1(\bar{N})$ holds in $\bar{W}[g]$.

By the argument we gave above, $\bar{\mathcal{T}}$ is a short tree on $\bar{N}$ in W.

Let $\bar{b}$ denote the cofinal branch through $\mathcal{T}^*$ which exists inside W and is defined as follows. We have that a branch through the iteration tree $\mathcal{T}^*$ can be considered as a branch through $\bar{\mathcal{T}}$, where the latter is a tree on $\bar{N}$, and vice versa. Since by the Weak Condensation Lemma 3.1.3 we have that

$$\bar{N} = \bar{W}|(\bar{\delta}^+)^{\bar{W}} \lhd W|\omega_1^W = M_{2n}(N|\delta_N)|\omega_1^{M_{2n}(N|\delta_N)} = N|\omega_1^N,$$

there exists a cofinal well-founded branch $\bar{b} \in W$ through $\bar{\mathcal{T}}$ by property (4) in the definition of pre-n-suitability (see Definition 3.1.1). We also consider this branch $\bar{b}$ as a branch through $\mathcal{T}^*$.

Assume first that there is a drop along the branch $\bar{b}$. Then there exists a $\mathcal{Q}$-structure $\mathcal{Q}(\bar{\mathcal{T}}) \unlhd \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$. Consider the statement

$$\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}})) \equiv \text{"there is a cofinal branch } b \text{ through } \bar{\mathcal{T}} \text{ such that } \mathcal{Q}(\bar{\mathcal{T}}) \unlhd \mathcal{M}_b^{\bar{\mathcal{T}}}\text{"}.$$

This statement $\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}))$ is Σ^1_1-definable uniformly in any code for the parameters $\bar{\mathcal{T}}$ and $\mathcal{Q}(\bar{\mathcal{T}})$ and holds in the model W as witnessed by the branch $\bar{b}$. Now a $\boldsymbol{\Sigma}^1_1$-absoluteness argument as the one given in the proof of Lemma 2.2.8 yields that this statement $\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}))$ also holds in $\bar{W}[g]$, which contradicts the fact that $\bar{\mathcal{T}}$ witnesses in $\bar{W}[g]$ that $\bar{N}$ is not short tree iterable.

Therefore we can assume that $\bar{b}$ does not drop.

Since $\bar{\mathcal{T}}$ witnesses that $\phi_1(\bar{N})$ holds in $\bar{W}[g]$, we have that there exists a $\mathcal{Q}$-structure $\mathcal{Q}_\lambda$ for $\bar{\mathcal{T}}$ as in $\phi_1(\bar{N})$. In particular $\mathcal{Q}_\lambda$ is $2n$-small above $\delta(\mathcal{T})$ and Π^1_{2n+1}-iterable above $\delta(\mathcal{T})$ in $\bar{W}[g]$.

Case 1.1. $\delta(\bar{\mathcal{T}}) = i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$.

Consider the comparison of $\mathcal{Q}_\lambda$ with $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ inside W.

This comparison takes place above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta}) = \delta(\bar{\mathcal{T}})$ and the premouse $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ is ω_1-iterable above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$ in W using property (4) in Definition 3.1.1 because $\mathcal{T}^*$ is an iteration tree on $\bar{W} \lhd W|\omega_1^W = N|\omega_1^N$ (using the Weak Condensation Lemma 3.1.3 again). Furthermore we have that $\bar{W}$ is $2n$-small above $\bar{\delta}$ and therefore $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ is $2n$-small above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$.

Moreover $\mathcal{Q}_\lambda$ is Π^1_{2n+1}-iterable above $\delta(\mathcal{T})$ in $\bar{W}[g]$ thus by $\boldsymbol{\Sigma}^1_{2n+1}$-correctness also inside W. The statement $\phi_1(\bar{N})$ yields that $\mathcal{Q}_\lambda$ is $2n$-small above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$.

We have that $\bar{W}$ is sound by construction and thus the non-dropping iterate $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ is sound above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$. Moreover we have that $\rho_\omega(\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}) \leq i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$. In addition $\mathcal{Q}_\lambda$ is also sound above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$ and we have that $\rho_\omega(\mathcal{Q}_\lambda) \leq \delta(\bar{\mathcal{T}}) = i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$. Hence Lemma 2.2.9 implies that

$$\mathcal{Q}_\lambda \lhd \mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \text{ or } \mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \unlhd \mathcal{Q}_\lambda.$$

So we again distinguish two different cases.

Case 1.1.1. $\mathcal{Q}_\lambda \lhd \mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$.

By assumption $\bar{\delta}$ is a Woodin cardinal in $\bar{W}$, because N is pre-$(2n+1)$-suitable and thus δ_N is a Woodin cardinal in W. Therefore we have by elementarity that

$$\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \vDash \text{“}i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta}) \text{ is Woodin”}.$$

But since $\mathcal{Q}_\lambda$ is a $\mathcal{Q}$-structure for $\mathcal{T}$, we have that $\delta(\mathcal{T}) = i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$ is not definably Woodin over $\mathcal{Q}_\lambda$. This contradicts $\mathcal{Q}_\lambda \lhd \mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$.

Case 1.1.2. $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \unlhd \mathcal{Q}_\lambda$.

In this case we have that

$$\mathcal{Q}_\lambda \cap \mathrm{Ord} < \bar{W} \cap \mathrm{Ord} \leq \mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \cap \mathrm{Ord} \leq \mathcal{Q}_\lambda \cap \mathrm{Ord},$$

where the first inequality holds true since $\mathcal{Q}_\lambda \in \bar{W}[g]$. This contradiction finishes Case 1.1.

Case 1.2. $\delta(\bar{\mathcal{T}}) < i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$.

In this case we have that

$$\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \vDash \text{“}\delta(\bar{\mathcal{T}}) \text{ is not Woodin”},$$

because otherwise $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ would not be $(2n+1)$-small. This implies that $\mathcal{Q}_\lambda = \mathcal{Q}(\bar{\mathcal{T}}) \lhd \mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ and therefore we have that

$$\mathcal{Q}(\bar{\mathcal{T}}) \unlhd \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}.$$

Now we can again consider the statement

$$\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}})) \equiv \text{ “there is a cofinal branch } b \text{ through } \bar{\mathcal{T}} \text{ such that } \mathcal{Q}(\bar{\mathcal{T}}) \unlhd \mathcal{M}_b^{\bar{\mathcal{T}}}\text{”}.$$

Again $\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}))$ holds in the model W as witnessed by the branch $\bar{b}$. By an absoluteness argument as above we have that it also holds in $\bar{W}[g]$, which contradicts the fact that $\bar{\mathcal{T}}$ witnesses in $\bar{W}[g]$ that $\bar{N}$ is not short tree iterable.

Case 2. $\bar{\mathcal{T}}$ witnesses that $\phi_2(\bar{N})$ holds in $\bar{W}[g]$.

In this case $\bar{\mathcal{T}}$ is a tree of length $\bar{\lambda}+1$ for some ordinal $\bar{\lambda}$.
Since $\phi_2(\bar{N})$ holds true in $\bar{W}[g]$, there exists a model $\bar{\mathcal{P}} \unrhd \mathcal{M}_{\bar{\lambda}}^{\mathcal{T}^*} | i_{0\bar{\lambda}}^{\mathcal{T}^*}(\bar{\delta})$, which is $2n$-small above $i_{0\bar{\lambda}}^{\mathcal{T}^*}(\bar{\delta})$, sound above $i_{0\bar{\lambda}}^{\mathcal{T}^*}(\bar{\delta})$ and Π^1_{2n+1}-iterable above $i_{0\bar{\lambda}}^{\mathcal{T}^*}(\bar{\delta})$ in $\bar{W}[g]$. Moreover $i_{0\bar{\lambda}}^{\mathcal{T}^*}(\bar{\delta})$ is not definably Woodin over $\bar{\mathcal{P}}$ and we have that $\rho_\omega(\bar{\mathcal{P}}) \leq i_{0\bar{\lambda}}^{\mathcal{T}^*}(\bar{\delta})$.

Consider the comparison of $\bar{\mathcal{P}}$ with $\mathcal{M}_\lambda^{\mathcal{T}^*}$ inside the model W. The comparison takes place above $i_{0\lambda}^{\mathcal{T}^*}(\bar{\delta})$ and we have that $\mathcal{M}_\lambda^{\mathcal{T}^*}$ is $2n$-small above $i_{0\lambda}^{\mathcal{T}^*}(\bar{\delta})$ because W is $2n$-small above δ_N. The premouse $\mathcal{M}_\lambda^{\mathcal{T}^*}$ is ω_1-iterable above $i_{0\lambda}^{\mathcal{T}^*}(\bar{\delta})$ in W by the same argument we gave above in Case 1 using property (4) in Definition 3.1.1. Therefore the coiteration is successful using Lemma 2.2.9 by the following argument.

We have that $\bar{\mathcal{P}}$ is Π^1_{2n+1}-iterable inside the model $\bar{W}[g]$ and thus by $\mathbf{\Sigma}^1_{2n+1}$-correctness also inside W. The statement $\phi_2(\bar{N})$ yields that $\bar{\mathcal{P}}$ is also $2n$-small above $i_{0\lambda}^{\mathcal{T}^*}(\bar{\delta})$. We have that $\bar{W}$ is sound by construction and thus the non-dropping iterate $\mathcal{M}_\lambda^{\mathcal{T}^*}$ is sound above $i_{0\lambda}^{\mathcal{T}^*}(\bar{\delta})$. Moreover we have that $\rho_\omega(\mathcal{M}_\lambda^{\mathcal{T}^*}) \leq i_{0\lambda}^{\mathcal{T}^*}(\bar{\delta})$. In addition $\bar{\mathcal{P}}$ is also sound above $i_{0\lambda}^{\mathcal{T}^*}(\bar{\delta})$ and we have that $\rho_\omega(\bar{\mathcal{P}}) \leq i_{0\lambda}^{\mathcal{T}^*}(\bar{\delta})$ because of $\phi_2(\bar{N})$. Hence Lemma 2.2.9 implies that

$$\bar{\mathcal{P}} \lhd \mathcal{M}_\lambda^{\mathcal{T}^*} \text{ or } \mathcal{M}_\lambda^{\mathcal{T}^*} \unlhd \bar{\mathcal{P}}.$$

So we consider two different cases.

Case 2.1. $\bar{\mathcal{P}} \lhd \mathcal{M}_\lambda^{\mathcal{T}^*}$.

By assumption $\bar{\delta}$ is a Woodin cardinal in $\bar{W}$, because N is pre-$(2n+1)$-suitable and thus δ_N is a Woodin cardinal in W. Therefore we have by elementarity that

$$\mathcal{M}_\lambda^{\mathcal{T}^*} \vDash \text{“}i_{0\lambda}^{\mathcal{T}^*}(\bar{\delta}) \text{ is Woodin”}.$$

Moreover we have by the statement $\phi_2(\bar{N})$ that $i_{0\lambda}^{\mathcal{T}^*}(\bar{\delta})$ is not definably Woodin over $\bar{\mathcal{P}}$. This is a contradiction to $\bar{\mathcal{P}} \lhd \mathcal{M}_\lambda^{\mathcal{T}^*}$.

Case 2.2. $\mathcal{M}_\lambda^{\mathcal{T}^*} \unlhd \bar{\mathcal{P}}$.

In this case we have that

$$\bar{\mathcal{P}} \cap \mathrm{Ord} < \bar{W} \cap \mathrm{Ord} \leq \mathcal{M}_\lambda^{\mathcal{T}^*} \cap \mathrm{Ord} \leq \bar{\mathcal{P}} \cap \mathrm{Ord},$$

where the first inequality holds since $\bar{\mathcal{P}} \in \bar{W}[g]$. This is a contradiction.

Therefore we proved that

$$M_{2n}(\mathcal{M}_\lambda^{\mathcal{T}} | \delta_{\mathcal{M}_\lambda^{\mathcal{T}}}) \vDash \text{“}\delta_{\mathcal{M}_\lambda^{\mathcal{T}}} \text{ is Woodin”},$$

if $\mathcal{T}$ is as in Case 2 above, that means if $\mathcal{T}$ is a short iteration tree on N of length $\lambda + 1$ which is non-dropping on the main branch.

This shows that there is an ordinal $\delta < \omega_1^V$ such that properties (1) and (2) in Definition 3.1.1 hold for the premouse $\mathcal{M}_\lambda^{\mathcal{T}}$. That property (3) holds for $\mathcal{M}_\lambda^{\mathcal{T}}$ follows from property (3) for the pre-$(2n+1)$-suitable premouse N by a similar argument and property (4) follows from the corresponding property for N as well. Thus $\mathcal{M}_\lambda^{\mathcal{T}}$ is pre-$(2n+1)$-suitable, as desired.

Case 3. $\bar{\mathcal{T}}$ witnesses that $\phi_3(\bar{N})$ holds in $\bar{W}[g]$.

Let $\bar{\mathcal{P}} \rhd \mathcal{M}(\bar{\mathcal{T}})$ witness that $\phi_3(\bar{N})$ holds inside $\bar{W}[g]$. We have that inside W there exists a cofinal well-founded branch $\bar{b}$ through the iteration tree $\mathcal{T}^*$ by property (4) in the definition of pre-$(2n+1)$-suitability for N as above in Case 1.

Case 3.1. There is a drop along $\bar{b}$.

Then we have as in Case 1 that there exists a $\mathcal{Q}$-structure $\mathcal{Q}(\bar{\mathcal{T}}) \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$ for $\bar{\mathcal{T}}$. Consider the statement

$$\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}})) \equiv \text{ "there is a cofinal branch } b \text{ through } \bar{\mathcal{T}} \text{ such that } \mathcal{Q}(\bar{\mathcal{T}}) \trianglelefteq \mathcal{M}_b^{\bar{\mathcal{T}}}\text{"}.$$

This statement $\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}))$ is Σ^1_1-definable from the parameters $\bar{\mathcal{T}}$ and $\mathcal{Q}(\bar{\mathcal{T}})$ and holds in the model W as witnessed by the branch $\bar{b}$. By an absoluteness argument as above it follows that it also holds in $\bar{W}[g]$, which contradicts the fact that $\bar{\mathcal{T}}$ witnesses in $\bar{W}[g]$ that $\phi_3(\bar{N})$ holds.

Case 3.2. There is no drop along $\bar{b}$.

Then we can consider the coiteration of $\bar{\mathcal{P}}$ and $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ inside the model W. We have that both premice are $2n$-small above $\delta(\bar{\mathcal{T}})$. Moreover this coiteration takes place above $\delta(\bar{\mathcal{T}})$ since we have that $\bar{\mathcal{P}} \rhd \mathcal{M}(\bar{\mathcal{T}})$. Therefore the coiteration is successful inside W using Lemma 2.2.9 by the same argument as the one we gave in Cases 1 and 2, because in W we have that $\bar{\mathcal{P}}$ is Π^1_{2n+1}-iterable above $\delta(\bar{\mathcal{T}})$ and $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ is ω_1-iterable above $\delta(\bar{\mathcal{T}})$ in W using property (4) in Definition 3.1.1. That means we have that

$$\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \trianglelefteq \bar{\mathcal{P}} \text{ or } \bar{\mathcal{P}} \trianglelefteq \mathcal{M}_{\bar{b}}^{\mathcal{T}^*}.$$

Case 3.2.1. $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \trianglelefteq \bar{\mathcal{P}}$.

In this case we have that

$$\bar{\mathcal{P}} \cap \mathrm{Ord} < \bar{W} \cap \mathrm{Ord} \leq \mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \cap \mathrm{Ord} \leq \bar{\mathcal{P}} \cap \mathrm{Ord},$$

where the first inequality holds true since $\bar{\mathcal{P}} \in \bar{W}[g]$. This is a contradiction.

Case 3.2.2. $\bar{\mathcal{P}} \trianglelefteq \mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$.

Then we have that in fact

$$\bar{\mathcal{P}} \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}},$$

because $\delta(\bar{\mathcal{T}})$ is the largest cardinal in $\bar{\mathcal{P}}$. This contradicts $\phi_3(\bar{N})$.

Therefore it follows for an iteration tree $\mathcal{T}$ as in Case 3 that there exists a cofinal well-founded branch through $\mathcal{T}$ and if there exists a non-dropping such branch b, then the premouse $\mathcal{M}_b^{\mathcal{T}}$ is pre-$(2n+1)$-suitable analogous to the argument at the end of Case 2 above.

Now the argument we just gave for Case 3 shows that in this case we have that in fact

$$\mathcal{M}_b^{\mathcal{T}} = M_{2n}(\mathcal{M}(\mathcal{T}))|(\delta(\mathcal{T})^+)^{M_{2n}(\mathcal{M}(\mathcal{T}))}.$$

□

3.2. Correctness for n-suitable Premice

In the following lemmas we prove some correctness results for suitable premice in the sense of Definition 3.1.8. We are stating these lemmas only for the levels $(2n+1)$ at which we proved that there exists a $(2n+1)$-suitable premouse (see Lemma 3.1.9).

Lemma 3.2.1. *Let $n \geq 0$ and $z \in {}^\omega\omega$. Assume that $M_{2n}^\#(x)$ exists for all $x \in {}^\omega\omega$ and let N be a $(2n+1)$-suitable z-premouse. Let φ be an arbitrary Σ^1_{2n+3}-formula and let $a \in N \cap {}^\omega\omega$ be arbitrary. Then we have*

$$\varphi(a) \;\leftrightarrow\; \Vdash^N_{\mathrm{Col}(\omega,\delta_N)} \varphi(a),$$

where δ_N as usual denotes the largest cardinal in N.

Here we again write a for the standard name $\check{a}$ for a real $a \in N$.

Proof. Let $n \geq 0$ and $z \in {}^\omega\omega$ be arbitrary and let N be a $(2n+1)$-suitable z-premouse. Let $\varphi(a)$ be a Σ^1_{2n+3}-formula for a parameter $a \in N \cap {}^\omega\omega$. That means

$$\varphi(a) \equiv \exists x \forall y\, \psi(x,y,a)$$

for a Σ^1_{2n+1}-formula $\psi(x,y,a)$. We first want to prove the downward implication, that means we want to prove that if $\varphi(a)$ holds in V, then

$$\Vdash^N_{\mathrm{Col}(\omega,\delta_N)} \varphi(a).$$

Let $x^* \in V$ be a witness for the fact that $\varphi(a)$ holds in V. That means x^* is a real such that

$$V \vDash \forall y\, \psi(x^*,y,a).$$

Use Corollary 1.8 from [Ne95] to make the real x^* generic over an iterate of N for the collapse of the image of δ_N. Since N is $(2n+1)$-suitable, we have enough iterability to apply Corollary 1.8 from [Ne95], so there exists a non-dropping iterate N^* of N such that N^* is $2n$-iterable and whenever g is $\mathrm{Col}(\omega,\delta_{N^*})$-generic over N^*, then $x^* \in N^*[g]$. Moreover let $i : N \rightarrow N^*$ denote the corresponding iteration embedding. Since N is $(2n+1)$-suitable and the iteration from N to N^* is non-dropping, we have that N^* is pre-$(2n+1)$-suitable.

Let $g \in V$ be $\mathrm{Col}(\omega,\delta_{N^*})$-generic over N^*. Since we have that $N^* = M_{2n}(N^*|\delta_{N^*})|(\delta^+_{N^*})^{M_{2n}(N^*|\delta_{N^*})}$, it follows that g is also $\mathrm{Col}(\omega,\delta_{N^*})$-generic over the proper class model $M_{2n}(N^*|\delta_{N^*})$. Moreover we can construe the model $M_{2n}(N^*|\delta_{N^*})[g]$ as a y-premouse for some real y (in fact $y = z \oplus x^*$, see for example [SchSt09] for the fine structural details) which has $2n$ Woodin cardinals. This yields by Lemma 1.3.4 that the premouse $M_{2n}(N^*|\delta_{N^*})[g]$

is $\mathbf{\Sigma}^1_{2n+2}$-correct in V, because $M_{2n}(N^*|\delta_{N^*})[g]$ construed as a y-premouse is ω_1-iterable above δ_{N^*}. Thus we have that

$$M_{2n}(N^*|\delta_{N^*})[g] \vDash \forall y \psi(x^*, y, a)$$

as $x^* \in M_{2n}(N^*|\delta_{N^*})[g]$. This in fact can be obtained using only $\mathbf{\Sigma}^1_{2n+1}$-correctness of $M_{2n}(N^*|\delta_{N^*})[g]$ and downward absoluteness.

Since the premice $M_{2n}(N^*|\delta_{N^*})[g]$ and $N^*[g]$ agree on their reals it follows that

$$N^*[g] \vDash \forall y \psi(x^*, y, a).$$

By homogeneity of the forcing $\mathrm{Col}(\omega, \delta_{N^*})$ we now obtain that

$$\Vdash^{N^*}_{\mathrm{Col}(\omega,\delta_{N^*})} \exists x \forall y\, \psi(x, y, a).$$

Since a is a real in N it follows by elementarity that

$$\Vdash^{N}_{\mathrm{Col}(\omega,\delta_{N})} \exists x \forall y\, \psi(x, y, a),$$

as desired.

For the upward implication let $\varphi(a) \equiv \exists x \forall y\, \psi(x, y, a)$ again be a Σ^1_{2n+3}-formula for a real a in N and a Σ^1_{2n+1}-formula $\psi(x, y, a)$ and assume that we have

$$\Vdash^{N}_{\mathrm{Col}(\omega,\delta_{N})} \exists x \forall y\, \psi(x, y, a).$$

Let g be $\mathrm{Col}(\omega, \delta_N)$-generic over the premouse $M_{2n}(N|\delta_N)$ and pick a real $x^* \in M_{2n}(N|\delta_N)[g]$ such that

$$M_{2n}(N|\delta_N)[g] \vDash \forall y\, \psi(x^*, y, a).$$

Since $M_{2n}(N|\delta_N)|(\delta_N^+)^{M_{2n}(N|\delta_N)} = N$ is countable in V, we can pick $g \in V$ and then get that $x^* \in V$. As above we can consider $M_{2n}(N|\delta_N)[g]$ as an ω_1-iterable y-premouse for some real y and therefore we have by Lemma 1.3.4 again that

$$M_{2n}(N|\delta_N)[g] \prec_{\mathbf{\Sigma}^1_{2n+2}} V.$$

Hence we have that

$$V \vDash \exists x \forall y\, \psi(x, y, a),$$

witnessed by the real x^*, because "$\forall y\, \psi(x^*, y, a)$" is a Π^1_{2n+2}-formula. $\square$

Lemma 3.2.2. *Let $n \geq 0$ and assume that $M^{\#}_{2n}(x)$ exists for all $x \in {}^\omega\omega$. Let N be a $(2n+1)$-suitable z-premouse for some $z \in {}^\omega\omega$. Then $N|\delta_N$ is closed under the operation*

$$A \mapsto M^{\#}_{2n}(A).$$

Proof. It is enough to consider sets A of the form $N|\xi$ for some ordinal $\xi < \delta_N$ for the following reason. Let $A \in N|\delta_N$ be arbitrary. Then there exists an ordinal $\xi < \delta_N$ such that $A \in N|\xi$. Assume first that the ordinal ξ is not overlapped by an extender on the N-sequence, that means there is no extender E on the N-sequence such that $\mathrm{crit}(E) \leq \xi < \mathrm{lh}(E)$. We will consider the case that ξ is overlapped by an extender on the N-sequence later. Moreover assume that we already proved that

$$M_{2n}^{\#}(N|\xi) \lhd N|\delta_N.$$

Then we also have that $M_{2n}^{\#}(A) \in N|\delta_N$ by the following argument. Consider the model $M_{2n}(N|\xi)$. Let $L[E](A)^{M_{2n}(N|\xi)}$ denote the result of a fully backgrounded extender construction above A in the sense of [MS94] with the smallness hypothesis weakened inside the model $M_{2n}(N|\xi)$. Then add the top measure of the active premouse $M_{2n}^{\#}(N|\xi)$ (intersected with $L[E](A)^{M_{2n}(N|\xi)}$) to an initial segment of $L[E](A)^{M_{2n}(N|\xi)}$ as described in Section 2 of [FNS10]. The main result in [FNS10] yields that the model we obtain from this construction is again ω_1-iterable and not $2n$-small. Thus it follows that the ω_1-iterable premouse $M_{2n}^{\#}(A)$ exists inside $N|\delta_N$ as $M_{2n}^{\#}(N|\xi) \lhd N|\delta_N$.

So let $\xi < \delta_N$ be an ordinal and assume as above first that ξ is not overlapped by an extender on the N-sequence. Then we consider the premouse $M_{2n}^{\#}(N|\xi)$, which exists in V by assumption and is not $2n$-small above ξ because ξ is countable in V. In this case we are left with showing that

$$M_{2n}^{\#}(N|\xi) \lhd N|\delta_N.$$

Let x be a real in V which codes the countable premice $M_{2n}^{\#}(N|\xi)$ and $N|\delta_N$. Work inside the model $M_{2n}^{\#}(x)$ and coiterate $M_{2n}^{\#}(N|\xi)$ with $N|\delta_N$. We have that $N|\delta_N$ is short tree iterable inside $M_{2n}^{\#}(x)$, because for a pre-$(2n+1)$-suitable premouse short tree iterability is a Π^1_{2n+2}-definable statement by Lemma 3.1.6 and the model $M_{2n}^{\#}(x)$ is $\boldsymbol{\Sigma}^1_{2n+2}$-correct in V by Lemma 1.3.4. In fact Lemma 1.3.4 implies that N has an iteration strategy which is fullness preserving in the sense of Definition 3.1.8 inside the model $M_{2n}^{\#}(x)$ by the proof of Lemma 3.1.9.

Note that the coiteration takes place above $N|\xi$ and that $M_{2n}^{\#}(N|\xi)$ is ω_1-iterable above $N|\xi$ in V by definition. Therefore Lemma 2.2.8 (2) implies that the comparison inside $M_{2n}^{\#}(x)$ cannot fail on this side of the coiteration. Say that the coiteration yields an iteration tree $\mathcal{T}$ on $M_{2n}^{\#}(N|\xi)$ and an iteration tree $\mathcal{U}$ on $N|\delta_N$.

We have that $\mathcal{U}$ is a short tree on $N|\delta_N$, because the $M_{2n}^{\#}(N|\xi)$-side of the coiteration provides $\mathcal{Q}$-structures. So the coiteration terminates successfully and there is an iterate M^* of $M_{2n}^{\#}(N|\xi)$ via $\mathcal{T}$ and an iterate R of $N|\delta_N$ via $\mathcal{U}$.

Claim 1. *The $M_{2n}^{\#}(N|\xi)$-side does not move in the coiteration with $N|\delta_N$.*

Proof. Assume toward a contradiction that the $M_{2n}^{\#}(N|\xi)$-side moves in the coiteration. Then we have that M^* is not sound as the coiteration takes place above ξ and thus

$$M^* \trianglerighteq R.$$

Furthermore we can consider $\mathcal{U}$ as a tree on the premouse $N \rhd N|\delta_N$ with final model N^* such that there is an ordinal δ^* with

$$N^*|\delta^* = R.$$

Then $\mathcal{U}$ is a short tree of length $\lambda + 1$ for some ordinal λ and there is no drop on the main branch on the $N|\delta_N$-side of the coiteration. So we have that

$$\delta^* = i^{\mathcal{U}}_{0\lambda}(\delta_N),$$

where $i^{\mathcal{U}}_{0\lambda} : N \rightarrow N^*$ denotes the corresponding iteration embedding. Moreover recall that N is $(2n+1)$-suitable and so $\mathcal{U}$ is obtained using the iteration strategy for N which is fullness preserving for short trees which do not drop on their main branch in the sense of Definition 3.1.8. Therefore we have that $M^{\#}_{2n}(N^*|\delta^*)$ is not $2n$-small above δ^* and

$$M^{\#}_{2n}(N^*|\delta^*) \vDash \text{“}\delta^* \text{ is Woodin”}.$$

We have for the other side of the coiteration that $\rho_\omega(M^*) < \delta^*$, because $\rho_\omega(M^{\#}_{2n}(N|\xi)) \leq \xi < \delta^*$. Let $\mathcal{Q}$ be the least initial segment of M^* such that δ^* is not definably Woodin over $\mathcal{Q}$. Recall that this means that $\mathcal{Q} \trianglelefteq M^*$ is such that

$$\mathcal{Q} \vDash \text{“}\delta^* \text{ is Woodin”},$$

but if $\mathcal{Q} = J^{M^*}_\alpha$ for some $\alpha < M^* \cap \mathrm{Ord}$ then

$$J^{M^*}_{\alpha+1} \vDash \text{“}\delta^* \text{ is not Woodin”},$$

and if $\mathcal{Q} = M^*$ then $\rho_\omega(\mathcal{Q}) < \delta^*$ or there exists an $m < \omega$ and an $r\Sigma_m$-definable set $A \subset \delta^*$ such that there is no $\kappa < \delta^*$ such that κ is strong up to δ^* with respect to A as witnessed by extenders on the sequence of $\mathcal{Q}$. In the latter case we have that in particular $\rho_\omega(\mathcal{Q}) \leq \delta^*$. Moreover $\mathcal{Q}$ is $2n$-small above δ^*, because we have $\mathcal{Q} \trianglelefteq M^*$ and

$$\mathcal{Q} \vDash \text{“}\delta^* \text{ is Woodin”}.$$

Furthermore $\mathcal{Q}$ is ω_1-iterable above δ^*.

By construction we have that the premouse $M^{\#}_{2n}(N^*|\delta^*)$ is ω_1-iterable above δ^*. Consider the coiteration of $\mathcal{Q}$ and $M^{\#}_{2n}(N^*|\delta^*)$ inside the model $M^{\#}_{2n}(y)$, where y is a real coding the countable premice $\mathcal{Q}$ and $M^{\#}_{2n}(N^*|\delta^*)$. Since

$$N^*|\delta^* = R \trianglelefteq M^*$$

this coiteration takes place above δ^*. We have that $\rho_\omega(\mathcal{Q}) \leq \delta^*$ and

$$\rho_\omega(M^{\#}_{2n}(N^*|\delta^*)) \leq \delta^*.$$

Moreover $\mathcal{Q}$ and $M_{2n}^{\#}(N^*|\delta^*)$ are both sound above δ^*. Thus the comparison is successful by Lemma 2.2.8 and we have that

$$\mathcal{Q} \trianglelefteq M_{2n}^{\#}(N^*|\delta^*) \text{ or } M_{2n}^{\#}(N^*|\delta^*) \trianglelefteq \mathcal{Q}.$$

The premouse $M_{2n}^{\#}(N^*|\delta^*)$ is not $2n$-small above δ^* because N^* is pre-$(2n+1)$-suitable. Since $\mathcal{Q}$ is $2n$-small above δ^* we have in fact that

$$\mathcal{Q} \lhd M_{2n}^{\#}(N^*|\delta^*).$$

But this implies by our choice of the premouse $\mathcal{Q}$ that δ^* is not Woodin in $M_{2n}^{\#}(N^*|\delta^*)$, which is a contradiction because by fullness preservation we have that δ^* is a Woodin cardinal in $M_{2n}^{\#}(N^*|\delta^*)$.

Therefore we have that the $M_{2n}^{\#}(N|\xi)$-side does not move in the coiteration with $N|\delta_N$. □

Note that the proof of Claim 1 also shows that the $M_{2n}^{\#}(N|\xi)$-side cannot win the comparison with $N|\delta_N$.

CLAIM 2. *The $N|\delta_N$-side does not move in the coiteration with $M_{2n}^{\#}(N|\xi)$.*

PROOF. Assume toward a contradiction that the $N|\delta_N$-side moves in this coiteration. Since the coiteration takes place above ξ this means that there is an ordinal $\gamma > \xi$ such that there is an extender E_γ^N indexed at γ on the N-sequence which is used in the coiteration. In particular γ is a cardinal in the iterate R of $N|\delta_N$. This implies regardless of whether the iteration from $N|\delta_N$ to R drops or not, that γ is a cardinal in $M_{2n}^{\#}(N|\xi)$ since we have that $M_{2n}^{\#}(N|\xi) \lhd R$ using Claim 1. In fact we have, regardless of whether the iteration from $N|\delta_N$ to R drops or not, that there is a model N' such that γ is a cardinal in N' and $M_{2n}^{\#}(N|\xi) \lhd N'$. This is a contradiction because

$$\rho_\omega(M_{2n}^{\#}(N|\xi)) \leq \xi < \gamma.$$

Therefore the $N|\delta_N$-side also does not move in the comparison. □

By Claims 1 and 2 we finally have that

$$M_{2n}^{\#}(N|\xi) \lhd N|\delta_N.$$

We now have to consider the case that $A \in N|\xi$ for an ordinal $\xi < \delta_N$ such that ξ is overlapped by an extender E on the N-sequence. That means there is an extender E on the N-sequence such that $\text{crit}(E) \leq \xi < \text{lh}(E)$. Let E be the least such extender, that means the index of E is minimal among all critical points of extenders on the N-sequence overlapping ξ. By the definition of a "fine extender sequence" (see Definition 2.4 in [St10]) we have that $A \in \text{Ult}(N; E)$ and that the ordinal ξ is no longer overlapped by an extender on the $\text{Ult}(N; E)$-sequence. Let $M = \text{Ult}(N; E)$ and consider the premouse $M|\xi$, so we have in particular that $A \in M|\xi$. The same argument as above for $N|\xi$ replaced by $M|\xi$ proves that

$$M_{2n}^{\#}(M|\xi) \lhd N|\delta_N$$

and therefore we finally have that $M_{2n}^{\#}(A) \in N|\delta_N$ by repeating the argument we already gave at the beginning of this proof. □

From Lemma 3.2.2 we can obtain the following lemma as a corollary.

LEMMA 3.2.3. *Let $n \geq 0$ and $z \in {}^{\omega}\omega$. Assume that $M_{2n}^{\#}(x)$ exists for all $x \in {}^{\omega}\omega$ and let N be a $(2n+1)$-suitable z-premouse. Then N is $\mathbf{\Sigma}^1_{2n+2}$-correct in V for real parameters in N and we write*

$$N \prec_{\mathbf{\Sigma}^1_{2n+2}} V.$$

PROOF. Let a be a real in N. By Lemma 3.2.2 we have that

$$M_{2n}^{\#}(a) \in N.$$

Moreover we have by Lemma 1.3.4 that $M_{2n}^{\#}(a)$ is $\mathbf{\Sigma}^1_{2n+2}$-correct in V. Therefore it follows that N is $\mathbf{\Sigma}^1_{2n+2}$-correct in V. □

Note that we could have also proven Lemma 3.2.3 by an argument similar to the one we gave in the proof of Lemma 3.2.1 but with additionally using the uniformization property as in the proof of Lemma 1.3.4 (1).

3.3. Outline of the Proof

Our main goal for the rest of this chapter is to give a proof of the following theorem.

THEOREM 3.3.1. *Let $n \geq 0$ and assume there is no $\mathbf{\Sigma}^1_{n+2}$-definable ω_1-sequence of pairwise distinct reals. Then the following are equivalent.*

(1) $\mathbf{\Pi}^1_n$ determinacy and Π^1_{n+1} determinacy,
(2) for all $x \in {}^{\omega}\omega$, $M_{n-1}^{\#}(x)$ exists and is ω_1-iterable, and $M_n^{\#}$ exists and is ω_1-iterable,
(3) $M_n^{\#}$ exists and is ω_1-iterable.

For $n = 0$ this is due to L. Harrington (see [Ha78]) and D. A. Martin (see [Ma70]).

The results that (3) implies (2) and that (2) implies (1) for ω_1-iterable premice $M_n^{\#}$ are due to Woodin for odd n (unpublished) and due to Neeman for even $n > 0$ (see Theorem 2.14 in [Ne02]), building on work of Martin and Steel (see [MaSt89]). Moreover the results that (3) implies (2) and that (2) implies (1) hold without the background hypothesis that every $\mathbf{\Sigma}^1_{n+2}$-definable sequence of pairwise distinct reals is countable.

We will focus on the proof of the following theorem, which is the implication "(1) ⇒ (3)" in Theorem 3.3.1 and due to W. Hugh Woodin (unpublished).

Theorem 3.3.2. *Let $n \geq 1$ and assume there is no $\boldsymbol{\Sigma}^1_{n+2}$-definable ω_1-sequence of pairwise distinct reals. Moreover assume that $\boldsymbol{\Pi}^1_n$ determinacy and Π^1_{n+1} determinacy hold. Then $M_n^\#$ exists and is ω_1-iterable.*

Remark. Let $n \geq 0$. Then we say that there exists a $\boldsymbol{\Sigma}^1_{n+2}$*-definable ω_1-sequence of pairwise distinct reals* iff there exists a well-order $\leq^*$ of ordertype ω_1 for reals such that if we let $X_{\leq^*} = field(\leq^*)$, that means if we have for all $y \in {}^\omega\omega$ that

$$y \in X_{\leq^*} \;\Leftrightarrow\; \exists x \; (x \leq^* y \vee y \leq^* x),$$

then there exists a $\boldsymbol{\Sigma}^1_{n+2}$-definable relation R_1 such that we have for all $x, y \in {}^\omega\omega$,

$$R_1(x,y) \;\Leftrightarrow\; x,y \in X_{\leq^*} \;\wedge\; x \leq^* y,$$

and there exists a $\boldsymbol{\Sigma}^1_{n+2}$-definable relation R_2 such that we have for all $x, w \in {}^\omega\omega$,

$$R_2(x,w) \;\Leftrightarrow\; w \in \mathrm{WO} \wedge x \text{ is the } ||w|| - \text{th element according to } \leq^*,$$

where $\mathrm{WO} = \{w \in {}^\omega\omega \mid w \text{ codes a well-ordering on } \mathbb{N}\}$.

In fact it is not necessary to demand the existence of R_1 above as it follows from the existence of R_2. We decided to explicitly phrase it here to be able to emphasize later what exactly is needed in a specific argument.

In the next section we will first prove Theorem 3.3.1 for odd levels, i.e. for the case where we construct an ω_1-iterable premouse $M_{2n-1}^\#$ which is not $(2n-1)$-small, under a slightly stronger hypothesis (see Theorem 3.4.1). Then we will show in Section 3.5 how boldface determinacy for a level of the projective hierarchy can be used to prove that every sequence of pairwise distinct reals which is projective at the next level of the projective hierarchy is in fact countable.

This will enable us to conclude Theorem 2.1.1 from Theorem 3.3.2, that means from the results shown in Sections 3.6 and 3.7. The odd levels of Theorem 3.3.2 will finally be proven in Section 3.6 and the even levels in Section 3.7. As mentioned before the proof for the odd and the even levels of the projective hierarchy will be different because of the periodicity in the projective hierarchy in terms of uniformization and correctness.

The proof of Theorem 3.3.2 in Sections 3.6 and 3.7 will be performed simultaneously by an induction on n. That means during the proof of Theorem 3.3.2 for n we will assume that Theorem 3.3.2 holds for all $m < n$.

3.4. $M_{2n-1}^\#(x)$ from a Slightly Stronger Hypothesis

In this section we will give a proof of the following theorem, which is slightly weaker than Theorem 3.3.2 for odd levels n. We are assuming for the construction of the ω_1-iterable premouse $M_n^\#$ for an odd natural number n that there is no $\boldsymbol{\Sigma}^1_{n+3}$-definable ω_1-sequence of pairwise distinct reals instead of just assuming that there is no $\mathbf{\Sigma}^1_{n+2}$-definable ω_1-sequence of pairwise distinct reals as in the statement of Theorem 3.3.2. The proof of this theorem will serve as a "warm up" for the proof of Theorem 3.3.2 since many of the techniques used in the proof are also important in Sections 3.6 and 3.7 for the proof of Theorem 3.3.2. Moreover some of the arguments given in this section are going to be reused there. In particular Theorem 3.6.1 will be a refinement of the implication "(1) $\Rightarrow$ (3)" in Theorem 3.4.1.

THEOREM 3.4.1. *Let $n \geq 1$ and assume there is no $\mathbf{\Sigma}^1_{2n+2}$-definable ω_1-sequence of pairwise distinct reals. Then the following are equivalent.*

(1) $\mathbf{\Pi}^1_{2n-1}$ determinacy and Π^1_{2n} determinacy,
(2) for all $x \in {}^\omega\omega$, $M^{\#}_{2n-2}(x)$ exists and is ω_1-iterable, and $M^{\#}_{2n-1}$ exists and is ω_1-iterable,
(3) $M^{\#}_{2n-1}$ exists and is ω_1-iterable.

REMARK. We will prove in Section 3.5 that $\mathbf{\Pi}^1_{2n+1}$-determinacy implies our background hypothesis that there is no $\mathbf{\Sigma}^1_{2n+2}$-definable ω_1-sequence of pairwise distinct reals.

Before we are ready to give a proof of Theorem 3.4.1 we need the following preliminary lemma about the uniformization property.

LEMMA 3.4.2. *Let $n < \omega$.*

(1) Assume that $M^{\#}_{2n-1}(x)$ exists for all $x \in {}^\omega\omega$ if $n \geq 1$. Then the pointclass Π^1_{2n+1} has the uniformization property, that means every Π^1_{2n+1}-definable set can be uniformized by a Π^1_{2n+1}-definable set.
(2) Assume that $M^{\#}_{2n}(x)$ exists for all $x \in {}^\omega\omega$. Then every Π^1_{2n+2}-definable set can be uniformized by a Π^1_{2n+3}-definable set.

PROOF. The odd case (part (1) of Lemma 3.4.2) follows from Theorem $6C.5$ in [Mo09] together with the implication "(3) $\Rightarrow$ (1)" in Theorem 3.4.1. The implication "(3) $\Rightarrow$ (1)" in Theorem 3.4.1, which is due to Woodin for odd n, follows for example also from [Ne95] (see also the proof of Theorem 3.4.1 below).
The even case (part (2) of Lemma 3.4.2) follows from Moschovakis' Second Periodicity Theorem (see Theorem $6C.3$ in [Mo09]) and the results in [Ne95] by the following argument. We thank Itay Neeman for telling us the proof of part (2) of Lemma 3.4.2.
Consider the pointclass

$$\Gamma = \Game^{(2n)}(< \omega^2 - \Pi^1_1),$$

where $\Game$ denotes the game quantifier[2]. By Theorem 2.5 in [Ne95] we have that (boldface) $\mathbf{\Gamma}$-determinacy holds, because $M_{2n}^{\#}(x)$ exists for all reals x. The Second Periodicity Theorem (see Theorem $6C.3$ in [Mo09]) implies inductively that $\forall^{\mathbb{R}}\Gamma$-definable sets admit $\forall^{\mathbb{R}}\exists^{\mathbb{R}}\Gamma$-definable scales. Moreover we have the following basic facts about the game quantifier (see Theorem $6D.2$ in [Mo09]).

(1) $\exists^{\mathbb{R}}\Gamma \subseteq \Game\Gamma$,
(2) $\forall^{\mathbb{R}}\Gamma \subseteq \Game\Gamma$,
(3) $\Game\Gamma \subseteq \exists^{\mathbb{R}}\forall^{\mathbb{R}}\Gamma$, and
(4) $\Game\Gamma \subseteq \forall^{\mathbb{R}}\exists^{\mathbb{R}}\Gamma$, since Γ is determined.

Therefore we have that

$$\Sigma^1_{2n+1} \subseteq \Game^{(2n)}(< \omega^2 - \Pi^1_1) \subseteq \Game^{(2n)}\Delta^1_2 \subseteq \Sigma^1_{2n+2}.$$

This implies that in particular every Π^1_{2n+2}-definable set admits a Π^1_{2n+3}-definable scale, because $\Pi^1_{2n+2} \subseteq \forall^{\mathbb{R}}\Gamma$ and $\forall^{\mathbb{R}}\exists^{\mathbb{R}}\Gamma \subseteq \Pi^1_{2n+3}$. Using the Uniformization Lemma (see Lemma $4E.3$ in [Mo09]) it follows that Π^1_{2n+2}-definable sets can be uniformized by Π^1_{2n+3}-definable sets. □

In the proof of Theorem 3.4.1 we aim to construct models by expanding the usual definition of Gödels constructible universe L by adding additional elements at the successor steps of the construction. Therefore recall the following definition which is used in the definition of Jensens J-hierarchy for L (see §1 and §2 in [Je72]).

Definition 3.4.3. *Let X be an arbitrary set. Then* $\mathrm{TC}(X)$ *denotes the transitive closure of X and $rud(X)$ denotes the closure of* $\mathrm{TC}(X) \cup \{\mathrm{TC}(X)\}$ *under rudimentary functions (see for example Definition* 1.1 *in [SchZe10] for the definition of a rudimentary function).*

Proof of theorem 3.4.1. The proof is organized by induction on n. We first give a direct argument for (3) implies (2) using our background hypothesis that there is no $\mathbf{\Sigma}^1_{2n+2}$-definable ω_1-sequence of pairwise distinct reals as an example how this background hypothesis can be used. The proof for this direction works for both odd and even levels, i.e. assuming that $M_{2n-1}^{\#}$ or $M_{2n}^{\#}$ exists, under the appropriate background hypothesis. Moreover we will in fact only use the weaker background hypothesis that every $\mathbf{\Sigma}^1_{2n+1}$-definable sequence of pairwise distinct reals is countable.

Using Neeman's genericity iteration (see Corollary 1.8 in [Ne95]) it is possible to show that (3) implies (2) by a similar argument without using the background hypothesis that there is no $\mathbf{\Sigma}^1_{2n+1}$-definable ω_1-sequence of pairwise distinct reals, because his methods only use the $(2n-1)$-iterability of

[2]See Section $6D$ in [Mo09] for a definition and some basic facts about the game quantifier "$\Game$". We let "$\Game^{(n)}$" denote n successive applications of the game quantifier $\Game$. For the definition of the difference hierarchy and in particular of the pointclass $\alpha - \Pi^1_1$ for an ordinal $\alpha < \omega_1$ see for example Section 31 in [Ka08].

$M^{\#}_{2n-1}$ to make a real x generic over an iterate of $M^{\#}_{2n-1}$ (see for example the proof of Lemma 1.3.4 where this kind of genericity iteration was already used).

We will present the direct proof for (3) implies (2) using the background hypothesis that there is no $\mathbf{\Sigma}^1_{2n+1}$-definable ω_1-sequence of pairwise distinct reals here, because it serves as a warm up for the proof of (1) implies (3).

(3) $\Rightarrow$ (2)**:** So assume that the premouse $M^{\#}_{2n-1}$ exists and is ω_1-iterable and let $x \in {}^\omega\omega$ be arbitrary. Let Σ be the ω_1-iteration strategy for $M^{\#}_{2n-1}$ which is guided by $\mathcal{Q}$-structures (see Definiton 2.2.2). Here we also have that the branch $b = \Sigma(\mathcal{T})$ with $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$ is unique, because $\mathcal{T}$ is an iteration tree on $M^{\#}_{2n-1}$ and therefore $\mathcal{Q}$-structures for $\mathcal{T}$ are in fact not more complicated than active premice $\mathcal{Q}$ such that every proper initial segment of $\mathcal{Q}$ is $(2n-2)$-small above $\delta(\mathcal{T})$. Thus the uniqueness follows as argued in the remark after Definition 2.2.2.

With $L_{\omega_1^V}[x, M^{\#}_{2n-1}, \Sigma]$ we denote the model of height ω_1^V which is closed under Σ and constructed as follows above x and a fixed real $x_{M^{\#}_{2n-1}}$ coding the countable premouse $M^{\#}_{2n-1}$. During the construction we will also define the order of construction for elements of the model $L_{\omega_1^V}[x, M^{\#}_{2n-1}, \Sigma]$.

Successor steps: At each successor step $L_{\alpha+1}[x, M^{\#}_{2n-1}, \Sigma]$ for an ordinal $\alpha < \omega_1^V$ we add the pair $(\mathcal{T}, \Sigma(\mathcal{T}))$ to the model $L_\alpha[x, M^{\#}_{2n-1}, \Sigma]$ for all iteration trees $\mathcal{T} \in L_\alpha[x, M^{\#}_{2n-1}, \Sigma]$ on $M^{\#}_{2n-1}$ of limit length. Note that $\mathcal{T}$ is an iteration tree of length $< \omega_1^V$ and so $\Sigma(\mathcal{T})$ exists in V.

Afterwards we close the new model under rudimentary functions as in the usual construction of L. That means we let

$$\begin{aligned} L_{\alpha+1}[x, M^{\#}_{2n-1}, \Sigma] = rud(L_\alpha[x, M^{\#}_{2n-1}, \Sigma] \cup \{(\mathcal{T}, \Sigma(\mathcal{T})) \mid \\ \mathcal{T} \in L_\alpha[x, M^{\#}_{2n-1}, \Sigma] \text{ is an iteration tree} \\ \text{of limit length on } M^{\#}_{2n-1}\}), \end{aligned}$$

where the operation $X \mapsto rud(X)$ is as defined in Definition 3.4.3.

Order of construction: For two iteration trees $\mathcal{T}, \mathcal{U} \in L_\alpha[x, M^{\#}_{2n-1}, \Sigma]$ we assume inductively that we already defined the order of construction for elements of $L_\alpha[x, M^{\#}_{2n-1}, \Sigma]$ and say that $(\mathcal{T}, \Sigma(\mathcal{T}))$ is constructed before $(\mathcal{U}, \Sigma(\mathcal{U}))$ if $\mathcal{T}$ is constructed before $\mathcal{U}$.

For elements added by the closure under rudimentary functions we define the order of construction analogous to the order of construction for L (see Lemma 5.26 in [Sch14]).

Limit steps: At a limit step $\lambda \leq \omega_1^V$ of the construction we let

$$L_\lambda[x, M^{\#}_{2n-1}, \Sigma] = \bigcup_{\alpha < \lambda} L_\alpha[x, M^{\#}_{2n-1}, \Sigma].$$

Order of construction: The order of construction at the limit steps is defined analogous to the order of construction for L (see Lemma 5.26 in [Sch14]).

Now the construction of the model $L_{\omega_1^V}[x, M_{2n-1}^\#, \Sigma]$ immediatly gives the following claim.

Claim 1. *The model* $L_{\omega_1^V}[x, M_{2n-1}^\#, \Sigma]$ *as constructed above is closed under the operation*

$$\mathcal{T} \mapsto \Sigma(\mathcal{T})$$

for all iteration trees $\mathcal{T}$ *on* $M_{2n-1}^\#$ *of limit length* $< \omega_1^V$.

The background hypothesis that there is no $\mathbf{\Sigma}^1_{2n+1}$-definable ω_1-sequence of pairwise distinct reals enables us to prove the following claim.

Claim 2. $L_{\omega_1^V}[x, M_{2n-1}^\#, \Sigma] \vDash \text{ZFC}$.

Proof. Assume not. Then the power set axiom has to fail. So let γ be a countable ordinal such that

$$\mathcal{P}(\gamma) \cap L_{\omega_1^V}[x, M_{2n-1}^\#, \Sigma] \notin L_{\omega_1^V}[x, M_{2n-1}^\#, \Sigma].$$

Then the set $\mathcal{P}(\gamma) \cap L_{\omega_1^V}[x, M_{2n-1}^\#, \Sigma]$ has size $\aleph_1$.

Fix a real a in V which codes the countable set $L_\gamma[x, M_{2n-1}^\#, \Sigma]$. That means $a \in {}^\omega\omega$ codes a set $E \subset \omega \times \omega$ such that there is an isomorphism $\pi : (\omega, E) \to (L_\gamma[x, M_{2n-1}^\#, \Sigma], \in)$.

If it exists, we let A_ξ for $\gamma < \xi < \omega_1^V$ be the smallest subset of γ in $L_{\xi+1}[x, M_{2n-1}^\#, \Sigma] \setminus L_\xi[x, M_{2n-1}^\#, \Sigma]$ according to the order of construction. Moreover we let X be the set of all ξ with $\gamma < \xi < \omega_1^V$ such that A_ξ exists. Then X is cofinal in ω_1^V.

Finally we let a_ξ be a real coding A_ξ relative to the code a for $L_\gamma[x, M_{2n-1}^\#, \Sigma]$. For $\xi \in X$ we have that $A_\xi \in \mathcal{P}(\gamma) \cap L_{\omega_1^V}[x, M_{2n-1}^\#, \Sigma]$ and thus $A_\xi \subseteq L_\gamma[x, M_{2n-1}^\#, \Sigma]$, so the canonical code a_ξ for A_ξ relative to a exists. That means $a_\xi \in {}^\omega\omega$ codes the real a together with some set $a' \subset \omega$ such that $b \in a'$ iff $\pi(b) \in A_\xi$, where π is the isomorphism given by a as above.

Now consider the following ω_1^V-sequence of reals

$$A = (a_\xi \in {}^\omega\omega \mid \xi \in X).$$

We aim to show that A is a $\mathbf{\Sigma}^1_{2n+1}$-definable ω_1^V-sequence of reals as defined in the remark after the statement of Theorem 3.3.2.

As the construction of the model $L_{\omega_1^V}[x, M_{2n-1}^\#, \Sigma]$ is defined in a unique way, we have that for some $\xi < \omega_1^V$ a real z codes an element of the model $L_{\xi+1}[x, M_{2n-1}^\#, \Sigma] \setminus L_\xi[x, M_{2n-1}^\#, \Sigma]$ iff there is a sequence of countable models $(M_\beta \mid \beta \leq \xi + 1)$ and an element $Z \in M_{\xi+1}$ such that

(1) $M_0 = \{x, x_{M^{\#}_{2n-1}}\}$,
(2) $M_{\beta+1}$ is constructed from M_β as described in the construction above for all $\beta \leq \xi$,
(3) $M_\lambda = \bigcup_{\beta<\lambda} M_\beta$ for all limit ordinals $\lambda \leq \xi$,
(4) z does not code an element of M_ξ, and
(5) z codes Z.

We can compute the branch $\Sigma(\mathcal{T})$ from an iteration tree $\mathcal{T}$ of limit length $< \omega_1^V$ on $M^{\#}_{2n-1}$ in V as in the successor steps of the construction defined above in a $\mathbf{\Sigma}^1_{2n+1}$-definable way in the codes because the iteration strategy Σ is guided by $\mathcal{Q}$-structures which are Π^1_{2n}-iterable above $\delta(\mathcal{T})$. We have that Π^1_{2n}-iterability suffices here to identify the $\mathcal{Q}$-structures, because for an iteration tree $\mathcal{T}$ of limit length on $M^{\#}_{2n-1}$ the $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{T})$ is in fact not more complicated than an active premouse which satisfies that every proper initial segment is $(2n-2)$-small above $\delta(\mathcal{T})$.

So this argument shows that for $\gamma < \xi < \omega_1^V$ the statement "a_ξ is a real coding the smallest subset of γ in $L_{\xi+1}[x, M^{\#}_{2n-1}, \Sigma] \setminus L_\xi[x, M^{\#}_{2n-1}, \Sigma]$ relative to the real a" is Σ^1_{2n+1}-definable in the parameters a, x, $x_{M^{\#}_{2n-1}}$ and reals coding the countable ordinals ξ and γ.

Therefore the sequence $A = (a_\xi \in {}^\omega\omega \mid \xi \in X)$ as defined above is Σ^1_{2n+1}-definable in the parameters a, x and $x_{M^{\#}_{2n-1}}$, i.e. there are $\mathbf{\Sigma}^1_{2n+1}$-definable binary relations R_1 and R_2 as in the remark after the statement of Theorem 3.3.2. Hence this sequence A contradicts the assumption that every $\mathbf{\Sigma}^1_{2n+1}$-definable sequence of pairwise distinct reals is countable. $\square$

Claim 2 yields that in particular $\omega_1^{L_{\omega_1^V}[x, M^{\#}_{2n-1}, \Sigma]}$ exists and we have that

$$\omega_1^{L_{\omega_1^V}[x, M^{\#}_{2n-1}, \Sigma]} < \omega_1^V.$$

Work inside the model $L_{\omega_1^V}[x, M^{\#}_{2n-1}, \Sigma]$. Then we can use the extender algebra to make the real x generic over an iterate N of $M^{\#}_{2n-1}$ using Woodin's genericity iteration (see Theorem 7.14 in [St10]), iterating below the least Woodin cardinal of $M^{\#}_{2n-1}$. Since we have that $\omega_1^{L_{\omega_1^V}[x, M^{\#}_{2n-1}, \Sigma]} < \omega_1^V$, the ω_1^V-iterability of $M^{\#}_{2n-1}$, which is witnessed by Σ, is enough to show that this genericity iteration terminates inside the model $L_{\omega_1^V}[x, M^{\#}_{2n-1}, \Sigma]$. So we have that $x \in N[g]$ where g is generic for the image of the extender algebra under the iteration embedding and $N[g]$ is an active premouse with $2n-2$ Woodin cardinals. Let N^* denote the premouse which is obtained from $N[g]$ by iterating the top measure out of the universe. A fully backgrounded extender construction $L[E](x)^{N^*}$ inside the model N^* above x in the sense of [MS94] with the smallness hypothesis weakened gives an ω_1-iterable x-premouse which is not $(2n-2)$-small by adding the top measure of $N[g]$

(intersected with $L[E](x)^{N^*}$) to an initial segment of $L[E](x)^{N^*}$ as described in Section 2 of [FNS10]. Therefore the premouse $M^\#_{2n-2}(x)$ exists and is ω_1-iterable, using the main result in [FNS10].

This finishes the proof of (3) $\Rightarrow$ (2).

(1) $\Rightarrow$ (3)**:** Let $n \geq 1$ and assume that every $\boldsymbol{\Pi}^1_{2n-1}$- and every Π^1_{2n}-definable set of reals is determined. Let N be a $(2n-1)$-suitable premouse with $N \cap \mathrm{Ord} < \omega_1^V$. Such a premouse N exists under our hypotheses by Lemmas 3.1.2 and 3.1.9 because we inductively assume that $\boldsymbol{\Pi}^1_{2n-1}$ determinacy implies that $M^\#_{2n-2}(x)$ exists for all reals x.

Let $z \in {}^\omega\omega$ be arbitrary. Using Lemma 3.4.2 and the inductive existence of $M^\#_{2n-2}(x)$ in V for every real x we can alternately close under Skolem functions for $\boldsymbol{\Sigma}^1_{2n+1}$-formulas (obtained from uniformization) and the operation $a \mapsto M^\#_{2n-2}(a)$ to construct a transitive model M_z of height ω_1^V over the countable premouse N and the real z. Moreover we will define the order of construction for elements of the model M_z along the way. The construction is done in the following way.

We construct a sequence of models
$$(W_\alpha \mid \alpha < \omega_1^V)$$
and the model $M_z = W_{\omega_1^V}$ level-by-level in a construction of length ω_1^V, starting from the premouse N and the real z and taking unions at limit steps of the construction. So we let $W_0 = \{N, z\}$.

Before we are describing this construction in more detail, we fix a Π^1_{2n}-definable set U which is universal for the pointclass $\boldsymbol{\Pi}^1_{2n}$. We pick the universal set U such that we have $U_{\ulcorner\varphi\urcorner\frown a} = A_{\varphi,a}$ for every Π^1_{2n}-formula φ and every $a \in {}^\omega\omega$, where $\ulcorner\varphi\urcorner$ denotes the Gödel number of the formula φ and
$$A_{\varphi,a} = \{x \mid \varphi(x,a)\}.$$
By Lemma 3.4.2 (2) we can uniformize the set U with a Π^1_{2n+1}-definable function F. So we have for all $x \in \mathrm{dom}(F)$ that
$$(x, F(x)) \in U,$$
where $\mathrm{dom}(F) = \{x \mid \exists y\, (x,y) \in U\}$.

Odd successor steps: Now let $\alpha < \omega_1^V$ be an even successor ordinal, $\alpha = 0$, or let α be a limit ordinal and assume that we already constructed the model W_α together with an order of construction. Let $a \in W_\alpha$ be such that $M^\#_{2n-2}(a)$ does not exist in W_α. The a-premouse $M^\#_{2n-2}(a)$ exists in V and is $(2n-1)$-small and ω_1-iterable there, because we inductively assume that this follows from our hypothesis that $\boldsymbol{\Pi}^1_{2n-1}$ determinacy holds, i.e. we inductively assume that Theorem 3.3.1 holds. Let $\mathcal{M}$ be a countable a-premouse in V with the following properties.

(i) $\mathcal{M}$ is $(2n-1)$-small, but not $(2n-2)$-small,
(ii) all proper initial segments of $\mathcal{M}$ are $(2n-2)$-small,

(*iii*) $\mathcal{M}$ is a-sound and $\rho_\omega(\mathcal{M}) = a$, and
(*iv*) $\mathcal{M}$ is Π^1_{2n}-iterable.

These properties uniquely determine the a-premouse $M^{\#}_{2n-2}(a)$ in V. We let $W_{\alpha+1}$ be the model obtained by taking the closure under rudimentary functions of W_α together with all such a-premice $\mathcal{M}$ as above for all $a \in W_\alpha$.

Order of construction: For an a-premouse $\mathcal{M}_a$ and a b-premouse $\mathcal{M}_b$ satifying properties $(i) - (iv)$ for $a, b \in W_\alpha$, we say that $\mathcal{M}_a$ is defined before $\mathcal{M}_b$ if a is defined before b in the order of construction for elements of W_α, which exists inductively. For elements added by the closure under rudimentary functions we define the order of construction analogous to the order of construction for L.

Even successor steps: For the even successor levels of the construction let $\beta < \omega_1^V$ be an odd successor ordinal and assume that we already constructed the model W_β together with an order of construction. We aim to close the model W_β under Skolem functions for Σ^1_{2n+1}-formulas. That means if φ is a Π^1_{2n}-formula with a real parameter a from W_β such that $\exists x \varphi(x, a)$ holds in V but not in W_β, then we uniformly want to add a real $x_{\varphi,a}$ to $W_{\beta+1}$ such that $\varphi(x_{\varphi,a}, a)$ holds. Afterwards we again close the new model under rudimentary functions as in the usual construction of L. In fact we want to perform this construction uniformly, that means we do not add the reals $x_{\varphi,a}$ individually, but we close under the function F we fixed above.

Therefore we add $F(x)$ for all $x \in \mathrm{dom}(F) \cap W_\beta$ to the current model W_β. We will show in Claim 4 that this procedure adds reals $x_{\varphi,a}$ as described above in a Π^1_{2n+1}-definable way from a to the model $W_{\beta+1}$.

So let φ_F be a Π^1_{2n+1}-formula such that for all $x, y \in {}^\omega\omega$,

$$F(x) = y \text{ iff } \varphi_F(x, y).$$

Then we let

$$W_{\beta+1} = rud(W_\beta \cup \{y \in {}^\omega\omega \mid \exists x \in W_\beta \cap {}^\omega\omega \ \varphi_F(x, y)\}).$$

Order of construction: We inductively define the order of construction for elements of $W_{\beta+1}$ as follows. First we say for $F(x) \neq F(x')$ with $x, x' \in \mathrm{dom}(F) \cap W_\beta$ that $F(x)$ is constructed before $F(x')$ iff x is constructed before x' in the order of construction for elements of W_β where x and x' are the minimal (according to the order of construction in W_β) reals y and y' in $\mathrm{dom}(F) \cap W_\beta$ such that $F(y) = F(x)$ and $F(y') = F(x')$. Then we define for elements added by the closure under rudimentary functions the order of construction analogous to the order of construction for L.

Limit steps: At a limit step of the construction we let

$$W_\lambda = \bigcup_{\alpha<\lambda} W_\alpha$$

for all limit ordinals $\lambda < \omega_1^V$ and we finally let

$$M_z = W_{\omega_1^V} = \bigcup_{\alpha < \omega_1^V} W_\alpha.$$

Order of construction: The order of construction at the limit steps is defined analogous to the order of construction for L.

This finishes the construction of the model M_z.

Now we can prove analogous to Claim 2 that M_z is a model of ZFC from the background hypothesis that there is no $\mathbf{\Sigma}^1_{2n+2}$-definable ω_1-sequence of pairwise distinct reals by the following argument.

CLAIM 3. $M_z \vDash$ ZFC.

PROOF. Assume not. Then the power set axiom has to fail. So let γ be a countable ordinal such that

$$\mathcal{P}(\gamma) \cap M_z \notin M_z.$$

As in the proof of Claim 2 this yields that the set $\mathcal{P}(\gamma) \cap M_z$ has size $\aleph_1$.

Let $W_\gamma = M_z | \gamma$ be the γ-th level in the construction of M_z. Then we can fix a real a in V which codes the countable set W_γ.

If it exists, we let A_ξ for $\gamma < \xi < \omega_1^V$ be the smallest subset of γ in

$$M_z|(\xi + 1) \setminus M_z|\xi$$

according to the order of construction. Moreover we let X be the set of all ξ with $\gamma < \xi < \omega_1^V$ such that A_ξ exists. Then X is cofinal in ω_1^V.

Finally we let a_ξ again be a real coding the set A_ξ relative to the code a for W_γ. For $\xi \in X$ we have that $A_\xi \in \mathcal{P}(\gamma) \cap M_z$ and thus $A_\xi \subseteq W_\gamma$, so the canonical code a_ξ for A_ξ relative to a exists.

Now consider the following ω_1^V-sequence of reals

$$A = (a_\xi \in {}^\omega\omega \mid \xi \in X).$$

As in the proof of Claim 2 we have that a real y codes an element of the model $M_z|(\xi+1) \setminus M_z|\xi$ for some $\xi < \omega_1^V$ iff there is a sequence of countable models $(W_\beta \mid \beta \leq \xi + 1)$ and an element $Y \in W_{\xi+1}$ such that

(1) $W_0 = \{z, N\}$,
(2) $W_{\beta+1}$ is constructed from W_β as described in the construction above for all $\beta \leq \xi$,
(3) $W_\lambda = \bigcup_{\beta<\lambda} W_\beta$ for all limit ordinals $\lambda \leq \xi$,
(4) y does not code an element of W_ξ, and
(5) y codes Y.

This states that the real y codes an element of the model $M_z|(\xi+1) \setminus M_z|\xi$ as the construction of the levels of the model M_z is defined in a unique way. Now we have to argue that this statement is definable enough for our purposes.

For the odd successor levels of the construction we have that the properties $(i) - (iv)$ as in the construction are Π^1_{2n}-definable uniformly in any code for the countable premouse $\mathcal{M}$.

For the even successor levels let $\beta < \omega_1^V$ be an odd successor ordinal. Recall that U is a Π^1_{2n+1}-definable set and that F is a Π^1_{2n+1}-definable function uniformizing U. Hence the even successor levels $W_{\alpha+1}$ can be computed in a Σ^1_{2n+2}-definable way from W_α.

This argument shows that the statement "a_ξ codes the smallest subset of γ in $M_z|(\xi+1) \setminus M_z|\xi$" is Σ^1_{2n+2}-definable uniformly in a, z and any code for N, ξ and γ.

Therefore the ω_1-sequence A as defined above is Σ^1_{2n+2}-definable uniformly in a, z and any code for N in the sense of the remark after the statement of Theorem 3.3.2. Thus A contradicts the assumption that every $\boldsymbol{\Sigma}^1_{2n+2}$-definable sequence of pairwise distinct reals is countable. $\square$

From the construction we also get the following claim.

Claim 4. *The model M_z as constructed above has the following properties.*

(1) $M_z \cap \mathrm{Ord} = \omega_1^V$, $z, N \in M_z$,

(2) $M_z \prec_{\boldsymbol{\Sigma}^1_{2n+1}} V$,

(3) M_z *is closed under the operation*

$$a \mapsto M^{\#}_{2n-2}(a),$$

and moreover $M^{\#}_{2n-2}(a)$ *is* ω_1*-iterable in* M_z *for all* $a \in M_z$.

If we write "$M_z \prec_{\boldsymbol{\Sigma}^1_{2n+1}} V$*" we always mean that* M_z *is correct in* V *with respect to* $\boldsymbol{\Sigma}^1_{2n+1}$*-formulas with parameters from* $M_z \cap {}^\omega\omega$.

Proof. Property (1) immediately follows from the construction.

Proof of (2)**:** The proof is organized as an induction on $m < 2n+1$. We have that M_z is $\boldsymbol{\Sigma}^1_2$-correct in V using Shoenfield's Absoluteness Theorem (see for example Theorem 13.15 in [Ka08]). We assume inductively that M_z is $\boldsymbol{\Sigma}^1_m$-correct in V and prove that M_z is $\boldsymbol{\Sigma}^1_{m+1}$-correct in V. Since the upward implication follows easily as in the proof of Lemma 1.3.4, we focus on the proof of the downward implication.

So let ψ be a Σ^1_{m+1}-formula and let $a \in M_z \cap {}^\omega\omega$ be such that $\psi(a)$ holds in V. Say

$$\psi(a) \equiv \exists x \varphi(x, a)$$

for a Π^1_m-formula φ.

Let $y = \ulcorner \varphi \urcorner ^\frown a \in {}^\omega\omega$. Then we have that $y \in \mathrm{dom}(F)$ since $\psi(a)$ holds in V and $m+1 \leq 2n+1$. Therefore $\varphi_F(y, F(y))$ holds and $F(y)$ is added to the model M_z at an even successor level of the construction because we have that $y \in M_z$. Recall that $F(y)$ is choosen such that $(y, F(y)) \in U$, that means we have that

$$F(y) \in U_y = U_{\ulcorner\varphi\urcorner \frown a} = \{x \mid \varphi(x, a)\},$$

by our choice of U. Now the inductive hypothesis implies that

$$M_z \vDash \varphi(F(y), a)$$

and therefore it follows that $M_z \vDash \psi(a)$, as desired.

Proof of (3)**:** Let $a \in M_z$ be arbitrary. Then we have that $a \in W_\alpha$ for some ordinal $\alpha < \omega_1^V$. Recall that by our assumptions the ω_1-iterable a-premouse $M^\#_{2n-2}(a)$ exists in V as a is countable in V. Moreover it satisfies properties $(i) - (iv)$ from the construction, i.e.

- (i) $M^\#_{2n-2}(a)$ is $(2n-1)$-small, but not $(2n-2)$-small,
- (ii) all proper initial segments of $M^\#_{2n-2}(a)$ are $(2n-2)$-small,
- (iii) $M^\#_{2n-2}(a)$ is a-sound and $\rho_\omega(M^\#_{2n-2}(a)) = a$, and
- (iv) $M^\#_{2n-2}(a)$ is Π^1_{2n}-iterable.

Therefore the a-premouse $\mathcal{M} = M^\#_{2n-2}(a)$, which is ω_1-iterable in V, has been added to the model M_z at some odd successor level of the construction. We want to show that this a-premouse $\mathcal{M}$ is ω_1-iterable inside M_z via the iteration strategy Σ which is guided by $\mathcal{Q}$-structures (see Definition 2.2.2).

Let $\mathcal{T}$ be an iteration tree of length λ on the a-premouse $\mathcal{M}$ for some limit ordinal $\lambda < \omega_1$ in M_z such that $\mathcal{T}$ is according to the iteration strategy Σ. Then $\mathcal{T}$ is guided by $\mathcal{Q}$-structures which are $(2n-2)$-small above $\delta(\mathcal{T} \upharpoonright \gamma)$, ω_1-iterable above $\delta(\mathcal{T} \upharpoonright \gamma)$ and thus also Π^1_{2n-1}-iterable above $\delta(\mathcal{T} \upharpoonright \gamma)$ in M_z for all limit ordinals $\gamma \leq \lambda$.

By (2) we have that M_z is $\mathbf{\Sigma}^1_{2n+1}$-correct in V for real parameters in M_z. Therefore it follows that the $\mathcal{Q}$-structures $\mathcal{Q}(\mathcal{T} \upharpoonright \gamma)$ which are $(2n-2)$-small above $\delta(\mathcal{T} \upharpoonright \gamma)$ and guiding the iteration tree $\mathcal{T}$ in M_z are Π^1_{2n-1}-iterable above $\delta(\mathcal{T} \upharpoonright \gamma)$ for all limit ordinals $\gamma \leq \lambda$ in V. Since $M^\#_{2n-2}(x)$ exists in V for all $x \in {}^\omega\omega$ by our assumptions and in particular $M^\#_{2n-2}(a)$ exists in V, we have by Lemma 2.2.10 that these $\mathcal{Q}$-structures also witness that $\mathcal{T}$ is according to the $\mathcal{Q}$-structure iteration strategy Σ in V. Therefore there exists a unique cofinal well-founded branch b through $\mathcal{T}$ in V such that we have $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$. By an absoluteness argument as given several times before (see for example the proof of Lemma 2.2.8), it follows that the unique cofinal well-founded branch b through $\mathcal{T}$ in V for which $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$ holds, is also contained in M_z since we have that $\mathcal{T}, \mathcal{Q}(\mathcal{T}) \in M_z$.

Therefore $\mathcal{M} = M^\#_{2n-2}(a)$ exists and is ω_1-iterable in M_z via the iteration strategy Σ. □

Now we aim to show that a K^c-construction in M_z reaches $M^\#_{2n-1}$, meaning that the premouse $(K^c)^{M_z}$ is not $(2n-1)$-small. Here and in what follows we consider a K^c-construction in the sense of [MSch04] as this construction does not assume any large cardinals in the background model. The following claim is also going to be used in the proof of Theorem 3.6.1 later.

Claim 5. *$(K^c)^{M_z}$ is not $(2n-1)$-small.*

Proof. We work inside the model M_z and distinguish numerous different cases. Moreover we assume toward a contradiction that $(K^c)^{M_z}$ is $(2n-1)$-small.
First we show the following subclaim.

Subclaim 1. *Let δ be a cutpoint in $(K^c)^{M_z}$. If $(K^c)^{M_z}$ does not have a Woodin cardinal above δ, then $(K^c)^{M_z}$ is fully iterable above δ in M_z.*

We allow $\delta = \omega$, i.e. the case that $(K^c)^{M_z}$ has no Woodin cardinals, in the statement of Subclaim 1.

Proof of Subclaim 1. $(K^c)^{M_z}$ is iterated via the $\mathcal{Q}$-structure iteration strategy Σ (see Definition 2.2.2). That means for an iteration tree $\mathcal{U}$ of limit length on $(K^c)^{M_z}$ we let

$$\Sigma(\mathcal{U}) = b \text{ iff } \mathcal{Q}(b,\mathcal{U}) = \mathcal{Q}(\mathcal{U}) \text{ and } \mathcal{Q}(\mathcal{U}) \trianglelefteq M^{\#}_{2n-2}(\mathcal{M}(\mathcal{U})),$$

since $(K^c)^{M_z}$ is assumed to be $(2n-1)$-small.
It is enough to show that $(K^c)^{M_z}$ is ω_1-iterable above δ inside M_z, because then an absoluteness argument as the one we gave in the proof of Lemma 2.2.8 yields that $(K^c)^{M_z}$ is fully iterable above δ inside M_z as the iteration strategy for $(K^c)^{M_z}$ is guided by $\mathcal{Q}$-structures.
Assume toward a contradiction that $(K^c)^{M_z}$ is not ω_1-iterable above δ in M_z. Then there exists an iteration tree $\mathcal{T}$ on $(K^c)^{M_z}$ above δ of limit length $< \omega_1$ inside M_z such that there exists no $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{T})$ for $\mathcal{T}$ with $\mathcal{Q}(\mathcal{T}) \trianglelefteq M^{\#}_{2n-2}(\mathcal{M}(\mathcal{T}))$ and hence

$$M^{\#}_{2n-2}(\mathcal{M}(\mathcal{T})) \vDash \text{“}\delta(\mathcal{T}) \text{ is Woodin”}.$$

The premouse $M^{\#}_{2n-2}(\mathcal{M}(\mathcal{T}))$, constructed in the sense of Definition 2.2.6, exists in M_z and is not $(2n-2)$-small above $\delta(\mathcal{T})$, because otherwise it would already provide a $\mathcal{Q}$-structure for $\mathcal{T}$.
Let $\bar{M}$ be the Mostowski collapse of a countable substructure of M_z containing the iteration tree $\mathcal{T}$. That means we choose a large enough natural number m and let $\bar{M}, X$ and σ be such that

$$\bar{M} \overset{\sigma}{\cong} X \prec_{\Sigma_m} M_z,$$

where

$$\sigma : \bar{M} \rightarrow M_z$$

denotes the uncollapse map such that we have a model $\bar{K}$ in $\bar{M}$ with $\sigma(\bar{K}|\gamma) = (K^c)^{M_z}|\sigma(\gamma)$ for every ordinal $\gamma < \bar{M} \cap \mathrm{Ord}$, and we have an iteration tree $\bar{\mathcal{T}}$ on $\bar{K}$ above an ordinal $\bar{\delta}$ in $\bar{M}$ with $\sigma(\bar{\mathcal{T}}) = \mathcal{T}$ and $\sigma(\bar{\delta}) = \delta$. Moreover we have that

$$M^{\#}_{2n-2}(\mathcal{M}(\bar{\mathcal{T}})) \vDash \text{“}\delta(\bar{\mathcal{T}}) \text{ is Woodin”}.$$

By the iterability proof of Chapter 9 in [St96] (adapted as in [MSch04]) applied inside M_z, there exists a cofinal well-founded branch b through the iteration tree $\bar{\mathcal{T}}$ on $\bar{K}$ in M_z such that we have a final model $\mathcal{M}^{\bar{\mathcal{T}}}_b$.

Consider the coiteration of $\mathcal{M}^{\bar{\mathcal{T}}}_b$ with $M^{\#}_{2n-2}(\mathcal{M}(\bar{\mathcal{T}}))$ and note that it takes place above $\delta(\bar{\mathcal{T}})$. Since $M^{\#}_{2n-2}(\mathcal{M}(\bar{\mathcal{T}}))$ is ω_1-iterable above $\delta(\bar{\mathcal{T}})$ inside M_z by definition and $\mathcal{M}^{\bar{\mathcal{T}}}_b$ is iterable in M_z by the iterability proof of Chapter 9 in [St96] (adapted as in [MSch04]) applied inside the model M_z, the coiteration is successful using Lemma 2.2.8 (2).

If there is no drop along the branch b, then $\mathcal{M}^{\bar{\mathcal{T}}}_b$ cannot loose the coiteration, because otherwise there exists a non-dropping iterate R^* of $\mathcal{M}^{\bar{\mathcal{T}}}_b$ and an iterate $\mathcal{M}^*$ of $M^{\#}_{2n-2}(\mathcal{M}(\bar{\mathcal{T}}))$ such that $R^* \trianglelefteq \mathcal{M}^*$. But we have that there is no Woodin cardinal in $\mathcal{M}^{\bar{\mathcal{T}}}_b$ above $\bar{\delta}$ by elementarity and at the same time we have that $\bar{\delta} < \delta(\bar{\mathcal{T}})$ and

$$M^{\#}_{2n-2}(\mathcal{M}(\bar{\mathcal{T}})) \vDash \text{“}\delta(\bar{\mathcal{T}}) \text{ is Woodin”}.$$

If there is a drop along b, then $\mathcal{M}^{\bar{\mathcal{T}}}_b$ also has to win the coiteration, because we have $\rho_\omega(\mathcal{M}^{\bar{\mathcal{T}}}_b) < \delta(\bar{\mathcal{T}})$ and $\rho_\omega(M^{\#}_{2n-2}(\mathcal{M}(\bar{\mathcal{T}}))) = \delta(\bar{\mathcal{T}})$.

That means in both cases there is an iterate R^* of $\mathcal{M}^{\bar{\mathcal{T}}}_b$ and a non-dropping iterate $\mathcal{M}^*$ of $M^{\#}_{2n-2}(\mathcal{M}(\bar{\mathcal{T}}))$ such that $\mathcal{M}^* \trianglelefteq R^*$. We have that $\mathcal{M}^*$ is not $(2n-1)$-small, because $M^{\#}_{2n-2}(\mathcal{M}(\bar{\mathcal{T}}))$ is not $(2n-1)$-small as argued above and the iteration from $M^{\#}_{2n-2}(\mathcal{M}(\bar{\mathcal{T}}))$ to $\mathcal{M}^*$ is non-dropping. Therefore it follows that R^* is not $(2n-1)$-small and thus $\mathcal{M}^{\bar{\mathcal{T}}}_b$ is not $(2n-1)$-small. By the iterability proof of Chapter 9 in [St96] (adapted as in [MSch04]) applied inside M_z we can re-embed the model $\mathcal{M}^{\bar{\mathcal{T}}}_b$ into a model of the $(K^c)^{M_z}$-construction. This yields that $(K^c)^{M_z}$ is not $(2n-1)$-small, contradicting our assumption that it is $(2n-1)$-small. □

Now we distinguish the following cases.

Case 1. Assume that

$$(K^c)^{M_z} \vDash \text{ “there is a Woodin cardinal”}.$$

Then we can assume that there is a largest Woodin cardinal in $(K^c)^{M_z}$, because otherwise $(K^c)^{M_z}$ has infinitely many Woodin cardinals and is therefore not $(2n-1)$-small. So let δ denote the largest Woodin cardinal in $(K^c)^{M_z}$.

Consider the premouse $\mathcal{M} = M^{\#}_{2n-2}((K^c)^{M_z}|\delta)$ in the sense of Definition 2.2.6 and note that $\mathcal{M}$ exists inside M_z by Claim 4. Now try to coiterate $(K^c)^{M_z}$ with $\mathcal{M}$ inside M_z.

Since the comparison takes place above δ and the premouse

$$\mathcal{M} = M^{\#}_{2n-2}((K^c)^{M_z}|\delta)$$

is ω_1-iterable above δ, the coiteration is successful, using Lemma 2.2.8 (2) and Subclaim 1 since the iteration strategies for the premice $(K^c)^{M_z}$ and $M^{\#}_{2n-2}((K^c)^{M_z}|\delta)$ above δ are guided by $\mathcal{Q}$-structures which are $(2n-2)$-small above $\delta(\mathcal{T})$ for any iteration tree $\mathcal{T}$ of limit length.

So there is an iterate R of $(K^c)^{M_z}$ and an iterate $\mathcal{M}^*$ of $\mathcal{M}$ such that the coiteration terminates with $R \trianglelefteq \mathcal{M}^*$ or $\mathcal{M}^* \trianglelefteq R$. By universality of the model $(K^c)^{M_z}$ inside M_z (see Section 3 in [MSch04]) we have that $\mathcal{M}^* \trianglelefteq R$ and that there is no drop on the $\mathcal{M}$-side of the coiteration. This implies that the premouse $\mathcal{M} = M^{\#}_{2n-2}((K^c)^{M_z}|\delta)$ in the sense of Definition 2.2.6 is not $(2n-2)$-small above δ, as otherwise we have that $\mathcal{M}$ is not fully sound and since $\mathcal{M}^* \trianglelefteq R$ this yields a contradiction because of soundness.

Therefore $\mathcal{M}$ is not $(2n-1)$-small and thus $\mathcal{M}^*$ is also not $(2n-1)$-small. That means R and thereby $(K^c)^{M_z}$ is not $(2n-1)$-small, which is the desired contradiction.

This finishes the proof of Claim 5 in the case that there is a Woodin cardinal in $(K^c)^{M_z}$.

Case 2. Assume that

$$(K^c)^{M_z} \nvDash \text{ ``there is a Woodin cardinal''}.$$

Recall that we fixed N to be a $(2n-1)$-suitable premouse and try to coiterate $(K^c)^{M_z}$ with N inside M_z. We again iterate $(K^c)^{M_z}$ via the $\mathcal{Q}$-structure iteration strategy Σ as in Subclaim 1.

Using Subclaim 1 the coiteration cannot fail on the $(K^c)^{M_z}$-side.

Recall that the premouse N is $(2n-1)$-suitable in V. Since the statement "N is pre-$(2n-1)$-suitable" is Π^1_{2n}-definable uniformly in any code for N, it follows that N is pre-$(2n-1)$-suitable inside M_z, because M_z is $\boldsymbol{\Sigma}^1_{2n+1}$-correct in V and closed under the operation $a \mapsto M^{\#}_{2n-2}(a)$. Therefore Lemma 3.1.9 implies that N is $(2n-1)$-suitable inside M_z. In particular N has an iteration strategy which is fullness preserving for non-dropping iteration trees.

So we have that the coiteration of $(K^c)^{M_z}$ with N is successful. Let $\mathcal{T}$ and $\mathcal{U}$ be the resulting trees on $(K^c)^{M_z}$ and N of length $\lambda+1$ for some ordinal λ. It follows that $(K^c)^{M_z}$ wins the comparison by universality of the premouse $(K^c)^{M_z}$ inside M_z (see Section 3 in [MSch04]). So we have that there exists an iterate $R = \mathcal{M}^{\mathcal{T}}_{\lambda}$ of $(K^c)^{M_z}$ and an iterate $N^* = \mathcal{M}^{\mathcal{U}}_{\lambda}$ of N such that

$N^* \trianglelefteq R$. Moreover there is no drop on the main branch on the N-side of the coiteration.

Since N is $(2n-1)$-suitable we have that $M^{\#}_{2n-2}(N^*|\delta_{N^*})$ is a premouse with $2n-1$ Woodin cardinals, which is ω_1-iterable above δ_{N^*} and not $(2n-2)$-small above δ_{N^*}. So we can consider the coiteration of R with $M^{\#}_{2n-2}(N^*|\delta_{N^*})$. This coiteration is successful using Lemma 2.2.8 (2) because we have

$$N^*|\delta_{N^*} = R|\delta_{N^*}$$

and R and $M^{\#}_{2n-2}(N^*|\delta_{N^*})$ are both iterable above δ_{N^*}, using Subclaim 1 for the iterate R of $(K^c)^{M_z}$. If there is no drop on the main branch in $\mathcal{T}$, then $R = \mathcal{M}^{\mathcal{T}}_{\lambda}$ wins the comparison by universality of $(K^c)^{M_z}$ inside M_z (again see Section 3 in [MSch04]). If there is a drop on the main branch in $\mathcal{T}$, then R also wins the comparison, because in this case we have that

$$\rho_\omega(R) < \delta_{N^*} \text{ and } \rho_\omega(M^{\#}_{2n-2}(N^*|\delta_{N^*})) = \delta_{N^*}.$$

Therefore we have that there exists an iterate R^* of R and an iterate M^* of $M^{\#}_{2n-2}(N^*|\delta_{N^*})$ such that we have $M^* \trianglelefteq R^*$ in both cases and the iteration from $M^{\#}_{2n-2}(N^*|\delta_{N^*})$ to M^* is non-dropping. Since $M^{\#}_{2n-2}(N^*|\delta_{N^*})$ is not $(2n-1)$-small, we have that M^* and therefore R^* is not $(2n-1)$-small. Thus R is not $(2n-1)$-small. But R is an iterate of $(K^c)^{M_z}$ and therefore this implies that $(K^c)^{M_z}$ is not $(2n-1)$-small, contradicting our assumption.

This finishes the proof of Claim 5. □

Claim 5 now implies that $M^{\#}_{2n-1}$ exists in M_z as the minimal ω_1-iterable premouse which is not $(2n-1)$-small.

Work in V now and let $z \in {}^\omega\omega$ be arbitrary. Then

$$M_z \models \text{“}(M^{\#}_{2n-1})^{M_z} \text{ is } \omega_1\text{-iterable”}.$$

Hence

$$M_z \models \text{“}(M^{\#}_{2n-1})^{M_z} \text{ is } \Pi^1_{2n+1}\text{-iterable”}.$$

Since M_z is $\boldsymbol{\Sigma}^1_{2n+1}$-correct in V we have that

$$V \models \text{“}(M^{\#}_{2n-1})^{M_z} \text{ is } \Pi^1_{2n+1}\text{-iterable”}.$$

By $\boldsymbol{\Sigma}^1_{2n+1}$-correctness again we get for every real y such that $(M^{\#}_{2n-1})^{M_z}$ is a countable premouse in M_y that

$$M_y \models \text{“}(M^{\#}_{2n-1})^{M_z} \text{ is } \Pi^1_{2n+1}\text{-iterable”}.$$

By Lemma 2.2.9 we have that the comparison of two countable $2n$-small premice which are sound and project to ω terminates successfully, if one of the premice is Π^1_{2n+1}-iterable and the other one is ω_1-iterable. Thus the comparison of $(M^{\#}_{2n-1})^{M_z}$ and $(M^{\#}_{2n-1})^{M_y}$ inside the model M_y is successful by Lemma 2.2.9 for all reals y as above, since $(M^{\#}_{2n-1})^{M_y}$ is ω_1-iterable inside

M_y and both premice $(M_{2n-1}^{\#})^{M_z}$ and $(M_{2n-1}^{\#})^{M_y}$ are sound and project to ω.

Therefore $(M_{2n-1}^{\#})^{M_z}$ and $(M_{2n-1}^{\#})^{M_y}$ coiterate to the same model inside M_y and are in fact equal. This yields that all these premice of the form $(M_{2n-1}^{\#})^{M_x}$ for reals x are equal in V. Call this unique model $M_{2n-1}^{\#}$.

We now finally show that this premouse $M_{2n-1}^{\#}$ is ω_1-iterable in V via the iteration strategy Σ which is guided by $\mathcal{Q}$-structures as introduced in Definition 2.2.2.

So let $\mathcal{T}$ be an iteration tree on $M_{2n-1}^{\#}$ via the iteration strategy Σ in V of limit length $\lambda < \omega_1^V$. Choose $z \in {}^\omega\omega$ such that $M_{2n-1}^{\#}, \mathcal{T} \in M_z$ and $\mathrm{lh}(\mathcal{T}) < \omega_1^{M_z}$.

Since $\mathcal{T}$ is an iteration tree on $M_{2n-1}^{\#}$, all $\mathcal{Q}$-structures for $\mathcal{T} \upharpoonright \lambda$ for limit ordinals $\lambda \leq \mathrm{lh}(\mathcal{T})$ are $(2n-1)$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$. In fact these $\mathcal{Q}$-structures are not more complicated than the least active premice which are not $(2n-2)$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$ and Π^1_{2n}-iterability above $\delta(\mathcal{T} \upharpoonright \lambda)$ for them is enough to determine a cofinal well-founded branch b through $\mathcal{T}$. Therefore we have by correctness of M_z in V that $\mathcal{T}$ is an iteration tree on $M_{2n-1}^{\#}$ which is guided by $\mathcal{Q}$-structures in M_z. Since

$$M_z \vDash \text{“}M_{2n-1}^{\#} \text{ is } \omega_1\text{-iterable via the iteration strategy } \Sigma\text{”},$$

it follows that M_z can find a cofinal well-founded branch b through $\mathcal{T}$ which is determined by $\mathcal{Q}$-structures. That means we have that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$ and we have in particular that $\mathcal{Q}(\mathcal{T})$ is ω_1-iterable and thus Π^1_{2n}-iterable inside M_z. So $\mathcal{Q}(\mathcal{T})$ is also Π^1_{2n}-iterable in V and it follows that b is also the unique cofinal well-founded branch through $\mathcal{T}$ in V determined by the same $\mathcal{Q}$-structures as in M_z. Therefore we finally have that

$$V \vDash \text{“}M_{2n-1}^{\#} \text{ is } \omega_1\text{-iterable”}.$$

□

3.5. Getting the Right Hypothesis

In this section we will provide one ingredient for the proof of Theorem 2.1.1. We will see that the results from Sections 3.6 and 3.7, which are a strengthening of Theorem 3.4.1, can actually be obtained from boldface determinacy at the right level of the projective hierarchy. The following lemma makes this precise.

Lemma 3.5.1. *Let $n \geq 0$. Then $\boldsymbol{\Pi}^1_{n+1}$ determinacy implies that every $\boldsymbol{\Sigma}^1_{n+2}$-definable sequence of pairwise distinct reals is countable.*

Proof. Again for periodicity reasons we give a completely different proof for the even and the odd levels of the projective hierarchy. We start with the even levels, for which we will give an inner model theoretic argument using the results in Sections 3.1 and 3.2.

So let $n \geq 1$ and assume that every $\mathbf{\Pi}^1_{2n}$-definable set of reals is determined. Assume further toward a contradiction that

$$(z_\alpha \mid \alpha < \omega_1)$$

is a Σ^1_{2n+1}-definable ω_1-sequence of pairwise distinct reals as defined in the remark after the statement of Theorem 3.3.2. So in particular there are Σ^1_{2n+1}-definable relations R_1 and R_2 as defined in the remark after the statement of Theorem 3.3.2. The following proof will generalize to $\Sigma^1_{2n+1}(x)$-definable sequences of pairwise distinct reals for $x \in {}^\omega\omega$, but we will assume that $x = 0$ to simplify the notation.

By Lemmas 3.1.2 and 3.1.9 there exists a countable $(2n-1)$-suitable premouse N. We aim to show that $z_\alpha \in N$ for all $\alpha < \omega_1^V$ to derive a contradiction.

For this purpose we fix an arbitrary ordinal $\alpha < \omega_1^V$ and let N^* be the model which is obtained from N by iterating the least measure in N and its images $(\alpha^+)^N$ times.

Claim 1. $z_\alpha \in N^*$.

Proof. Since N is $(2n-1)$-suitable and thus has a fullness preserving iteration strategy for non-dropping iteration trees, the same holds for the premouse N^*, and we have by Lemma 3.2.1 that for any Σ^1_{2n+1}-formula φ with a parameter $a \in N^* \cap {}^\omega\omega$,

$$\varphi(a) \;\leftrightarrow\; \Vdash^{N^*}_{\mathrm{Col}(\omega,\delta_{N^*})} \varphi(a).$$

Pick mutual generics g and h for $\mathrm{Col}(\omega,\alpha)$ over N^* such that

$$N^*[g] \cap N^*[h] = N^*.$$

Since $\alpha < \delta_{N^*}$ we still have by Lemma 3.2.1 that for any Σ^1_{2n+1}-formula φ and any parameter $b \in N^*[g] \cap {}^\omega\omega$,

$$\varphi(b) \;\leftrightarrow\; \Vdash^{N^*[g]}_{\mathrm{Col}(\omega,\delta_{N^*})} \varphi(b)$$

and similarly for $N^*[h]$.

In $N^*[g]$, the ordinal $\alpha < \omega_1^V$ can be coded by a real $a \in \mathrm{WO}$ such that $||a|| = \alpha$. This yields that $z_\alpha \in N^*[g]$, because the sequence $(z_\alpha \mid \alpha < \omega_1^V)$ is Σ^1_{2n+1}-definable and thus z_α is a $\Sigma^1_{2n+1}(a)$-singleton, i.e. $\{z_\alpha\}$ is Σ^1_{2n+1}-definable from the parameter $a \in N^*[g] \cap {}^\omega\omega$ using the binary relation R_2 as defined in the remark after the statement of Theorem 3.3.2. Analogously we have that $z_\alpha \in N^*[h]$.

Therefore it follows that $z_\alpha \in N^*[g] \cap N^*[h] = N^*$, as desired. □

The premice N and N^* have the same reals since the iteration from N to N^* was obtained by iterating the least measure in N and its images $(\alpha^+)^N$ times and therefore does not drop. This yields that in fact

$$z_\alpha \in N.$$

Since $\alpha < \omega_1^V$ was arbitrary and the z_α are pairwise distinct reals this is a contradiction to the fact that N is countable. This finishes the proof for the even levels of the projective hierarchy.

Note that we in fact proved an effective version of Lemma 3.5.1 for the even levels of the projective hierarchy. That means for any $n \geq 1$ and $x \in {}^\omega\omega$, under the hypothesis that every $\boldsymbol{\Pi}^1_{2n-1}$-definable set of reals is determined and every $\Pi^1_{2n}(x)$-definable set of reals is determined, there exists no $\Sigma^1_{2n+1}(x)$-definable ω_1-sequence of pairwise distinct reals.

Now assume for the odd levels of the projective hierarchy that every $\boldsymbol{\Pi}^1_{2n+1}$-definable set of reals is determined. The following proof for the odd levels is completely descriptive set theoretic and uses uniformization and determinacy for the Davis game. Assume toward a contradiction that there exists a $\boldsymbol{\Sigma}^1_{2n+2}$-definable ω_1-sequence of pairwise distinct reals. That means in particular that there exists a well-order $\leq^*$ of ordertype ω_1 for reals as in the remark after the statement of Theorem 3.3.2 such that if we let $X_{\leq^*} = field(\leq^*)$, i.e. if we have for all $y \in {}^\omega\omega$,

$$y \in X_{\leq^*} \;\Leftrightarrow\; \exists x \, (x \leq^* y \vee y \leq^* x),$$

then there exists a $\boldsymbol{\Sigma}^1_{2n+2}$-definable relation R_1 such that we have for all $x, y \in {}^\omega\omega$,

$$R_1(x,y) \;\Leftrightarrow\; x, y \in X_{\leq^*} \;\wedge\; x \leq^* y.$$

Let A be a $\boldsymbol{\Pi}^1_{2n+1}$-definable relation such that we have for all $x, y \in {}^\omega\omega$,

$$R_1(x,y) \;\Leftrightarrow\; \exists z \, A(z,x,y).$$

Moreover consider the relation A_2 such that for all $u \in ({}^\omega\omega)^2$ and for all $y \in {}^\omega\omega$,

$$A_2(y,u) \;\Leftrightarrow\; A((u)_0,(u)_1,y).$$

By $\boldsymbol{\Pi}^1_{2n+1}$-uniformization (see Theorem $6C.5$ in [Mo09] or Lemma 3.4.2 (1) in this thesis) there exists a $\boldsymbol{\Pi}^1_{2n+1}$-definable function F which uniformizes the set A_2, that means we have

$$(y, F(y)) \in A_2$$

for all $y \in \mathrm{dom}(F)$, where $\mathrm{dom}(F) = \{x \in {}^\omega\omega \mid \exists z \, (y,z) \in A_2\}$. We have that $X_{\leq^*} \subseteq \mathrm{dom}(F)$. So the relation A^* defined by

$$A^*(z,x,y,u) \;\Leftrightarrow\; A(z,x,y) \wedge z = (u)_0 \wedge x = (u)_1 \wedge u = F(y)$$

is $\boldsymbol{\Pi}^1_{2n+1}$-definable. Let A^* denote the set of all tuples (z,x,y,u) such that $A^*(z,x,y,u)$ holds.

Consider the following game $G^p(A^*)$ which is due to M. Davis (see [Da64] or Theorem 12.11 in [Sch14]). Here we as usual identify reals (or elements of the Baire space ${}^\omega\omega$), in particular elements of the set A^*, with elements of the Cantor space ${}^\omega 2$.

I	s_0		s_1		$\dots$	
II		n_0		n_1		$\dots$

Player I plays finite $0-1$-sequences $s_i \in {}^{<\omega}2$ (allowing $s_i = \emptyset$), player II responds with $n_i \in \{0,1\}$ and the game lasts ω steps. We say that player I wins the game $G^p(A^*)$ iff

$$s_0{}^\frown n_0{}^\frown s_1{}^\frown n_1{}^\frown \dots \in A^*.$$

Otherwise player II wins. We may code the game $G^p(A^*)$ into a "usual" Gale-Stewart game $G(A')$ as in Definition 1.1.1 for some $A' \subset {}^\omega\omega$ such that A' is $\mathbf{\Pi}^1_{2n+1}$-definable. So we have by assumption that the game $G(A')$ is determined and thus the same holds for the Davis game $G^p(A^*)$.

As the set A^* is uncountable, the proof of Theorem 12.11 in [Sch14] gives that the set A^* has a perfect subset. Since by definition of A^* we have that if $A^*(z,x,y,u)$ holds, the real y uniquely determines the reals u, z and x, we immediately have the following claim.

Claim 2. *The set*

$$B = \{y \in {}^\omega\omega \mid \exists z \exists x \exists u\ A^*(z,x,y,u)\}$$

has a perfect subset.

For $y \in B$ we have that there exists a real x such that $R_1(x,y)$ holds. So we have in particular that $y \in X_{\leq^*} = field(\leq^*)$.

Therefore there exists a continuous function $f : \mathbb{R} \to B$ such that we can consider the following order $\leq$ on the reals. We say for two reals x and y that

$$x \leq y \Leftrightarrow f(x) \leq^* f(y).$$

Then we immediately get that the following claim holds.

Claim 3. *The order $\leq$ is a $\mathbf{\Sigma}^1_{2n+2}$-definable well-order of the reals.*

Now we can obtain the following claim using Bernstein's argument.

Claim 4. *There are two disjoint $\mathbf{\Sigma}^1_{2n+2}$-definable sets D and D' of size $\aleph_1$ without perfect subsets.*

In fact we only need one of the sets D and D' in what follows.

Proof of Claim 4. We have that CH holds, i.e. we have $2^{\aleph_0} = \aleph_1$, so there are $\aleph_1$-many perfect sets of reals, and let

$$(P_\alpha \mid \alpha < \omega_1)$$

enumerate all perfect sets of reals using the well-ordering of the reals $\leq$ as defined above. Using Claim 3 we can now pick sequences $(x_\alpha \mid \alpha < \omega_1)$ and $(y_\alpha \mid \alpha < \omega_1)$ of pairwise distinct reals in a $\mathbf{\Sigma}^1_{2n+2}$-definable way following

Bernstein's argument such that for all $\alpha < \omega_1$ the real x_α is picked such that $x_\alpha \in P_\alpha$ is least according to the well-order $\leq$ with

$$x_\alpha \notin \{y_\beta \mid \beta < \alpha\} \cup \{x_\beta \mid \beta < \alpha\}.$$

Similarly we pick $y_\alpha \in P_\alpha$ for all $\alpha < \omega_1$ such that y_α is least according to the well-order $\leq$ with

$$y_\alpha \notin \{x_\beta \mid \beta \leq \alpha\} \cup \{y_\beta \mid \beta < \alpha\}.$$

Now let

$$D = \{x_\alpha \mid \alpha < \omega_1\},$$

and

$$D' = \{y_\alpha \mid \alpha < \omega_1\}.$$

Then it is easy to see that D and D' are disjoint and both do not contain a perfect subset. Moreover we have by construction that they are both $\mathbf{\Sigma}^1_{2n+2}$-definable by Claim 3. □

Let E be a $\mathbf{\Pi}^1_{2n+1}$-definable set such that

$$x \in D \;\Leftrightarrow\; \exists y \; (x, y) \in E.$$

By $\mathbf{\Pi}^1_{2n+1}$-uniformization (see Theorem 6C.5 in [Mo09] or Lemma 3.4.2 (1) in this thesis) there exists a $\mathbf{\Pi}^1_{2n+1}$-definable partial function $C : \mathbb{R} \rightarrow \mathbb{R}$ uniformizing E, that means we have that

$$\exists y \, (x, y) \in E \Leftrightarrow (x, C(x)) \in E.$$

The set C' which is defined as

$$C' = \{(x, C(x)) \mid x \in D\} = C \cap E$$

is $\mathbf{\Pi}^1_{2n+1}$-definable and consider the Davis game $G^p(C')$. Since we can as before "code" this game into a Gale-Stewart game $G(C_0)$ for a $\mathbf{\Pi}^1_{2n+1}$-definable set of reals C_0, the games $G(C_0)$ and $G^p(C')$ are determined by assumption. As above this yields that C' has a perfect subset $P \subset C'$, because D and thus also C' is uncountable. Since P is a perfect set and D does not contain a perfect subset, there are reals x, y and y' such that $y \neq y'$ and $(x, y) \in P$ and $(x, y') \in P$. But we have $y = C(x) = y'$. This is a contradiction. □

3.6. $M^{\#}_{2n-1}(x)$ from Boldface Π^1_{2n} Determinacy

The goal of this section is to prove Theorem 3.3.2 in the case that n is odd. The proof which is presented in the following argument only works if n is odd because of the periodicity in the projective hierarchy in terms of uniformization (see Lemma 3.4.2). The even levels in the statement of Theorem 3.3.2 have to be treated differently (see Section 3.7). So we are going to prove the following theorem.

THEOREM 3.6.1. *Let $n \geq 1$ and assume that every $\mathbf{\Pi}^1_{2n-1}$-definable set of reals and every Π^1_{2n}-definable set of reals is determined. Moreover assume that there is no $\mathbf{\Sigma}^1_{2n+1}$-definable ω_1-sequence of pairwise distinct reals. Then the premouse $M^{\#}_{2n-1}$ exists and is ω_1-iterable.*

The proof of Theorem 3.6.1 uses the following Determinacy Transfer Theorem, which is due to A. S. Kechris and W. H. Woodin (see [KW08]). In the version as stated below it follows from [KW08] using [Ne95] and that Theorem 3.3.1 holds below $2n-1$ inductively.

THEOREM 3.6.2 (Determinacy Transfer Theorem). *Let $n \geq 1$. Assume determinacy for every $\mathbf{\Pi}^1_{2n-1}$- and every Π^1_{2n}-definable set of reals. Then we have determinacy for all $\Game^{(2n-1)}(<\omega^2-\Pi^1_1)$-definable sets of reals.*

PROOF. The lightface version of Theorem 1.10 in [KW08] (see p. 369 in [KW08]) gives that for all $n \geq 1$,

$$\mathrm{Det}(\Game^{(2n-2)}(<\omega^2-\mathbf{\Pi}^1_1)) \rightarrow [\mathrm{Det}(\Delta^1_{2n}) \leftrightarrow \mathrm{Det}(\Game^{(2n-1)}(<\omega^2-\Pi^1_1))].$$

As in the addendum §5 in [KW08] we need to argue that $\mathbf{\Pi}^1_{2n-1}$ determinacy implies $\Game^{(2n-2)}(<\omega^2-\mathbf{\Pi}^1_1)$ determinacy to obtain that it implies the right-hand side of the implication, i.e. "$\mathrm{Det}(\Delta^1_{2n}) \leftrightarrow \mathrm{Det}(\Game^{(2n-1)}(<\omega^2-\Pi^1_1))$". Recall that we assume inductively that Theorem 3.3.1 holds for all $m < 2n-1$. Together with Lemma 3.5.1 this implies that the premouse $M^{\#}_{2n-2}(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$ from $\mathbf{\Pi}^1_{2n-1}$ determinacy. By Theorem 2.5 in [Ne95] this yields $\Game^{(2n-2)}(<\omega^2-\mathbf{\Pi}^1_1)$ determinacy. Therefore we have that

$$\mathrm{Det}(\mathbf{\Pi}^1_{2n-1}) \rightarrow [\mathrm{Det}(\Delta^1_{2n}) \leftrightarrow \mathrm{Det}(\Game^{(2n-1)}(<\omega^2-\Pi^1_1))].$$

We have

$$\mathrm{Det}(\Delta^1_{2n}) \leftrightarrow \mathrm{Det}(\Pi^1_{2n})$$

by Theorem 5.1 in [KS85] which is due to Martin (see [Ma73]). So in particular $\mathbf{\Pi}^1_{2n-1}$ determinacy and Π^1_{2n} determinacy together imply that $\Game^{(2n-1)}(<\omega^2-\Pi^1_1)$ determinacy holds. □

Martin proves in [Ma08] that under the assumption that $x^{\#}$ exists for every real x,

$$A \in \Game(<\omega^2-\Pi^1_1)$$

iff there is a formula ϕ such that for all $x \in {}^\omega\omega$

$$x \in A \text{ iff } L[x] \vDash \phi[x, \gamma_1, \ldots, \gamma_k],$$

where $\gamma_1, \ldots, \gamma_k$ are Silver indiscernibles for x. In the light of this result (see also Definition 2.7 in [Ne02] for the general case) we can obtain the following corollary of the Determinacy Transfer Theorem 3.6.2.

Corollary 3.6.3. *Let $n \geq 1$. Assume that $\mathbf{\Pi}^1_{2n-1}$ determinacy and Π^1_{2n} determinacy hold. Suppose Q is a set of reals such that there is an $m < \omega$ and a formula ϕ such that for all $x \in {}^\omega\omega$*

$$x \in Q \text{ iff } M_{2n-2}(x) \vDash \phi(x, E, \gamma_1, \dots, \gamma_m),$$

where E is the extender sequence of $M_{2n-2}(x)$ and $\gamma_1, \dots, \gamma_m$ are the first m indiscernibles of $M_{2n-2}(x)$. Then Q is determined.

This follows from the Determinacy Transfer Theorem 3.6.2 as the set Q defined in Corollary 3.6.3 is $\Game^{(2n-1)}(< \omega^2 - \Pi^1_1)$-definable. Moreover note in the statement of Corollary 3.6.3 that $\mathbf{\Pi}^1_{2n-1}$ determinacy inductively implies that the premouse $M^{\#}_{2n-2}(x)$ exists for every real x.
Now we are ready to prove Theorem 3.6.1.

Proof of Theorem 3.6.1. Let $n \geq 1$. We assume inductively that Theorem 3.3.2 holds for $2n-2$, that means we assume that $\mathbf{\Pi}^1_{2n-1}$ determinacy implies that $M^{\#}_{2n-2}(x)$ exists and is ω_1-iterable for all reals x. These even levels will be proven in Theorem 3.7.1. Recall that then our hypothesis implies by Lemmas 3.1.2 and 3.1.9 that there exists a $(2n-1)$-suitable premouse.
Let $x \in {}^\omega\omega$ and consider the simultaneous comparison of all $(2n-1)$-suitable premice N such that N is coded by a real $y_N \leq_T x$, using the iterability stated in Definition 3.1.8. That means analogous to a usual comparison as in Theorem 3.14 in [St10] we iterate away the least disagreement that exists between any two of the models we are considering.
These premice successfully coiterate to a common model since they are all $(2n-1)$-suitable and call this common iterate N_x. We could have performed this simultaneous comparison inside an inner model of V of height ω_1^V which contains x and is closed under the operation $a \mapsto M^{\#}_{2n-2}(a)$ and therefore the resulting premouse N_x is countable in V.
Since N_x results from a successful comparison using iterability in the sense of Definition 3.1.8, there either exists a $(2n-1)$-suitable premouse N and a non-dropping iteration from N to N_x via a short iteration tree or there exists a maximal iteration tree $\mathcal{T}$ on a $(2n-1)$-suitable premouse such that

$$N_x = M_{2n-2}(\mathcal{M}(\mathcal{T})) \,|\, (\delta(\mathcal{T})^+)^{M_{2n-2}(\mathcal{M}(\mathcal{T}))}.$$

Let δ_{N_x} as usual denote the largest cardinal in N_x. Then we have that in the first case N_x is a $(2n-1)$-suitable premouse again by fullness preservation and so in particular the model $M_{2n-2}(N_x|\delta_{N_x})$ constructed in the sense of Definition 2.2.6 is a well-defined proper class premouse with $2n-1$ Woodin cardinals. In the second case we have by maximality of $\mathcal{T}$ that

$$M_{2n-2}(N_x|\delta_{N_x}) \vDash \text{“}\delta(\mathcal{T}) \text{ is Woodin”},$$

with $\delta_{N_x} = \delta(\mathcal{T})$, and $M_{2n-2}(N_x|\delta_{N_x})$ is again a well-defined premouse with $2n-1$ Woodin cardinals.

For each formula ϕ and each $m < \omega$ let Q_m^ϕ be the set of all $x \in {}^\omega\omega$ such that

$$M_{2n-2}(N_x|\delta_{N_x}) \vDash \phi(E, \gamma_1, \dots, \gamma_m),$$

where E is the extender sequence of $M_{2n-2}(N_x|\delta_{N_x})$ and $\gamma_1, \dots, \gamma_m$ are indiscernibles of $M_{2n-2}(N_x|\delta_{N_x})$.

Claim 1. *For all formulas ϕ and for all $m < \omega$ the set Q_m^ϕ is determined.*

Proof. We aim to reduce determinacy for the set Q_m^ϕ to determinacy for a set Q as in Corollary 3.6.3.
Recall that by Lemma 3.1.9 we have that a premouse N is $(2n-1)$-suitable iff it is pre-$(2n-1)$-suitable, that means iff it satisfies the following properties for an ordinal δ_0.

(1) $N \vDash$ "$\mathrm{ZFC}^- + \delta_0$ is the largest cardinal",

$$N = M_{2n-2}(N|\delta_0) \,|\, (\delta_0^+)^{M_{2n-2}(N|\delta_0)},$$

and for every $\gamma < \delta_0$,

$$M_{2n-2}(N|\gamma) \,|\, (\gamma^+)^{M_{2n-2}(N|\gamma)} \lhd N,$$

(2) $M_{2n-2}(N|\delta_0)$ is a proper class model and

$$M_{2n-2}(N|\delta_0) \vDash \text{"}\delta_0 \text{ is Woodin"},$$

(3) for every $\gamma < \delta_0$, $M_{2n-2}(N|\gamma)$ is a set, or

$$M_{2n-2}(N|\gamma) \nvDash \text{"}\gamma \text{ is Woodin"},$$

and

(4) for every $\eta < \delta_0$, $M_{2n-2}(N|\delta_0) \vDash$ "$N|\delta_0$ is η-iterable".

Formally the definition of pre-$(2n-1)$-suitability requires that in the background universe the premouse $M_{2n-2}^\#(z)$ exists for all $z \in {}^\omega\omega$. But for a premouse N which is coded by a real $y_N \leq_T x$, the model $M_{2n-2}(x)$ can compute if N is pre-n-suitable in V, i.e. if N satisfies properties (1) - (4) above in V, by considering a fully backgrounded extender construction in the sense of [MS94] above $N|\delta_0$, where δ_0 denotes the largest cardinal in N. Therefore we can make sense of these $(2n-1)$-suitable premice inside the model $M_{2n-2}(x)$. For the same reason the simultaneous comparison of all such $(2n-1)$-suitable premice which are coded by a real $y_N \leq_T x$ as introduced above can be computed inside $M_{2n-2}(x)$ and therefore we have that $N_x \in M_{2n-2}(x)$.

Now consider the following formula ψ_ϕ, where ϕ as above is an arbitrary formula.

$$\begin{aligned}\psi_\phi(x, E, \gamma_1, \dots, \gamma_m) \equiv \text{“}&\text{Write } N_x \text{ for the result of the simultaneous}\\ &\text{comparison of all } (2n-1)\text{-suitable premice } N\\ &\text{which are coded by a real } y_N \leq_T x \text{ and}\\ &\delta_{N_x} \text{ for the largest cardinal in } N_x, \text{ then}\\ &\quad L[\bar{E}](N_x|\delta_{N_x}) \vDash \phi(E^*, \gamma_1, \dots, \gamma_m),\\ &\text{where } E^* \text{ denotes the extender sequence}\\ &\text{of } L[\bar{E}](N_x|\delta_{N_x}), \text{which in turn is computed from } E\\ &\text{via a fully backgrounded extender construction}\\ &\quad \text{over } N_x|\delta_{N_x} \text{ as in [MS94].”}\end{aligned}$$

Now let E denote the extender sequence of the x-premouse $M_{2n-2}(x)$ and let $\gamma_1, \dots, \gamma_m$ denote indiscernibles of the model $L[M_{2n-2}(x)|\delta] = M_{2n-2}(x)$, where δ is the largest Woodin cardinal in $M_{2n-2}(x)$. Then we have in particular that $\gamma_1, \dots, \gamma_m$ are indiscernibles of the model $L[\bar{E}](N_x|\delta_{N_x})$, constructed via a fully backgrounded extender construction inside $M_{2n-2}(x)$ as defined in the formula $\psi_\phi(x, E, \gamma_1, \dots, \gamma_m)$ above.

Therefore we have that

$$x \in \mathcal{Q}^\phi_m \Leftrightarrow M_{2n-2}(x) \vDash \psi_\phi(x, E, \gamma_1, \dots, \gamma_m),$$

because the premice $L[\bar{E}](N_x|\delta_{N_x})$ as defined above and $M_{2n-2}(N_x|\delta_{N_x})$ as in the definition of the set $\mathcal{Q}^\phi_m$ coiterate to the same model.

Thus Corollary 3.6.3 implies that $\mathcal{Q}^\phi_m$ is determined for any formula ϕ and any $m < \omega$. □

The sets Q^ϕ_m are Turing invariant, since the premouse N_x by definition only depends on the Turing degree of x.

Let $Th^{M_{2n-2}(N_x|\delta_{N_x})}$ denote the theory of $M_{2n-2}(N_x|\delta_{N_x})$ with indiscernibles (computed in V). That means

$$\begin{aligned}Th^{M_{2n-2}(N_x|\delta_{N_x})} = \{\phi \mid M_{2n-2}(N_x|\delta_{N_x}) \vDash \phi(E, \gamma_1, \dots, \gamma_m),\\ m < \omega, \phi \text{ formula}\},\end{aligned}$$

where as above E denotes the extender sequence of $M_{2n-2}(N_x|\delta_{N_x})$ and $\gamma_1, \dots, \gamma_m$ are indiscernibles of $M_{2n-2}(N_x|\delta_{N_x})$. Then we have that the theory of $M_{2n-2}(N_x|\delta_{N_x})$ stabilizes on a cone of reals x as in the following claim.

Claim 2. *There exists a real $x_0 \geq_T x$ such that for all reals $y \geq_T x_0$,*

$$Th^{M_{2n-2}(N_{x_0}|\delta_{N_{x_0}})} = Th^{M_{2n-2}(N_y|\delta_{N_y})}.$$

Proof. By Claim 1 the set Q^ϕ_m is determined and as argued above it is also Turing invariant for all formulas ϕ and all $m < \omega$. That means the set

Q_m^ϕ either contains a cone of reals or is completely disjoint from a cone of reals.

For each formula ϕ and each natural number m let $x_m^\phi \in {}^\omega\omega$ be such that either $y \in Q_m^\phi$ for all $y \geq_T x_m^\phi$ or else $y \notin Q_m^\phi$ for all $y \geq_T x_m^\phi$. Let

$$x_0 = \bigoplus \{x_m^\phi \mid \phi \text{ formula}, m < \omega\}.$$

Then we have by construction for all $y \geq_T x_0$ that

$$Th^{M_{2n-2}(N_{x_0}|\delta_{N_{x_0}})} = Th^{M_{2n-2}(N_y|\delta_{N_y})},$$

as desired. □

Let $x_0 \in {}^\omega\omega$ be as in Claim 2. We want to show that the unique theory $T = Th^{M_{2n-2}(N_x|\delta_{N_x})}$ of $M_{2n-2}(N_x|\delta_{N_x})$ with indiscernibles as defined above for $x \geq_T x_0$ in fact gives a candidate for the theory of the premouse $M_{2n-1}^\#$ in V to conclude that $M_{2n-1}^\#$ exists and is ω_1-iterable in V. By coding a formula ϕ by its unique Gödel number $\ulcorner\phi\urcorner$ we can code the theory T by a real x_T.

Fix a real z such that $z \geq_T x_0 \oplus x_T$. Moreover we can pick the real z such that it in addition codes a $(2n-1)$-suitable premouse using Lemmas 3.1.2 and 3.1.9.

Using the uniformization property of the pointclass Π_{2n-1}^1 (see Theorem 6C.5 in [Mo09] or Lemma 3.4.2 (1) in this thesis) and the existence of the premouse $M_{2n-2}^\#(x)$ in V for every real x, we can as described below alternately close under Skolem functions for $\mathbf{\Sigma}_{2n}^1$-formulas (obtained from Π_{2n-1}^1-uniformization) and the operation

$$a \mapsto M_{2n-2}^\#(a)$$

to construct a transitive model M_z from z similar as in the proof of Theorem 3.4.1 we gave earlier.

The fact that the model M_z is closed under $a \mapsto M_{2n-2}^\#(x)$ directly yields that M_z is $\mathbf{\Sigma}_{2n}^1$-correct. Therefore there is no need to close under Skolem functions for $\mathbf{\Sigma}_{2n}^1$-formulas manually. We only include this in the construction because it simplifies the discussion a bit and like this the construction is analogous to the one in the proof of Theorem 3.4.1 (where the closure under Skolem functions was necessary).

Here we only aim for a model M_z which is $\mathbf{\Sigma}_{2n}^1$-correct in V with parameters from M_z (instead of $\mathbf{\Sigma}_{2n+1}^1$-correct as in the proof of Theorem 3.4.1). This is the reason why we can reduce the hypothesis that every $\mathbf{\Sigma}_{2n+2}^1$-definable sequence of pairwise distinct reals is countable from Theorem 3.4.1 to the hypothesis that every $\mathbf{\Sigma}_{2n+1}^1$-definable sequence of pairwise distinct reals is countable as in the statement of Theorem 3.6.1.

We construct the model $M_z = W_{\omega_1^V}$ level-by-level in a construction of length ω_1^V, starting from z and taking unions at limit steps of the construction. So we let $W_0 = \{z\}$.

The order of construction for elements of the model M_z is defined exactly as in the proof of Theorem 3.4.1, so we do not give the details here again.

Before we are describing this construction in more detail, we fix a Π^1_{2n-1}-definable set U which is universal for the pointclass $\boldsymbol{\Pi}^1_{2n-1}$. Pick U such that we have $U_{\ulcorner\varphi\urcorner\frown a} = A_{\varphi,a}$ for every Π^1_{2n-1}-formula φ and every $a \in {}^\omega\omega$, where $\ulcorner\varphi\urcorner$ denotes the Gödel number of the formula φ and

$$A_{\varphi,a} = \{x \mid \varphi(x,a)\}.$$

Then the uniformization property (see Theorem $6C.5$ in [Mo09] or Lemma 3.4.2 (1) in this thesis) yields that there exists a Π^1_{2n-1}-definable function F uniformizing the universal set U. So we have for all $x \in \mathrm{dom}(F)$ that

$$(x, F(x)) \in U,$$

where $\mathrm{dom}(F) = \{x \mid \exists y\, (x,y) \in U\}$.

Odd successor steps: At an odd successor level $\alpha + 1$ of the construction we close the previous model W_α under the operation $a \mapsto M^{\#}_{2n-2}(a)$ before closing under rudimentary functions. More precisely let α be an even successor ordinal, $\alpha = 0$, or let α be a limit ordinal and assume that we already constructed W_α. Let $a \in W_\alpha$ be arbitrary. Then the a-premouse $M^{\#}_{2n-2}(a)$ exists in V because we again inductively assume that Theorem 3.3.1 holds for $2n-2$. As in the proof of Theorem 3.4.1 let $\mathcal{M}$ be a countable a-premouse in V with the following properties.

(*i*) $\mathcal{M}$ is $(2n-1)$-small, but not $(2n-2)$-small,
(*ii*) all proper initial segments of $\mathcal{M}$ are $(2n-2)$-small,
(*iii*) $\mathcal{M}$ is a-sound and $\rho_\omega(\mathcal{M}) = a$, and
(*iv*) $\mathcal{M}$ is Π^1_{2n}-iterable.

As in the proof of Theorem 3.4.1 these properties again uniquely determine the a-premouse $M^{\#}_{2n-2}(a)$ in V.

We add such a-premice $\mathcal{M}$ for all $a \in W_\alpha$ to $W_{\alpha+1}$ before closing under rudimentary functions as in the usual construction of L.

Even successor steps: At an even successor level $\beta+1$ of the construction, we close W_β under Skolem functions for Σ^1_{2n}-formulas. So assume that β is an odd successor ordinal and that we already constructed W_β. Whenever φ is a Π^1_{2n-1}-formula with a fixed parameter a from $W_\beta \cap {}^\omega\omega$ such that

$$\exists x\, \varphi(x,a)$$

holds in V but not in W_β, we add a real $x_{\varphi,a}$ such that $\varphi(x_{\varphi,a}, a)$ holds, obtained as described below, to the model $W_{\beta+1}$. Afterwards we again close the model under rudimentary functions as in the usual construction of L to obtain $W_{\beta+1}$. In fact we want to perform this construction uniformly,

that means we do not add the reals $x_{\varphi,a}$ individually, but we close under the function F which uniformizes the set U we fixed above.
Therefore we add $F(x)$ for all $x \in \mathrm{dom}(F) \cap W_\beta$ to the current model W_β. We will see in Claims 3 and 4 that this procedure adds reals $x_{\varphi,a}$ as above in a Π^1_{2n-1}-definable way from a to the model M_z.
So let φ_F be a Π^1_{2n-1}-formula such that for all $x, y \in {}^\omega\omega$,

$$F(x) = y \text{ iff } \varphi_F(x, y).$$

Then we let

$$W_{\beta+1} = rud(W_\beta \cup \{y \in {}^\omega\omega \mid \exists x \in W_\beta \cap {}^\omega\omega \; \varphi_F(x, y)\}).$$

Limit steps: At a limit step of the construction we let as usual

$$W_\lambda = \bigcup_{\alpha<\lambda} W_\alpha$$

for all limit ordinals $\lambda < \omega_1^V$ and we finally let

$$M_z = W_{\omega_1^V} = \bigcup_{\alpha<\omega_1^V} W_\alpha.$$

As before we get that M_z is a model of ZFC from the background hypothesis that there is no $\mathbf{\Sigma}^1_{2n+1}$-definable ω_1-sequence of pairwise distinct reals as in the following claim.

Claim 3. $M_z \vDash \mathrm{ZFC}$.

Proof. This claim follows with the same argument as the one we gave for Claim 3 in the proof of Theorem 3.4.1. In this case the background hypothesis that there is no $\mathbf{\Sigma}^1_{2n+1}$-definable ω_1-sequence of pairwise distinct reals suffices, because we closed the model M_z at the even successor levels of the construction under the function F which uniformizes the universal set U and is Π^1_{2n-1}-definable. Moreover properties $(i) - (iv)$ at the odd successor levels are again Π^1_{2n}-definable uniformly in any code for the countable a-premouse $\mathcal{M}$. □

Now we have the following claim.

Claim 4. *The resulting model M_z has the following properties.*

(1) $M_z \cap \mathrm{Ord} = \omega_1^V$, $z \in M_z$,
(2) $M_z \prec_{\mathbf{\Sigma}^1_{2n}} V$,
(3) M_z *is closed under the operation*

$$a \mapsto M^\#_{2n-2}(a),$$

and moreover $M^\#_{2n-2}(a)$ *is* ω_1*-iterable in* M_z *for all* $a \in M_z$.

PROOF. Property (1) immediately follows from the construction of M_z. Furthermore property (2) follows from the even successor levels of the construction exactly as it does in the proof of Claim 4 in the proof of Theorem 3.4.1. Finally property (3) follows from the proof of Claim 4 in the proof of Theorem 3.4.1, because $M_z \prec_{\mathbf{\Sigma}^1_{2n}} V$ is sufficient for the proof of property (3) given there. □

Now we can show exactly as in Claim 5 of the proof of Theorem 3.4.1 that a K^c-construction (in the sense of [MSch04]) inside the model M_z reaches the premouse $M_{2n-1}^{\#}$, meaning that the premouse $(K^c)^{M_z}$ is not $(2n-1)$-small. This uses the fact that, by our choice of the real z, there exists a $(2n-1)$-suitable premouse in M_z (which is coded by z). Thus we omit the proof here and have the following claim.

CLAIM 5. *$M_z \vDash$ "$M_{2n-1}^{\#}$ exists and is ω_1-iterable".*

Work inside the model M_z and let $x \in M_z$ be a real which codes x_0, the theory T and the premouse $(M_{2n-1}^{\#})^{M_z}$. Let

$$N^* = (M_{2n-1}^{\#} | (\delta_0^+)^{M_{2n-1}^{\#}})^{M_z}$$

denote the *suitable initial segment* of $(M_{2n-1}^{\#})^{M_z}$, where δ_0 denotes the least Woodin cardinal in $(M_{2n-1}^{\#})^{M_z}$. Then we have in particular that N^* is a $(2n-1)$-suitable premouse (in M_z).

Recall that by Lemma 3.1.9 we have that N^* is a $(2n-1)$-suitable premouse iff it is pre-$(2n-1)$-suitable, that means iff it satisfies properties $(1)-(4)$ listed in the proof of Claim 1. Therefore the statement "N^* is a $(2n-1)$-suitable premouse" is Π^1_{2n}-definable uniformly in any code for N^*. Hence by $\mathbf{\Sigma}^1_{2n}$-correctness of M_z in V it follows that N^* is also a $(2n-1)$-suitable premouse in V.

Recall that N_x is the common iterate of all $(2n-1)$-suitable premice N which are coded by a real y_N recursive in x. Since M_z is $\mathbf{\Sigma}^1_{2n}$-correct in V it follows that the premouse N_x is the same computed in M_z or in V. This yields by correctness of M_z in V again that the premouse $M_{2n-2}^{\#}(N_x|\delta_{N_x})$ is the same computed in M_z or in V by the following argument. The premouse $(M_{2n-2}^{\#}(N_x|\delta_{N_x}))^{M_z}$ is Π^1_{2n}-iterable above δ_{N_x} in M_z and by $\mathbf{\Sigma}^1_{2n}$-correctness of M_z also in V. Therefore we can successfully coiterate the premice $(M_{2n-2}^{\#}(N_x|\delta_{N_x}))^{M_z}$ and $(M_{2n-2}^{\#}(N_x|\delta_{N_x}))^{V}$ inside V by Lemma 2.2.9 since the latter premouse is ω_1-iterable in V above δ_{N_x} and the comparison takes place above δ_{N_x}. It follows that in fact

$$(M_{2n-2}^{\#}(N_x|\delta_{N_x}))^{M_z} = (M_{2n-2}^{\#}(N_x|\delta_{N_x}))^{V}.$$

As mentioned before, we have that $N^* = (M_{2n-1}^{\#} | (\delta_0^+)^{M_{2n-1}^{\#}})^{M_z}$ is a $(2n-1)$-suitable premouse in M_z and in V and since x codes the premouse $(M_{2n-1}^{\#})^{M_z}$, the premouse N^* is coded by a real recursive in x.

Consider the comparison of N^* with $N_x = (N_x)^{M_z}$ inside an inner model of M_z of height $\omega_1^{M_z}$ which is closed under the operation $a \mapsto M^\#_{2n-2}(a)$. The premouse $(M^\#_{2n-1})^{M_z}$ is ω_1-iterable in M_z and therefore it follows that N^* is ω_1-iterable in M_z. Thus arguments as in the proof of Lemma 2.2.8 yield that N_x is in fact a non-dropping iterate of N^*, because N^* is one of the models giving rise to N_x.

The same argument shows that N_x does not move in the comparison with $(M^\#_{2n-1})^{M_z}$. So in fact there is a non-dropping iterate M of $(M^\#_{2n-1})^{M_z}$ below $(\delta_0^+)^{(M^\#_{2n-1})^{M_z}}$ such that $N_x \trianglelefteq M$. Since the iteration from $(M^\#_{2n-1})^{M_z}$ to M is fullness preserving in the sense of Definition 3.1.8 and it takes place below $(\delta_0^+)^{(M^\#_{2n-1})^{M_z}}$ in M_z, we have that in fact

$$M = M^\#_{2n-2}(N_x|\delta_{N_x}),$$

because $(M^\#_{2n-1})^{M_z} = M^\#_{2n-2}(N^*|\delta_0)$. Therefore we have that $M^\#_{2n-2}(N_x|\delta_{N_x})$ is a non-dropping iterate of $M^\#_{2n-2}(N^*|\delta_0)$ and hence

$$Th^{M_{2n-2}(N_x|\delta_{N_x})} = Th^{M_{2n-2}(N^*|\delta_0)}.$$

Recall that by Claim 2 we picked the real x_0 such that $Th^{M_{2n-2}(N_x|\delta_{N_x})}$ and thus $Th^{M_{2n-2}(N^*|\delta_0)}$ is constant for all $x \geq_T x_0$. This now implies that the theory of $(M^\#_{2n-1})^{M_z}$ is constant for all $z \geq_T x_0 \oplus x_T$, where x_T is as above a real coding the theory T.

Thus if we now work in V and let $N = (M^\#_{2n-1})^{M_z}$ for $z \geq_T x_0 \oplus x_T$, then we have

$$(M^\#_{2n-1})^{M_z} = N = (M^\#_{2n-1})^{M_y}$$

for all $y \geq_T x_0 \oplus x_T$. We aim to show that

$$(M^\#_{2n-1})^V = N,$$

so in particular that $(M^\#_{2n-1})^V$ exists.

For this reason we inductively show that the premouse N is ω_1-iterable in V via the $\mathcal{Q}$-structure iteration strategy Σ (see Definition 2.2.2). So assume that $\mathcal{T}$ is an iteration tree via Σ of limit length $< \omega_1$ on N (in V). So we have that the branch b through the iteration tree $\mathcal{T} \upharpoonright \lambda$ is given by $\mathcal{Q}$-structures, i.e. $\mathcal{Q}(b, \mathcal{T} \upharpoonright \lambda) = \mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$, for every limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$.

Pick $z \in {}^\omega\omega$ with $z \geq_T x_0 \oplus x_T$ such that $\mathcal{T} \in M_z$ and $\mathrm{lh}(\mathcal{T}) < \omega_1^{M_z}$. Since $\mathcal{T}$ is an iteration tree on $N = (M^\#_{2n-1})^{M_z}$ according to the iteration strategy Σ in V, we have that for all limit ordinals $\lambda < \mathrm{lh}(\mathcal{T})$ the $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ exists in V and is $(2n-1)$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$. In fact $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ is not more complicated than the least active premouse which is not $(2n-2)$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$. So in this case we have that Π^1_{2n}-iterability for these $\mathcal{Q}$-structures is enough to determine a unique cofinal well-founded branch b through $\mathcal{T}$. Since M_z is $\boldsymbol{\Sigma}^1_{2n}$-correct in V it follows that $\mathcal{T}$ is also according to the $\mathcal{Q}$-structure iteration strategy Σ inside M_z. Moreover recall that

$$(M_{2n-1}^{\#})^{M_z} = N.$$

Therefore there exists a cofinal well-founded branch b through $\mathcal{T}$ in M_z. As above this branch is determined by $\mathcal{Q}$-structures $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ which are ω_1-iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$ and therefore Π^1_{2n}-iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$ in M_z for all limit ordinals $\lambda \leq \mathrm{lh}(\mathcal{T})$. That means in particular that we have $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$. Moreover $\mathcal{Q}(\mathcal{T})$ is also Π^1_{2n}-iterable in V and therefore it follows that b is the unique cofinal well-founded branch determined by these $\mathcal{Q}$-structures in V as well. So N is ω_1-iterable in V via the $\mathcal{Q}$-structure iteration strategy Σ.

Thus we now finally have that

$$V \vDash \text{“}M_{2n-1}^{\#} \text{ exists and is } \omega_1\text{-iterable”}.$$

This finishes the proof of Theorem 3.3.2 for odd $n < \omega$. □

3.7. $M_{2n}^{\#}(x)$ from Boldface Π^1_{2n+1} Determinacy

In this section we will finish the proof of Theorem 2.1.1 by proving Theorem 3.7.1 which will yield Theorem 3.3.2 and finally Theorem 2.1.1 for even levels n in the projective hierarchy (using Lemma 3.5.1). Therefore together with the previous section we will have Theorem 2.1.1 for arbitrary levels n.

THEOREM 3.7.1. *Let $n \geq 1$ and assume that there is no $\boldsymbol{\Sigma}^1_{2n+2}$-definable ω_1-sequence of pairwise distinct reals. Moreover assume that $\boldsymbol{\Pi}^1_{2n}$ determinacy and Π^1_{2n+1} determinacy hold. Then $M_{2n}^{\#}$ exists and is ω_1-iterable.*

Recall that Lemma 3.5.1 gives that $\boldsymbol{\Pi}^1_{2n+1}$ determinacy suffices to prove that every $\boldsymbol{\Sigma}^1_{2n+2}$-definable sequence of pairwise distinct reals is countable.

In order to prove Theorem 3.7.1 we are considering slightly different premice than before. The main advantage of these models is that they only contain partial extenders on their extender sequence and therefore behave nicer in some arguments to follow. The premice we want to consider are defined as follows.

DEFINITION 3.7.2. *Let A be an arbitrary countable transitive swo'd* [3] *set. With $Lp^n(A)$ we denote the unique model of height ω_1^V called* lower part model above A *which is given by the following recursive definition. We start with $N_0 = A$. Assume that we already constructed N_α. Then we let $N_{\alpha+1}$ be the model theoretic union of all countable A-premice $M \trianglerighteq N_\alpha$ such that*

(1) *M is n-small above $N_\alpha \cap \mathrm{Ord}$,*
(2) *$\rho_\omega(M) \leq N_\alpha \cap \mathrm{Ord}$,*
(3) *M is sound above $N_\alpha \cap \mathrm{Ord}$,*

[3] i.e. self-wellordered as for example defined in Definition 3.1 in [SchT].

(4) $N_\alpha \cap \mathrm{Ord}$ *is a cutpoint of* M*, and*
(5) M *is* ω_1*-iterable above* $N_\alpha \cap \mathrm{Ord}$.

For the limit step let λ *be a limit ordinal and assume that we already defined* N_α *for all* $\alpha < \lambda$*. Then we let* N_λ *be the model theoretic union of all* A*-premice* N_α *for* $\alpha < \lambda$.
We finally let $Lp^n(A) = N_{\omega_1^V}$.

A special case of this is $Lp^n(x)$ for $x \in {}^\omega\omega$.

Remark. We have that every lower part model as defined above does not contain total extenders.

We will state the following lemmas for lower part models constructed above reals x instead of swo'd sets A as this will be our main application. But they also hold (with the same proofs) if we replace x with a countable transitive swo'd set A as in Definition 3.7.2.

Lemma 3.7.3. *Let* $n \geq 1$ *and assume that* $\mathbf{\Pi}^1_{2n}$ *determinacy holds. Moreover let* $x \in {}^\omega\omega$ *be arbitrary. Then the lower part model above* x*,* $Lp^{2n-1}(x)$*, is well-defined, that means* $Lp^{2n-1}(x)$ *is an* x*-premouse.*

Proof. Recall that we assume inductively that Theorem 3.6.1 holds. That means $\mathbf{\Pi}^1_{2n}$ determinacy implies that $M^\#_{2n-1}(x)$ exists for all $x \in {}^\omega\omega$. Therefore we have by Lemma 2.2.8 that whenever M and M' are two x-premice extending some x-premouse N_α (as in Definition 3.7.2) and satisfying properties (1)-(5) in Definition 3.7.2 for some $x \in {}^\omega\omega$, then we have that in fact $M \trianglelefteq M'$ or $M' \trianglelefteq M$. Therefore $Lp^{2n-1}(x)$ is a well-defined x-premouse. $\square$

Lemma 3.7.4. *Let* $n \geq 1$ *and assume that* $\mathbf{\Pi}^1_{2n}$ *determinacy holds. Moreover let* $x \in {}^\omega\omega$ *be arbitrary. Let* M *denote the* ω_1^V*-th iterate of* $M^\#_{2n-1}(x)$ *by its least measure and its images. Then*

$$M|\omega_1^V = Lp^{2n-1}(x).$$

Proof. Let $x \in {}^\omega\omega$ and let $(N_\alpha \mid \alpha \leq \omega_1^V)$ be the sequence of models from the definition of $Lp^{2n-1}(x)$ (see Definition 3.7.2). We aim to show inductively that

$$N_\alpha \trianglelefteq M$$

for all $\alpha < \omega_1^V$, where M denotes the ω_1^V-th iterate of $M^\#_{2n-1}(x)$ by its least measure and its images as in the statement of Lemma 3.7.4. Fix an $\alpha < \omega_1^V$ and assume inductively that

$$N_\beta \trianglelefteq M$$

for all $\beta \leq \alpha$. Let z be a real which codes the countable premice $N_{\alpha+1}$ and $M^\#_{2n-1}(x)$.

Since $M^\#_{2n-1}(x)$ is ω_1-iterable in V and has no definable Woodin cardinals, we have by Lemma 2.2.8 (2) that it is $(\omega_1 + 1)$-iterable inside the model

$M_{2n-1}(z)$. Therefore we have in particular that M is (ω_1+1)-iterable inside $M_{2n-1}(z)$.

Recall that $N_{\alpha+1}$ is by definition the model theoretic union of all countable x-premice $N \trianglerighteq N_\alpha$ such that

(1) N is $(2n-1)$-small above $N_\alpha \cap \mathrm{Ord}$,
(2) $\rho_\omega(N) \leq N_\alpha \cap \mathrm{Ord}$,
(3) N is sound above $N_\alpha \cap \mathrm{Ord}$,
(4) $N_\alpha \cap \mathrm{Ord}$ is a cutpoint of N, and
(5) N is ω_1-iterable above $N_\alpha \cap \mathrm{Ord}$.

In particular Lemma 2.2.8 (2) implies that all these x-premice $N \trianglerighteq N_\alpha$ satisfying properties $(1)-(5)$ are (ω_1+1)-iterable above $N_\alpha \cap \mathrm{Ord}$ inside the model $M_{2n-1}(z)$ since they have no definable Woodin cardinals above $N_\alpha \cap \mathrm{Ord}$. In particular $N_{\alpha+1}$ is well-defined as in Lemma 3.7.3 and it follows that $N_{\alpha+1}$ is (ω_1+1)-iterable above $N_\alpha \cap \mathrm{Ord}$ inside $M_{2n-1}(z)$.

Hence we can consider the comparison of the x-premice M and $N_{\alpha+1}$ inside the model $M_{2n-1}(z)$. This comparison is successful because by our inductive hypothesis it takes place above $N_\alpha \cap \mathrm{Ord}$. Now we distinguish the following cases.

Case 1. Both sides of the comparison move.

In this case both sides of the comparison have to drop since we have that $\rho_\omega(N) \leq N_\alpha \cap \mathrm{Ord}$ for all x-premice N occurring in the definition of $N_{\alpha+1}$, $N_{\alpha+1} \cap \mathrm{Ord} < \omega_1^V$ and M only has partial extenders on its sequence below ω_1^V. Let $\mathcal{T}$ and $\mathcal{U}$ be iteration trees of length $\lambda+1$ for some ordinal λ on M and $N_{\alpha+1}$ respectively resulting from the comparison. Moreover let E_β and F_γ be the first extenders used in the coiteration after the last drop along $[0,\lambda]_T$ and $[0,\lambda]_U$ respectively. Then by the proof of the Comparison Lemma (see Theorem 3.11 in [St10]) the extenders E_β and F_γ are compatible. Again by the proof of the Comparison Lemma this is a contradiction, so only one side of the coiteration can move.

Case 2. Only the M-side of the comparison moves.

As above we have in this case that the M-side drops. So there is an iterate M^* of M such that $N_{\alpha+1}$ is an initial segment of M^*. Let E_β for some ordinal β be the first extender used on the M-side in the coiteration of M with $N_{\alpha+1}$. In particular E_β is an extender indexed on the M-sequence above $N_\alpha \cap \mathrm{Ord}$ and below ω_1^V. Since E_β has to be a partial extender, there exists a $(2n-1)$-small sound countable x-premouse $N \triangleleft M$ such that E_β is a total extender on the N-sequence and $\rho_m(N) \leq crit(E_\beta)$ for some natural number m. Moreover we have that N is ω_1-iterable by the iterability of M. Furthermore we have that $crit(E_\beta)$ is a cardinal in $N_{\alpha+1}$, because the extender E_β is used in the coiteration and we have that

$$N_{\alpha+1} \trianglelefteq M^*.$$

Therefore the premouse N is contained in $N_{\alpha+1}$ by the definition of a lower part model. But this contradicts the assumption that E_β was used in the coiteration, because then we have that there is no disagreement between M and $N_{\alpha+1}$ at E_β.

Case 3. Only the $N_{\alpha+1}$-side of the comparison moves.

In this case there is an iterate N^* of $N_{\alpha+1}$ above $N_\alpha \cap \mathrm{Ord}$ such that the iteration from $N_{\alpha+1}$ to N^* drops and we have that

$$M \trianglelefteq N^*.$$

Recall that M denotes the ω_1^V-th iterate of $M_{2n-1}^\#(x)$ by its least measure and its images and is therefore in particular not $(2n-1)$-small above ω_1^V. But then the same holds for N^* and thus it follows that $N_{\alpha+1}$ is not $(2n-1)$-small above $N_\alpha \cap \mathrm{Ord}$, which is a contradiction, because $N_{\alpha+1}$ is the model theoretic union of premice which are $(2n-1)$-small above $N_\alpha \cap \mathrm{Ord}$.

This proves that

$$N_{\alpha+1} \trianglelefteq M$$

for all $\alpha < \omega_1^V$ and since $Lp^{2n-1}(x) \cap \mathrm{Ord} = N_{\omega_1^V} \cap \mathrm{Ord} = \omega_1^V$ we finally have that

$$Lp^{2n-1}(x) = M|\omega_1^V.$$

$\square$

Remark. Lemma 3.7.4 also implies that the lower part model $Lp^{2n-1}(x)$ is closed under the operation $a \mapsto M_{2n-2}^\#(a)$ by the proof of Lemma 2.2.8 (1) as this holds for $M|\omega_1^V$, where M again denotes the ω_1^V-th iterate of $M_{2n-1}^\#(x)$ by its least measure and its images.

Using this representation of lower part models we can also prove the following lemma.

Lemma 3.7.5. *Let $n \geq 1$ and assume that $\boldsymbol{\Pi}^1_{2n}$ determinacy holds. Let $x, y \in {}^\omega\omega$ be such that $x \in Lp^{2n-1}(y)$. Then we have that*

$$Lp^{2n-1}(x) \subseteq Lp^{2n-1}(y).$$

If we moreover have that $y \leq_T x$, then

$$Lp^{2n-1}(x) = (Lp^{2n-1}(x))^{Lp^{2n-1}(y)}.$$

Here $(Lp^{2n-1}(x))^{Lp^{2n-1}(y)}$ denotes the model of height ω_1^V which is constructed analogous to Definition 3.7.2, but with models M which are ω_1-iterable above $N_\alpha \cap \mathrm{Ord}$ inside $Lp^{2n-1}(y)$ instead of inside V.

Proof. Let $x, y \in {}^\omega\omega$ be such that $x \in Lp^{2n-1}(y)$. We first prove that

$$Lp^{2n-1}(x) \subseteq Lp^{2n-1}(y).$$

By Lemma 3.7.4 we have that $Lp^{2n-1}(x) = M(x)|\omega_1^V$, where $M(x)$ denotes the ω_1^V-th iterate of $M_{2n-1}^\#(x)$ by its least measure and its images. Moreover we have that $Lp^{2n-1}(y) = M(y)|\omega_1^V$, where $M(y)$ denotes the ω_1^V-th iterate of $M^{\#}_{2n-1}(y)$ by its least measure and its images. Let $M^*(y)$ denote the result of iterating the top measure of $M(y)$ out of the universe.

Consider the result of a fully backgrounded extender construction (in the sense of [MS94] but with the smallness hypothesis weakened) above x inside the model $M^*(y)$, which we denote by

$$L[E](x)^{M^*(y)}.$$

Moreover let $M_x^{\#}$ denote the model obtained from $L[E](x)^{M^*(y)}$ by adding the top measure (intersected with $L[E](x)^{M^*(y)}$) of the active premouse $M(y)$ to an initial segment of $L[E](x)^{M^*(y)}$ as in Section 2 of [FNS10].

We can successfully compare the active x-premice $M_x^{\#}$ and $M(x)$ inside the model $M_{2n-1}(z)$, where z is a real coding $M^{\#}_{2n-1}(y)$ and $M^{\#}_{2n-1}(x)$, by an argument using Lemma 2.2.8 as in the proof of Lemma 2.2.15. Therefore it follows that $M_x^{\#} = M(x)$ and thus we have that in fact

$$L[E](x)^{M^*(y)}|\omega_1^V = M_x^{\#}|\omega_1^V = M(x)|\omega_1^V.$$

Since $L[E](x)^{M^*(y)} \subseteq M^*(y)$ it follows that

$$Lp^{2n-1}(x) = M(x)|\omega_1^V = L[E](x)^{M^*(y)}|\omega_1^V \subseteq M^*(y)|\omega_1^V = Lp^{2n-1}(y).$$

Now we prove the "moreover" part of Lemma 3.7.5, so assume that we in addition have that $y \leq_T x$. Let $(N_\alpha \mid \alpha \leq \omega_1^V)$ be the sequence of models from the definition of $Lp^{2n-1}(x)$ in V (see Definition 3.7.2) and let $(N_\alpha^{Lp^{2n-1}(y)} \mid \alpha \leq \omega_1^V)$ denote the corresponding sequence of models from the definition of $(Lp^{2n-1}(x))^{Lp^{2n-1}(y)}$. Let $\alpha < \omega_1^V$ be such that $N_\alpha = N_\alpha^{Lp^{2n-1}(y)}$ and let $N \trianglerighteq N_\alpha$ be an x-premouse which satisfies properties $(1) - (5)$ in Definition 3.7.2 in V. So in particular we have that $N \in Lp^{2n-1}(x)$ and N is $(2n-1)$-small above $N_\alpha \cap \mathrm{Ord}$. As argued above we have that $N \in Lp^{2n-1}(y)$ and we first want to show that as N is ω_1-iterable above $N_\alpha \cap \mathrm{Ord}$ in V it follows that N is ω_1-iterable above $N_\alpha \cap \mathrm{Ord}$ inside $Lp^{2n-1}(y)$.

So assume that N is ω_1-iterable above $N_\alpha \cap \mathrm{Ord}$ in V and recall that the model $Lp^{2n-1}(y)$ is closed under the operation $a \mapsto M^{\#}_{2n-2}(a)$. Since N is $(2n-1)$-small above $N_\alpha \cap \mathrm{Ord}$ and has no definable Woodin cardinal above $N_\alpha \cap \mathrm{Ord}$, we have that for an iteration tree $\mathcal{T}$ on N of length $< \omega_1$ in $Lp^{2n-1}(y)$ above $N_\alpha \cap \mathrm{Ord}$ the iteration strategy Σ is guided by $\mathcal{Q}$-structures which are $(2n-2)$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$ for every limit ordinal $\lambda \leq \mathrm{lh}(\mathcal{T})$. Therefore the $\mathcal{Q}$-structures for $\mathcal{T}$ are contained in the model $Lp^{2n-1}(y)$ and we have that N is ω_1-iterable inside $Lp^{2n-1}(y)$ above $N_\alpha \cap \mathrm{Ord}$ if we argue analogous to the proof of Lemma 2.2.8 (2).

Assume now toward a contradiction that $Lp^{2n-1}(x) \neq (Lp^{2n-1}(x))^{Lp^{2n-1}(y)}$. That means there is an ordinal $\alpha < \omega_1^V$ such that $N_\alpha = N_\alpha^{Lp^{2n-1}(y)}$ and there exists a premouse $N \triangleright N_\alpha$ which satifies properties $(1) - (5)$ in the definition

of $(Lp^{2n-1}(x))^{Lp^{2n-1}(y)}$, so in particular N is ω_1-iterable above $N_\alpha \cap \mathrm{Ord}$ inside $Lp^{2n-1}(y)$, but N is not ω_1-iterable above $N_\alpha \cap \mathrm{Ord}$ in V.
Recall the model $M^*(y)$ from the first part of this proof. We have that $Lp^{2n-1}(x) = L[E](x)^{M^*(y)}|\omega_1^V$ and N are both sufficiently iterable in $M^*(y)$, so we can consider the coiteration of $Lp^{2n-1}(x)$ and N inside $M^*(y)$ and distinguish the following cases.

Case 1. Both sides of the comparison move.

In this case both sides of the comparison have to drop since we have that $\rho_\omega(N) \leq N_\alpha \cap \mathrm{Ord}$ and $Lp^{2n-1}(x)$ only has partial extenders on its sequence. As in Case 1 in the proof of Lemma 3.7.4 this yields a contradiction.

Case 2. Only the $Lp^{2n-1}(x)$-side of the comparison moves.

Then the $Lp^{2n-1}(x)$-side drops and we have that there is an iterate M of $Lp^{2n-1}(x)$ such that $N \trianglelefteq M$. But this would imply that N is ω_1-iterable in V, contradicting our choice of N.

Case 3. Only the N-side of the comparison moves.

In this case there exists an iterate N^* of N such that $Lp^{2n-1}(x) \trianglelefteq N^*$. In fact the iteration from N to N^* only uses measures of Mitchell order 0 as the iteration cannot leave any total measures behind. Since there are only finitely many drops along the main branch of an iteration tree, this implies that the whole iteration from N to N^* can be defined over the model N. As $N \triangleleft (Lp^{2n-1}(x))^{Lp^{2n-1}(y)}$ we therefore have that

$$Lp^{2n-1}(x) \subsetneq (Lp^{2n-1}(x))^{Lp^{2n-1}(y)} \subseteq Lp^{2n-1}(y).$$

This contradicts the following claim.

Claim 1. *For reals x, y such that $x \in Lp^{2n-1}(y)$ and $y \leq_T x$ we have that*

$$Lp^{2n-1}(x) = Lp^{2n-1}(y).$$

Proof. As $x \in Lp^{2n-1}(y)$ we have that $x \in M(y)$ and thus $x \in M^\#_{2n-1}(y)$, because $M(y)$ is obtained from $M^\#_{2n-1}(y)$ by iterating its least measure and its images.
As in the proof of Lemma 2.2.15 we can consider the premice $L[E](x)^{M_{2n-1}(y)}$ and $L[E](y)^{L[E](x)^{M_{2n-1}(y)}}$ and if we let κ denote the least measurable cardinal in $M_{2n-1}(y)$, then we get as in the proof of Lemma 2.2.15 that

$$V_\kappa^{M_{2n-1}(y)} = V_\kappa^{L[E](x)^{M_{2n-1}(y)}}.$$

Moreover if we let $(L[E](x)^{M_{2n-1}(y)})^\#$ denote the premouse obtained by adding the restriction of the top extender of $M^\#_{2n-1}(y)$ to an initial segment of the premouse $L[E](x)^{M_{2n-1}(y)}$ as in [FNS10], then another comparison argument analogous to Lemma 2.2.15 yields that $M^{\#}_{2n-1}(x) = (L[E](x)^{M_{2n-1}(y)})^{\#}$. Therefore we have that

$$V_\kappa^{M_{2n-1}(y)} = V_\kappa^{L[E](x)^{M_{2n-1}(y)}} = V_\kappa^{M_{2n-1}(x)}.$$

This implies that $Lp^{2n-1}(x) = Lp^{2n-1}(y)$, as desired. □

□

We also have a version of Lemma 2.1.3 for lower part models $Lp^{2n-1}(x)$ as follows.

LEMMA 3.7.6. *Let $n \geq 1$ and assume that $\mathbf{\Pi}^1_{2n}$ determinacy and Π^1_{2n+1} determinacy hold. Then there exists a real x such that we have for all reals $y \geq_T x$ that*

$$Lp^{2n-1}(y) \vDash \text{OD-}\textit{determinacy.}$$

PROOF. Recall that we assume inductively that $\mathbf{\Pi}^1_{2n}$ determinacy implies that $M^{\#}_{2n-1}(x)$ exists for all $x \in {}^\omega\omega$. Then Lemma 3.7.6 follows from Lemma 2.1.3 by the following argument. For $x \in {}^\omega\omega$ we have that

$$M_{2n-1}(x)|\delta_x \vDash \text{OD-determinacy}$$

implies that

$$M_{2n-1}(x)|\kappa \vDash \text{OD-determinacy},$$

where κ is the least measurable cardinal in $M_{2n-1}(x)$, because whenever a set of reals A is ordinal definable in $M_{2n-1}(x)|\kappa$, then it is also ordinal definable in $M_{2n-1}(x)|\delta_x$, since $M_{2n-1}(x)|\kappa$ and $M_{2n-1}(x)|\delta_x$ have the same sets of reals. This yields by elementarity that

$$M|\omega_1^V \vDash \text{OD-determinacy},$$

where M denotes the ω_1^V-th iterate of $M_{2n-1}(x)$ by its least measure and its images. By Lemma 3.7.4 we have that $M|\omega_1^V = Lp^{2n-1}(x)$, so it follows that

$$Lp^{2n-1}(x) \vDash \text{OD-determinacy}.$$

□

This immediately yields that we have the following variant of Theorem 2.4.3.

THEOREM 3.7.7. *Let $n \geq 1$ and assume that $\mathbf{\Pi}^1_{2n}$ determinacy and Π^1_{2n+1} determinacy hold. Then there exists a real x such that we have for all reals $y \geq_T x$ that*

$$\text{HOD}^{Lp^{2n-1}(y)} \vDash \text{“}\omega_2^{Lp^{2n-1}(y)} \textit{ is inaccessible”.}$$

Now we can turn to the proof of Theorem 3.7.1, which is going to yield Theorem 3.3.2 in the case that n is even.

Proof of Theorem 3.7.1. We start with constructing a ZFC model M_x of height ω_1^V for some $x \in {}^\omega\omega$ such that M_x is $\boldsymbol{\Sigma}^1_{2n+2}$-correct in V for real parameters in M_x and closed under the operation $a \mapsto M^\#_{2n-1}(a)$. To prove that this construction yields a model of ZFC, we are as before going to use the hypothesis that there is no $\boldsymbol{\Sigma}^1_{2n+2}$-definable ω_1-sequence of pairwise distinct reals. The construction will be similar to the constructions we gave in Sections 3.4 and 3.6.

Fix an arbitrary $x \in {}^\omega\omega$ and construct a sequence of models $(W_\alpha \mid \alpha \leq \omega_1^V)$. So the model $M_x = W_{\omega_1^V}$ is build level-by-level in a construction of length ω_1^V. We are starting from $W_0 = \{x\}$ and are taking unions at limit steps of the construction. At an odd successor level $\alpha+1$ we will close the model W_α under Skolem functions for $\boldsymbol{\Sigma}^1_{2n+2}$-formulas. At the same time we will use the even successor levels $\alpha + 2$ to ensure that M_x will be closed under the operation $a \mapsto M^\#_{2n-1}(a)$. As before the order of construction for elements of the model M_x can be defined along the way analogous to the construction in the proof of Theorem 3.4.1, so we omit the details here.

Before we are describing this construction in more detail, we fix a Π^1_{2n+1}-definable set U which is universal for the pointclass $\boldsymbol{\Pi}^1_{2n+1}$. Pick the set U such that we have $U_{\ulcorner\varphi\urcorner\frown a} = A_{\varphi,a}$ for every Π^1_{2n+1}-formula φ and every $a \in {}^\omega\omega$, where $\ulcorner\varphi\urcorner$ denotes the Gödel number of the formula φ and

$$A_{\varphi,a} = \{x \mid \varphi(x, a)\}.$$

Then the uniformization property (see Theorem 6C.5 in [Mo09] or Lemma 3.4.2 (1) in this thesis) yields that there exists a Π^1_{2n+1}-definable function F uniformizing the universal set U. So we have for all $z \in \mathrm{dom}(F)$ that

$$(z, F(z)) \in U,$$

where $\mathrm{dom}(F) = \{z \mid \exists y\, (z, y) \in U\}$.

Odd successor steps: For the odd successor steps of the construction assume now that we already constructed the model W_α such that $\alpha + 1$ is odd and that there exists a $\boldsymbol{\Pi}^1_{2n+1}$-formula φ with a real parameter a from W_α such that $\exists x \varphi(x, a)$ holds in V but not in the model W_α. In this case we aim to add a real $x_{\varphi,a}$ constructed as described below to $W_{\alpha+1}$ such that $\varphi(x_{\varphi,a}, a)$ holds, analogous to the proof of Theorem 3.4.1. This real will witness that $\exists x \varphi(x, a)$ holds true inside $W_{\alpha+1}$.

We aim to build these levels of M_x in a Σ^1_{2n+2}-definable way, so we choose reals $x_{\varphi,a}$ carefully. Therefore we add $F(z)$ for all $z \in \mathrm{dom}(F) \cap W_\alpha$ to the current model W_α. We will see in Claims 1 and 2 that this procedure adds reals $x_{\varphi,a}$ as above in a Π^1_{2n+1}-definable way to the model M_x.

So let φ_F be a Π^1_{2n+1}-formula such that for all $x, y \in {}^\omega\omega$,

$$F(x) = y \text{ iff } \varphi_F(x, y).$$

Then we let

$$W_{\alpha+1} = rud(W_\alpha \cup \{y \in {}^\omega\omega \mid \exists x \in W_\alpha \cap {}^\omega\omega\ \varphi_F(x, y)\}).$$

Even successor steps: At an even successor level $\alpha+2$ of the construction we close the previous model $W_{\alpha+1}$ under the operation $a \mapsto M_{2n-1}^{\#}(a)$ before closing under rudimentary functions. Assume that we already constructed $W_{\alpha+1}$ and let $a \in W_{\alpha+1}$ be arbitrary. The a-premouse $M_{2n-1}^{\#}(a)$ exists in V because we as usual inductively assume that Theorem 3.3.1 holds for $2n-1$. Analogous to the proof of Theorem 3.6.1 let $\mathcal{M}$ be a countable a-premouse in V with the following properties.

(i) $\mathcal{M}$ is $2n$-small, but not $(2n-1)$-small,
(ii) all proper initial segments of $\mathcal{M}$ are $(2n-1)$-small,
(iii) $\mathcal{M}$ is a-sound and $\rho_\omega(\mathcal{M}) = a$, and
(iv) $\mathcal{M}$ is Π^1_{2n+1}-iterable.

We have that these properties $(i)-(iv)$ uniquely determine the a-premouse $M_{2n-1}^{\#}(a)$ in V.
We add such a-premice $\mathcal{M}$ for all $a \in W_{\alpha+1}$ to $W_{\alpha+2}$ before closing under rudimentary functions as in the usual construction of L.
Limit steps: Finally we let

$$W_\lambda = \bigcup_{\alpha<\lambda} W_\alpha$$

for limit ordinals $\lambda < \omega_1^V$ and

$$M_x = W_{\omega_1^V} = \bigcup_{\alpha<\omega_1^V} W_\alpha.$$

As in Sections 3.4 and 3.6 we are now able to show that this model M_x satisfies ZFC, using the background hypothesis that every $\boldsymbol{\Sigma}^1_{2n+2}$-definable sequence of pairwise distinct reals is countable. As the proof is analogous to the proof of Claim 3 in the proof of Theorem 3.4.1 we omit it here.

Claim 1. $M_x \vDash \mathrm{ZFC}$.

Moreover we can prove the following claim. The proof is similar to the proof of Claim 4 in the proof of Theorem 3.4.1 so we also omit it here.

Claim 2. *The model M_x as constructed above has the following properties.*

(1) $M_x \cap \mathrm{Ord} = \omega_1^V$,
(2) $x \in M_x$,
(3) *M_x is $\boldsymbol{\Sigma}^1_{2n+2}$-correct in V for real parameters in M_x, that means we have that*

$$M_x \prec_{\boldsymbol{\Sigma}^1_{2n+2}} V,$$

(4) *M_x is closed under the operation*

$$a \mapsto M_{2n-1}^{\#}(a),$$

and moreover $M_{2n-1}^{\#}(a)$ is ω_1-iterable in M_x for all $a \in M_x$.

The following additional property of the model M_x is a key point in proving that $M_{2n}^\#$ exists and is ω_1-iterable in V.

CLAIM 3. *For all $x \in {}^\omega\omega$ in the cone of reals given in Theorem 3.7.7,*

$$M_x \vDash \text{“}M_{2n}^\# \textit{ exists and is } \omega_1\textit{-iterable.”}$$

The proof of this claim is now different from the proof of the analogous claim in the previous section. The reason for this is that at the even levels we cannot assume that we have a $2n$-suitable premouse to compare the model K^c with (which at the odd levels was given by Lemmas 3.1.2 and 3.1.9). This is why we have to give a different argument here.

PROOF OF CLAIM 3. Assume this is not the case. Then $(K^c)^{M_x}$ is fully iterable inside M_x by a generalization of Theorem 2.11 in [St96], since M_x is closed under the operation $a \mapsto M_{2n-1}^\#(a)$. This yields that we can build the core model K^{M_x} inside M_x by a generalization of Theorem 1.1 in [JS13] due to Jensen and Steel. The core model K^{M_x} has to be $2n$-small, because otherwise we would have that

$$M_x \vDash \text{“There exists a model which is fully iterable and not } 2n\text{-small”}.$$

This would already imply that $M_{2n}^\#$ exists and is fully iterable inside M_x, so then there is nothing left to show.

SUBCLAIM 1. *K^{M_x} is closed under the operation*

$$a \mapsto M_{2n-1}^\#(a).$$

PROOF. We start with considering sets of the form $a = K^{M_x}|\xi$ where $\xi < K^{M_x} \cap \mathrm{Ord}$ is not overlapped by an extender on the K^{M_x}-sequence. That means there is no extender E on the K^{M_x}-sequence such that $\mathrm{crit}(E) \leq \xi < \mathrm{lh}(E)$. We aim to prove that in fact

$$M_{2n-1}^\#(K^{M_x}|\xi) \lhd K^{M_x}.$$

We have that the premouse $M_{2n-1}^\#(K^{M_x}|\xi)$ exists inside the model M_x since we have that $\xi < K^{M_x} \cap \mathrm{Ord} = M_x \cap \mathrm{Ord}$ and M_x is closed under the operation

$$a \mapsto M_{2n-1}^\#(a)$$

by property (4) in Claim 2. Consider the coiteration of the premice K^{M_x} and $M_{2n-1}^\#(K^{M_x}|\xi)$ inside M_x. This coiteration takes place above ξ and thus both premice are iterable enough such that the comparison is successful by Lemma 2.2.8 since $M_{2n-1}^\#(K^{M_x}|\xi)$ is ω_1-iterable above ξ in M_x. By universality of K^{M_x} inside M_x (by Lemma 3.5 in [St96] applied inside M_x), we have that there is an iterate M^* of $M_{2n-1}^\#(K^{M_x}|\xi)$ and an iterate K^* of

K^{M_x} such that $M^* \trianglelefteq K^*$ and the iteration from $M_{2n-1}^{\#}(K^{M_x}|\xi)$ to M^* is non-dropping on the main branch. Since

$$\rho_\omega(M_{2n-1}^{\#}(K^{M_x}|\xi)) \leq \xi$$

and since the coiteration takes place above ξ, we have that the iterate M^* of $M_{2n-1}^{\#}(K^{M_x}|\xi)$ is not sound, if any extender is used on this side of the coiteration. Therefore it follows that in fact

$$M_{2n-1}^{\#}(K^{M_x}|\xi) \lhd K^*.$$

Assume that the K^{M_x}-side moves in the coiteration, that means we have that $K^{M_x} \neq K^*$. Let α be an ordinal such that E_α is the first extender on the K^{M_x}-sequence which is used in the coiteration. Then we have that $\alpha > \xi$. We have in particular that α is a cardinal in K^*. But then since we have that $\rho_\omega(M_{2n-1}^{\#}(K^{M_x}|\xi)) \leq \xi < \alpha$ and $M_{2n-1}^{\#}(K^{M_x}|\xi) \lhd K^*$, this already implies that

$$\alpha > M_{2n-1}^{\#}(K^{M_x}|\xi) \cap \mathrm{Ord}\,.$$

Therefore there was no need to iterate K^{M_x} at all and we have that

$$M_{2n-1}^{\#}(K^{M_x}|\xi) \lhd K^{M_x}.$$

Now let $a \in K^{M_x}$ be arbitrary. Then there exists an ordinal $\xi < K^{M_x} \cap \mathrm{Ord}$ such that $a \in K^{M_x}|\xi$ and ξ is not overlapped by an extender on the K^{M_x}-sequence. We just proved that

$$M_{2n-1}^{\#}(K^{M_x}|\xi) \lhd K^{M_x}.$$

As we argued several times before, by performing a fully backgrounded extender construction (denoted by $L[E](a)^{M_{2n-1}(K^{M_x}|\xi)}$) above a inside the model $M_{2n-1}(K^{M_x}|\xi)$ in the sense of [MS94] (with the smallness hypothesis weakened) and adding the top extender of the active premouse $M_{2n-1}^{\#}(K^{M_x}|\xi)$ (intersected with $L[E](a)^{M_{2n-1}(K^{M_x}|\xi)}$) to an initial segment of the model $L[E](a)^{M_{2n-1}(K^{M_x}|\xi)}$ as described in Section 2 in [FNS10] we obtain that

$$M_{2n-1}^{\#}(a) \in K^{M_x},$$

as desired. □

Using the Weak Covering Lemma from [MSch95] (see also Theorem 1.1 in [JS13]) we can pick a cardinal $\gamma \in M_x$ such that γ is singular in M_x and γ^+ is computed correctly by K^{M_x} inside M_x. That means we pick γ such that we have

$$(\gamma^+)^{K^{M_x}} = (\gamma^+)^{M_x}.$$

For later purposes we want to pick γ such that it additionally satisfies

$$\mathrm{cf}(\gamma)^{M_x} \geq \omega_1^{M_x}.$$

Subclaim 2. *There exists a real $z \geq_T x$ such that*

(1) $(\gamma^+)^{M_x} = \omega_2^{Lp^{2n-1}(z)}$, *and*
(2) $K^{M_x}|(\gamma^+)^{M_x} \in Lp^{2n-1}(z)$.

Proof. We are going to produce the real z via a five step forcing using an almost disjoint coding. For an introduction into this kind of forcing see for example [FSS14] for a survey or [Sch00] where a similar argument is given.

We force over the ground model

$$Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x}).$$

We have that $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ is a definable subset of M_x because we have by property (4) in Claim 2 that

$$M_{2n-1}^{\#}(x, K^{M_x}|(\gamma^+)^{M_x}) \in M_x$$

and by Lemma 3.7.4 the lower part model $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ is obtained by iterating the least measure of $M_{2n-1}^{\#}(x, K^{M_x}|(\gamma^+)^{M_x})$ and its images ω_1^V times and cutting off at ω_1^V.

This implies that in particular

$$\mathrm{cf}(\gamma)^{Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})} \geq \omega_1^{M_x}.$$

Step 1: Write $V_0 = Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ for the ground model. We start with a preparatory forcing that collapses everything below $\omega_1^{M_x}$ to ω. Afterwards we collapse γ to $\omega_1^{M_x}$.

So let $G_0 \in V$ be $\mathrm{Col}(\omega, < \omega_1^{M_x})$-generic over V_0 and let

$$V_0' = V_0[G_0].$$

Moreover let $G_0' \in V$ be $\mathrm{Col}(\omega_1^{M_x}, \gamma)$-generic over V_0' and let

$$V_1 = V_0'[G_0'].$$

So we have that $\omega_1^{M_x} = \omega_1^{V_1}$ and by our choice of γ, i.e. $\mathrm{cf}(\gamma)^{V_0} \geq \omega_1^{M_x}$, we moreover have that $(\gamma^+)^{M_x} = (\gamma^+)^{K^{M_x}} = \omega_2^{V_1}$. We write $\omega_1 = \omega_1^{V_1}$ and $\omega_2 = \omega_2^{V_1}$.

Furthermore let A' be a set of ordinals coding G_0 and G_0', such that if we let $A \subset (\gamma^+)^{M_x}$ be a code for

$$x \oplus (K^{M_x}|(\gamma^+)^{M_x}) \oplus A',$$

then we have that $G_0, G_0' \in Lp^{2n-1}(A)$ and $K^{M_x}|(\gamma^+)^{M_x} \in Lp^{2n-1}(A)$.

We can in fact pick the set A such that we have $V_1 = Lp^{2n-1}(A)$ by the following argument: Recall that

$$Lp^{2n-1}(A) = M(A)|\omega_1^V,$$

where $M(A)$ denotes the ω_1^V-th iterate of $M_{2n-1}^{\#}(A)$ by its least measure and its images for a set A as above. Then we can consider G_0 as being generic over the model $M(x, K^{M_x}|(\gamma^+)^{M_x})$ and G_0' as being generic over the model $M(x, K^{M_x}|(\gamma^+)^{M_x})[G_0]$, where $M(x, K^{M_x}|(\gamma^+)^{M_x})$ denotes the ω_1^V-th iterate of $M_{2n-1}^{\#}(x, K^{M_x}|(\gamma^+)^{M_x})$ by its least measure and its images. Since both forcings in Step 1 take place below $(\gamma^+)^{M_x} < \omega_1^V$, it follows as in the proof of Theorem 2.5.1 that $M(x, K^{M_x}|(\gamma^+)^{M_x})[G_0][G_0'] = M(A)$ for a set $A \subset (\gamma^+)^{M_x}$ coding $x, K^{M_x}|(\gamma^+)^{M_x}, G_0$ and G_0' and thus we have that $V_1 = M(A)|\omega_1^V$ for this set A, as desired.

Step 2: Before we can perform the first coding using almost disjoint subsets of $\omega_1 = \omega_1^{V_1}$ we have to "reshape" the interval between $(\gamma^+)^{M_x} = \omega_2^{V_1}$ and ω_1 to ensure that the coding we will perform in Step 3 exists. Moreover we have to make sure that the reshaping forcing itself does not collapse ω_1 and $(\gamma^+)^{M_x}$. We are going to show this by proving that the reshaping forcing is $< (\gamma^+)^{M_x}$-distributive.

We are going to use the following notion of reshaping.

Definition 3.7.8. *Let η be a cardinal and let $X \subset \eta^+$. We say a function f is* (X, η^+)-reshaping *iff $f : \alpha \to 2$ for some $\alpha \leq \eta^+$ and moreover for all $\xi \leq \alpha$ with $\xi < \eta^+$ we have that*

(i) *$L[X \cap \xi, f \upharpoonright \xi] \vDash |\xi| \leq \eta$, or*
(ii) *there is a model N and a Σ_k-elementary embedding*

$$j : N \to Lp^{2n-1}(X)|\eta^{++}$$

for some large enough $k < \omega$ such that
(a) *$\mathrm{crit}(j) = \xi$, $j(\xi) = \eta^+$,*
(b) *$\rho_{k+1}(N) \leq \xi$, N is sound above ξ, and*
(c) *definably over N there exists a surjection $g : \eta \twoheadrightarrow \xi$.*

Now we denote with P_1 the forcing that adds an $(A, (\gamma^+)^{M_x})$-reshaping function for $(\gamma^+)^{M_x} = \omega_2^{V_1}$, defined inside our new ground model $V_1 = Lp^{2n-1}(A)$.

We let $p \in P_1$ iff p is an $(A, (\gamma^+)^{M_x})$-reshaping function with $\mathrm{dom}(p) < (\gamma^+)^{M_x}$ and we order two conditions p and q in P_1 by reverse inclusion, that means we let $p \leq_{P_1} q$ iff $q \subseteq p$.

First notice that the forcing P_1 is extendable, that means for every ordinal $\alpha < (\gamma^+)^{M_x}$ the set $D^\alpha = \{p \in P_1 \mid \mathrm{dom}(p) \geq \alpha\}$ is open and dense in P_1.

We now want to show that P_1 is $< (\gamma^+)^{M_x}$-distributive. For that we fix a condition $p \in P_1$ and open dense sets $(D_\beta \mid \beta < \omega_1)$. We aim to find a condition $q \leq_{P_1} p$ such that $q \in D_\beta$ for all $\beta < \omega_1$.

Consider, for some large enough fixed natural number k, transitive Σ_k-elementary substructures of the model $Lp^{2n-1}(A) = V_1$. More precisely we want to pick a continuous sequence

$$(N_\alpha, \pi_\alpha, \xi_\alpha \mid \alpha \leq \omega_1)$$

of transitive models N_α of size $|\omega_1^{V_1}|$ together with Σ_k-elementary embeddings

$$\pi_\alpha : N_\alpha \to Lp^{2n-1}(A)$$

and an increasing sequence of ordinals ξ_α such that we have $p \in N_0$, and for all $\alpha \leq \omega_1$

(1) $\mathrm{crit}(\pi_\alpha) = \xi_\alpha$ with $\pi_\alpha(\xi_\alpha) = (\gamma^+)^{M_x}$,
(2) for all ordinals $\alpha < \omega_1$ we have that $\rho_{k+1}(N_\alpha) \leq \xi_\alpha$ and N_α is sound above ξ_α, and
(3) $\{p\} \cup \{D_\beta \mid \beta < \omega_1\} \subset \mathrm{ran}(\pi_\alpha)$.

We can obtain N_α and π_α for all $\alpha \leq \omega_1$ with these properties inductively as follows. Let M_0 be the (uncollapsed) Σ_k-hull of

$$\gamma \cup \{p\} \cup \{D_\beta \mid \beta < \omega_1\}$$

taken inside $Lp^{2n-1}(A)$. Then let N_0 be the Mostowski collapse of M_0 and let

$$\pi_0 : N_0 \to M_0 \prec_{\Sigma_k} Lp^{2n-1}(A)$$

be the inverse of the embedding obtained from the Mostowski collapse with critical point ξ_0.

Now assume we already constructed $(N_\alpha, \pi_\alpha, \xi_\alpha)$ and M_α for some $\alpha < \omega_1$. Then we let $M_{\alpha+1}$ be the (uncollapsed) Σ_k-hull of

$$\gamma \cup \{p\} \cup \{D_\beta \mid \beta < \omega_1\} \cup M_\alpha \cup \{M_\alpha\}$$

taken inside $Lp^{2n-1}(A)$. Further let $N_{\alpha+1}$ be the Mostowski collapse of $M_{\alpha+1}$ and let

$$\pi_{\alpha+1} : N_{\alpha+1} \to M_{\alpha+1} \prec_{\Sigma_k} Lp^{2n-1}(A)$$

be the inverse of the embedding obtained from the Mostowski collapse with critical point $\xi_{\alpha+1}$. Note that we have $\xi_{\alpha+1} > \xi_\alpha$.

Moreover if we assume that $(N_\alpha, \pi_\alpha, \xi_\alpha)$ is already constructed for all $\alpha < \lambda$ for some limit ordinal $\lambda \leq \omega_1$, then we let

$$N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha,$$

$$\pi_\lambda = \bigcup_{\alpha<\lambda} \pi_\alpha,$$

and $\xi_\lambda = \mathrm{crit}(\pi_\lambda)$.

Recall that we fixed open dense sets $(D_\beta \mid \beta < \omega_1)$. We are now going to construct a sequence $(p_\alpha \mid \alpha \leq \omega_1)$ of conditions such that $p_{\alpha+1} \leq_{P_1} p_\alpha$ and $p_{\alpha+1} \in D_\alpha$ for all $\alpha < \omega_1$. Moreover we are going to construct these conditions such that we inductively maintain $p_\alpha \in \pi_\alpha^{-1}(P_1) \subset N_\alpha$.

We start with $p_0 = p \in N_0$. For the successor step suppose that we already defined $p_\alpha \in \pi_\alpha^{-1}(P_1) \subset N_\alpha$ for some $\alpha < \omega_1$. Then we have that $\mathrm{dom}(p_\alpha) < \xi_\alpha$ and $p_\alpha \in N_{\alpha+1}$ by the definition of the models and embeddings $(N_\alpha, \pi_\alpha \mid \alpha \leq \omega_1)$. By extendibility of the forcing $\pi_{\alpha+1}^{-1}(P_1)$ and the density of the set $\pi_{\alpha+1}^{-1}(D_\alpha) \subseteq D_\alpha$, there exists a condition $p_{\alpha+1} \leq_{P_1} p_\alpha$ such that we have $p_{\alpha+1} \in \pi_{\alpha+1}^{-1}(P_1) \subset N_{\alpha+1}$, $p_{\alpha+1} \in D_\alpha$ and $\mathrm{dom}(p_{\alpha+1}) \geq \xi_\alpha$ since $\pi_{\alpha+1}^{-1}(P_1) \subseteq P_1$.

For a limit ordinal $\lambda \leq \omega_1$ we simply let $p_\lambda = \bigcup_{\alpha<\lambda} p_\alpha$. We have that p_λ is a condition in the forcing P_1 by the following argument. We have that the sequence $(\xi_\alpha \mid \alpha < \lambda)$ of critical points of $(\pi_\alpha \mid \alpha < \lambda)$ is definable over N_λ since for $\alpha < \lambda$ the model N_α is equal to the transitive collapse of a Σ_k-elementary submodel of N_λ which is constructed inside N_λ exactly as it was constructed inside $Lp^{2n-1}(A)$ above. Therefore we have that

$$\mathrm{cf}^{N_\lambda}(\xi_\lambda) \leq \lambda \leq \omega_1 = \omega_1^{V_1}.$$

This implies that

$$N_\lambda \vDash |\xi_\lambda| \leq \omega_1^{V_1},$$

for all limit ordinals $\lambda \leq \omega_1$.

Now consider the function $q = p_{\omega_1}$. Then we have by construction that $\mathrm{dom}(q) = \bigcup_{\alpha<\omega_1} \xi_\alpha = \xi_{\omega_1}$. Moreover we have as above that there exists a model N as in (ii) in Definition 3.7.8 witnessing that q is $(A, (\gamma^+)^{M_x})$-reshaping and thus we have that $q \in P_1$.

This finally proves that the reshaping forcing P_1 is $< (\gamma^+)^{M_x}$-distributive and therefore does not collapse ω_1 and $(\gamma^+)^{M_x} = \omega_2$.

So let G_1 be P_1-generic over V_1 and let $V_2 = V_1[G_1]$. The extendability of the forcing P_1 yields that $\bigcup G_1$ is an $(A, (\gamma^+)^{M_x})$-reshaping function with domain $(\gamma^+)^{M_x}$. Let B' be a subset of $(\gamma^+)^{M_x}$ which codes the function $\bigcup G_1$, for example the subset of $(\gamma^+)^{M_x}$ which has $\bigcup G_1$ as its characteristic function. Finally let $B \subset (\gamma^+)^{M_x}$ be a code for $A \oplus B'$.

As at the end of Step 1 we can pick this code $B \subset (\gamma^+)^{M_x}$ such that the model V_2 is of the form $Lp^{2n-1}(B)$ by the following argument: Recall that

$$Lp^{2n-1}(B) = M(B)|\omega_1^V,$$

where as above $M(B)$ denotes the ω_1^V-th iterate of $M_{2n-1}^{\#}(B)$ by its least measure and its images. Therefore we can consider G_1 as being generic over $M(A)$. This yields analogous to the argument at the end of Step 1 that we can pick B such that $V_2 = M(B)|\omega_1^V$ because the "reshaping forcing" P_1 takes place below $(\gamma^+)^{M_x} < \omega_1^V$. Therefore we have that

$$V_2 = Lp^{2n-1}(B).$$

Step 3: Now we can perform the first coding using almost disjoint subsets of $\omega_1 = \omega_1^{V_2} = \omega_1^{V_1}$. Since B is "reshaped" we can inductively construct a sequence of almost disjoint subsets of ω_1,

$$(A_\xi \mid \xi < (\gamma^+)^{M_x}),$$

as follows. Let $\xi < (\gamma^+)^{M_x}$ be such that we already constructed a sequence $(A_\zeta \mid \zeta < \xi)$ of almost disjoint subsets of ω_1.

Case 1. $L[B \cap \xi] \models |\xi| \leq \omega_1^{V_2}$.

Then we let A_ξ be the least subset of ω_1 in $L[B \cap \xi]$ which is almost disjoint from any A_ζ for $\zeta < \xi$ and which satisfies that

$$|\omega_1 \setminus \bigcup_{\zeta \leq \xi} A_\zeta| = \aleph_1.$$

Case 2. Otherwise.

Let N be the least initial segment of $Lp^{2n-1}(A \cap \xi)^{Lp^{2n-1}(B)}$ such that $\rho_\omega(N) \leq \xi$, N is sound above ξ, ξ is the largest cardinal in N, and definably over N there exists a surjection $g : \omega_1^{V_2} \twoheadrightarrow \xi$. Now let A_ξ be the least subset of $\omega_1^{V_2}$ which is definable over N, almost disjoint from any A_ζ for $\zeta < \xi$ and which satisfies that $|\omega_1 \setminus \bigcup_{\zeta \leq \xi} A_\zeta| = \aleph_1$.

The existence of such a set A_ξ follows from the fact that the set $B \subset (\gamma^+)^{M_x}$ is "reshaped" by the following argument. As B is "reshaped" we have in Case 2 above that there exists a model N as in Definition 3.7.8 (ii). Then a comparison argument yields that $N \lhd Lp^{2n-1}(A \cap \xi)$. In general it need not be the case that $Lp^{2n-1}(A \cap \xi)^{Lp^{2n-1}(B)}$ is equal to $Lp^{2n-1}(A \cap \xi)$, but as ξ is the largest cardinal in N, it follows that in fact $N \lhd Lp^{2n-1}(A \cap \xi)^{Lp^{2n-1}(B)}$. Therefore we have that in Case 2 such a premouse N and thereby the set A_ξ exists.

Moreover the sequence $(A_\xi \mid \xi < (\gamma^+)^{M_x})$ is definable in $V_2 = Lp^{2n-1}(B)$.

Now let P_2 be the forcing for coding B by a subset of ω_1 using the almost disjoint sets $(A_\xi \mid \xi < (\gamma^+)^{M_x})$. That means a condition $p \in P_2$ is a pair (p_l, p_r) such that $p_l : \alpha \rightarrow 2$ for some $\alpha < \omega_1$ and p_r is a countable subset of $(\gamma^+)^{M_x}$. We say $p = (p_l, p_r) \leq_{P_2} (q_l, q_r) = q$ iff $q_l \subseteq p_l$, $q_r \subseteq p_r$, and for all $\xi \in q_r$ we have that if $\xi \in B$, then

$$\{\beta \in \text{dom}(p_l) \setminus \text{dom}(q_l) \mid p_l(\beta) = 1\} \cap A_\xi = \emptyset.$$

An easy argument shows that the $(\gamma^+)^{M_x}$-c.c. holds true for the forcing P_2. Moreover it is ω-closed and therefore no cardinals are collapsed.

Let G_2 be P_2-generic over V_2 and let

$$C' = \bigcup_{p \in G_2} \{\beta \in \text{dom}(p_l) \mid p_l(\beta) = 1\}.$$

Then $C' \subset \omega_1$ and we have that for all $\xi < (\gamma^+)^{M_x}$,

$$\xi \in B \text{ iff } |C' \cap A_\xi| \leq \aleph_0.$$

Finally let $V_3 = V_2[G_2]$. By the same argument as we gave at the end of Step 2 we can obtain that

$$V_3 = Lp^{2n-1}(C)$$

for some set $C \subset \omega_1$ coding C' and the real x, as the model $Lp^{2n-1}(C)$ can successfully decode the set $B \subset (\gamma^+)^{M_x}$ by the following argument. We show inductively that for every $\xi < (\gamma^+)^{M_x}$, $(A_\zeta \mid \zeta < \xi) \in Lp^{2n-1}(C)$ and $B \cap \xi \in Lp^{2n-1}(C)$. This yields that $B \in Lp^{2n-1}(C)$.

For the inductive step let $\xi < (\gamma^+)^{M_x}$ be an ordinal and assume inductively that we have

$$(A_\zeta \mid \zeta < \xi) \in Lp^{2n-1}(C).$$

Since for all $\zeta < \xi$,

$$\zeta \in B \text{ iff } |C' \cap A_\zeta| \leq \aleph_0,$$

we have that $B \cap \xi \in Lp^{2n-1}(C)$.

In Case 1, i.e. if $L[B \cap \xi] \vDash |\xi| \leq \omega_1^{V_2}$, the set A_ξ can easily be identified inside $Lp^{2n-1}(C)$. In Case 2 let N be the least initial segment of $Lp^{2n-1}(A \cap \xi)^{Lp^{2n-1}(C)}$ such that $\rho_\omega(N) \leq \xi$, N is sound above ξ, ξ is the largest cardinal in N, and definably over N there exists a surjection $g : \omega_1 \twoheadrightarrow \xi$. Then we have that in fact such an N with $N \lhd Lp^{2n-1}(A \cap \xi)$ and $N \lhd Lp^{2n-1}(A \cap \xi)^{Lp^{2n-1}(B)}$. This yields that also in this case the set A_ξ can be identified inside $Lp^{2n-1}(C)$. Finally the uniform definition of the sets A_ξ yields that

$$(A_\zeta \mid \zeta \leq \xi) \in Lp^{2n-1}(C).$$

Step 4: Before we can "code down to a real", that means before we can find a real z such that $K^{M_x}|(\gamma^+)^{M_x} \in Lp^{2n-1}(z)$, we have to perform another "reshaping" similar to the one in Step 2. So let P_3 be the forcing for adding a (C, ω_1)-reshaping function working in V_3 as the new ground model, where $\omega_1 = \omega_1^{V_3} = \omega_1^{V_2}$. That means we let $p \in P_3$ iff p is a (C, ω_1)-reshaping function with $\text{dom}(p) < \omega_1$. The order of two conditions p and q in P_3 is again by reverse inclusion, that means $p \leq_{P_3} q$ iff $q \subseteq p$.

The forcing P_3 is extendable and $< \omega_1$-distributive by the same arguments as we gave in Step 2 since we have that $V_3 = Lp^{2n-1}(C)$. Therefore P_3 does not collapse ω_1.

Let G_3 be P_3-generic over V_3 and let $V_4 = V_3[G_3]$. We again have that $\bigcup G_3$ is a (C, ω_1)-reshaping function with domain ω_1, because P_3 is extendable. Let D' be a subset of ω_1 which codes $\bigcup G_3$, for example the subset of ω_1 which has $\bigcup G_3$ as its characteristic function. Finally let $D \subset \omega_1$ code $C \oplus D'$.

By the same argument as the one we gave at the end of Step 2 we can obtain that in fact

$$V_4 = Lp^{2n-1}(D).$$

Step 5: Now we are ready to finally "code down to a real". Since D is "reshaped" we can consider a uniformly defined sequence

$$(B_\xi \mid \xi < \omega_1)$$

of almost disjoint subsets of ω analogous to Step 3, where $\omega_1 = \omega_1^{V_4} = \omega_1^{V_3}$.

Now we let P_4 be the forcing for coding D by a subset of ω using the almost disjoint sets $(B_\xi \mid \xi < \omega_1)$. That means a condition $p \in P_4$ is a pair (p_l, p_r) such that $p_l : \alpha \rightarrow 2$ for some $\alpha < \omega$ and p_r is a finite subset of ω_1. We say $p = (p_l, p_r) \leq_{P_4} (q_l, q_r) = q$ iff $q_l \subseteq p_l$, $q_r \subseteq p_r$, and for all $\xi \in q_r$ we have that if $\xi \in D$, then

$$\{\beta \in \operatorname{dom}(p_l) \setminus \operatorname{dom}(q_l) \mid p_l(\beta) = 1\} \cap B_\xi = \emptyset.$$

As in Step 3 above an easy argument shows that the forcing P_4 has the c.c.c. and therefore no cardinals are collapsed.

Finally let G_4 be P_4-generic over V_4 and let

$$E' = \bigcup_{p \in G_4} \{\beta \in \operatorname{dom}(p_l) \mid p_l(\beta) = 1\}.$$

Then $E' \subset \omega$ and we have that for all $\xi < \omega_1$,

$$\xi \in D \text{ iff } |E' \cap B_\xi| < \aleph_0.$$

Let $V_5 = V_4[G_4]$ and finally let z be a real coding E' and the real x. Analogous to the arguments given at the end of Step 3 we can pick the real $z \geq_T x$ such that we have

$$V_5 = Lp^{2n-1}(z)$$

and the model $Lp^{2n-1}(z)$ is able to successfully decode the set D and thereby also the set A.

This ultimately yields that we have a real $z \geq_T x$ such that

$$(\gamma^+)^{M_x} = \omega_2^{Lp^{2n-1}(z)}$$

and

$$K^{M_x}|(\gamma^+)^{M_x} \in Lp^{2n-1}(z).$$

□

Subclaim 3. *$K^{M_x}|(\gamma^+)^{M_x}$ is fully iterable inside $Lp^{2n-1}(z)$.*

Proof. It is enough to show that the $2n$-small premouse $K^{M_x}|(\gamma^+)^{M_x}$ is ω_1-iterable inside $Lp^{2n-1}(z)$ because once we showed this, an absoluteness argument, as for example similar to the one we already gave in the proof of Lemma 3.1.9, yields that $K^{M_x}|(\gamma^+)^{M_x}$ is in fact fully iterable inside $Lp^{2n-1}(z)$ since the iteration strategy for $K^{M_x}|(\gamma^+)^{M_x}$ is given by $\mathcal{Q}$-structures $\mathcal{Q}(\mathcal{T})$ for iteration trees $\mathcal{T}$ on $K^{M_x}|(\gamma^+)^{M_x}$ which are $(2n-1)$-small above $\delta(\mathcal{T})$ and such $\mathcal{Q}$-structures $\mathcal{Q}(\mathcal{T})$ are contained in every lower part model at the level $2n-1$, as for example $Lp^{2n-1}(z)$, by definition of the lower part model.

In a first step we show that there is a tree T such that $p[T]$ is a universal Π^1_{2n}-set in $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ and moreover for every forcing $\mathbb{P}$ of size at most $(\gamma^+)^{M_x}$ in $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ and every $\mathbb{P}$-generic G over $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ we have that

$$Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})[G] \vDash \text{“}p[T] \text{ is a universal } \Pi^1_{2n}\text{-set”}.$$

Let φ be a Π^1_{2n}-formula defining a universal Π^1_{2n}-set, i.e. $\{y \in {}^\omega\omega \mid \varphi(y)\}$ is a universal Π^1_{2n}-set. Then we let $T \in Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ be a tree of height ω searching for $y, H, \mathcal{M}, \sigma, \mathbb{Q}$ and g such that

(1) $y \in {}^\omega\omega$,
(2) $\mathcal{M}$ is a countable (x, H)-premouse,
(3) $\sigma : \mathcal{M} \rightarrow Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ is a sufficiently elementary embedding,
(4) $\sigma(H) = K^{M_x}|(\gamma^+)^{M_x}$, and
(5) $\mathbb{Q} \in \mathcal{M}$ is a partial order of size at most $H \cap \mathrm{Ord}$ in $\mathcal{M}$ and g is $\mathbb{Q}$-generic over $\mathcal{M}$ such that
$$\mathcal{M}[g] \vDash \varphi(y).$$

This tree T has the properties we claimed above by the following argument. The model $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ is closed under the operation $a \mapsto M_{2n-2}^{\#}(a)$ and is therefore $\boldsymbol{\Sigma}^1_{2n}$-correct in V. The same holds for the model $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})[G]$, where G is $\mathbb{P}$-generic over the model $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ for some forcing $\mathbb{P}$ of size at most $(\gamma^+)^{M_x}$ in the model $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$. Moreover we have that if $\mathcal{M}, H, \sigma, \mathbb{Q}$ and g are as searched by the tree T, then the forcing $\mathbb{Q} \in \mathcal{M}$ has size at most $H \cap \mathrm{Ord}$ in $\mathcal{M}$ and the embedding $\sigma : \mathcal{M} \rightarrow Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ is sufficiently elementary, so it follows that $\mathcal{M}[g]$ is $\boldsymbol{\Sigma}^1_{2n}$-correct in V. This easily yields that $p[T]$ is a universal Π^1_{2n}-set in $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})[G]$ for every generic set G as above, in fact

$$p[T] \cap Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})[G] = \\ \{y \in {}^\omega\omega \mid \varphi(y)\} \cap Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})[G].$$

In a second step we now define another tree U whose well-foundedness is going to witness that $K^{M_x}|(\gamma^+)^{M_x}$ is ω_1-iterable. So we define the tree $U \in Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ such that U is searching for $\bar{K}, \pi, \mathcal{T}, \mathcal{N}, \sigma$ and a sequence $(\mathcal{Q}_\lambda \mid \lambda \leq \mathrm{lh}(\mathcal{T}), \lambda \text{ limit})$ with the following properties.

(1) $\bar{K}$ is a countable premouse,
(2) $\pi : \bar{K} \rightarrow K^{M_x}|(\gamma^+)^{M_x}$ is an elementary embedding,
(3) $\mathcal{T}$ is a countable putative[4] iteration tree on $\bar{K}$ such that for all limit ordinals $\lambda < \mathrm{lh}(\mathcal{T})$,
$$\mathcal{Q}_\lambda \trianglelefteq \mathcal{M}_\lambda^{\mathcal{T}},$$
(4) for all limit ordinals $\lambda \leq \mathrm{lh}(\mathcal{T})$, $\mathcal{Q}_\lambda$ is a Π^1_{2n}-iterable (above $\delta(\mathcal{T} \upharpoonright \lambda)$) $\mathcal{Q}$-structure for $\mathcal{T} \upharpoonright \lambda$ and $\mathcal{Q}_\lambda$ is $(2n-1)$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$ (where this Π^1_{2n}-statement is witnessed using the tree T defined above),
(5) $\mathcal{N}$ is a countable model of ZFC^- such that either
$$\mathcal{N} \vDash \text{“}\mathcal{T} \text{ has a last ill-founded model”},$$

[4]Recall that we say that a tree $\mathcal{T}$ is a *putative iteration tree* if $\mathcal{T}$ satisfies all properties of an iteration tree, but we allow the last model of $\mathcal{T}$ to be ill-founded, in case $\mathcal{T}$ has a last model.

or else

$$\mathcal{N} \models \text{ ``lh}(\mathcal{T}) \text{ is a limit ordinal and there is no cofinal branch } b \text{ through } \mathcal{T} \text{ such that } \mathcal{Q}_{\text{lh}(\mathcal{T})} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}\text{''},$$

and

(6) $\sigma : \mathcal{N} \cap \text{Ord} \hookrightarrow \omega_1$.

Here a code for the sequence $(\mathcal{Q}_\lambda \mid \lambda \leq \text{lh}(\mathcal{T}), \lambda \text{ limit})$ of $\mathcal{Q}$-structures satisfying property (4) can be read off from $p[T]$.

Recall that we have that the core model K^{M_x} is $2n$-small and thereby fully iterable in the model M_x via an iteration strategy which is guided by $\mathcal{Q}$-structures $\mathcal{Q}(\mathcal{T})$ for iteration trees $\mathcal{T}$ on K^{M_x} of limit length which are $(2n-1)$-small above $\delta(\mathcal{T})$. This implies that the premouse $K^{M_x}|(\gamma^+)^{M_x}$ is iterable inside the model $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ (which as argued earlier is a definable subset of M_x) by an argument as in the proof of Lemma 2.2.8, because $K^{M_x}|(\gamma^+)^{M_x}$ is a $2n$-small premouse and the ω_1-iterable $\mathcal{Q}$-structures $\mathcal{Q}(\mathcal{T})$ for iteration trees $\mathcal{T}$ on $K^{M_x}|(\gamma^+)^{M_x}$ in $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ are contained in the model $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ as they are $(2n-1)$-small above $\delta(\mathcal{T})$.

We aim to show that the tree U defined above is well-founded inside the model $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$. So assume toward a contradiction that U is ill-founded in $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ and let $\bar{K}, \pi, \mathcal{T}, \mathcal{N}, \sigma$ and $(\mathcal{Q}_\lambda \mid \lambda \leq \text{lh}(\mathcal{T}), \lambda \text{ limit})$ be as above satisfying properties (1) – (6) in the definition of the tree U.

Assume that the iteration tree $\mathcal{T}$ has limit length since the other case is easier. Since as argued above the premouse $K^{M_x}|(\gamma^+)^{M_x}$ is countably iterable inside $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$, we have that in $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ there exists a cofinal well-founded branch b through the iteration tree $\mathcal{T}$ on $\bar{K}$ and ω_1-iterable $\mathcal{Q}$-structures $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ for all limit ordinals $\lambda \leq \text{lh}(\mathcal{T})$ such that we have $\mathcal{Q}(\mathcal{T}) = \mathcal{Q}(b, \mathcal{T})$ by an argument analogous to the one we gave in Case 1 in the proof of Lemma 3.1.9. Moreover every $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ is $(2n-1)$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$.

Since the model $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ is closed under the operation $a \mapsto M_{2n-2}^{\#}(a)$, the proofs of Lemmas 2.2.8 and 2.2.9 imply that we can successfully compare $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ and $\mathcal{Q}_\lambda$ for all limit ordinals $\lambda \leq \text{lh}(\mathcal{T})$. This implies that in fact

$$\mathcal{Q}(\mathcal{T} \upharpoonright \lambda) = \mathcal{Q}_\lambda.$$

Therefore it follows by an absoluteness argument as the one we already gave in the proof of Lemma 2.2.8 that

$$\mathcal{N} \models \text{ ``there exists a cofinal branch } b \text{ through } \mathcal{T} \text{ such that } \mathcal{Q}_{\text{lh}(\mathcal{T})} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}\text{''},$$

because we in particular have that $\mathcal{Q}_{\text{lh}(\mathcal{T})} = \mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$.

Therefore the tree U is well-founded in the model $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ and by absoluteness of well-foundedness this implies that the tree U is also well-founded inside $Lp^{2n-1}(z)$.

We have by construction of the tree T that a code for the sequence $(\mathcal{Q}_\lambda \mid \lambda \leq \mathrm{lh}(\mathcal{T}), \lambda \text{ limit})$ can still be read off from $p[T]$ in $Lp^{2n-1}(z)$ because the forcing we performed over the ground model $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$ to obtain the real z has size at most $(\gamma^+)^{M_x}$ in $Lp^{2n-1}(x, K^{M_x}|(\gamma^+)^{M_x})$. Therefore the well-foundedness of the tree U in $Lp^{2n-1}(z)$ implies that $K^{M_x}|(\gamma^+)^{M_x}$ is ω_1-iterable in $Lp^{2n-1}(z)$. □

Let

$$(K^c)^{Lp^{2n-1}(z)}$$

be the model obtained from a K^c-construction as defined in [MSch04] performed inside the model $Lp^{2n-1}(z)$ in the following sense. We aim to construct the premouse $(K^c)^{Lp^{2n-1}(z)}$ such that it does not depend on the specific choice of the real z. Therefore we consider the K^c-construction of [MSch04] inside the model $Lp^{2n-1}(z)$ with $V = L[A_z]$ for $A_z = z^\frown E_z \subset \mathrm{Ord}$, where we identify the real z with a set of ordinals and E_z codes the extender sequence of $Lp^{2n-1}(z)$.

Subclaim 4. $(K^c)^{Lp^{2n-1}(z)} \subset \mathrm{HOD}^{Lp^{2n-1}(z)}$.

Proof. The definition of *certified* in the construction of the model K^c in the sense of [MSch04] makes reference to some class of ordinals A such that $V = L[A]$. Therefore the model K^c constructed in this sense is in general not contained in HOD, because whether an extender is certified in the sense of Definition 1.6 in [MSch04] may depend on the choice of A. This is why we have to argue that in this situation the model $(K^c)^{Lp^{2n-1}(z)}$ as defined above does not depend on the choice of the real z.

So let $z' \in Lp^{2n-1}(z)$ be an arbitrary real with $z \leq_T z'$. That means we have $Lp^{2n-1}(z) = Lp^{2n-1}(z')$. Analogous as above let $A_{z'} = z'^\frown E_{z'} \subset \mathrm{Ord}$, where $E_{z'}$ codes the extender sequence of $Lp^{2n-1}(z')$. Then we need to show the following claim.

Claim 1. *An extender E is certified with respect to $L[A_z]$ iff E is certified with respect to $L[A_{z'}]$, in the sense of Definition* 1.6 *in [MSch04].*

Proof. It follows from Lemma 3.7.5 that

$$Lp^{2n-1}(z') = (Lp^{2n-1}(z'))^{Lp^{2n-1}(z)}$$

and we even have that the set $A_{z'} \cap \omega_1^{Lp^{2n-1}(z)}$ can be computed from $A_z \cap \omega_1^{Lp^{2n-1}(z)}$. Similarly we have that the set $A_z \cap \omega_1^{Lp^{2n-1}(z)}$ can be computed from $A_{z'} \cap \omega_1^{Lp^{2n-1}(z)}$. Moreover the extender sequences of $Lp^{2n-1}(z)$ and $Lp^{2n-1}(z')$, coded into E_z and $E_{z'}$, agree above $\omega_1^{Lp^{2n-1}(z)}$. Therefore it follows inductively that an extender E is certified with respect to $L[A_z]$ iff E is certified with respect to $L[A_{z'}]$. □

This yields that the model $(K^c)^{Lp^{2n-1}(z)}$ does not depend on the specific choice of the real z and thus it follows that

$$(K^c)^{Lp^{2n-1}(z)} \subset \mathrm{HOD}^{Lp^{2n-1}(z)},$$

as desired. □

In what follows we will also need the following notion of iterability.

Definition 3.7.9. *Let N be a countable premouse. Then we inductively define an iteration strategy Λ for N as follows. Assume that $\mathcal{T}$ is a normal iteration tree on N of limit length according to Λ such that there exists a $\mathcal{Q}$-structure $\mathcal{Q} \trianglerighteq \mathcal{M}(\mathcal{T})$ for $\mathcal{T}$ which is fully iterable above $\delta(\mathcal{T})$. Then we define that $\Lambda(\mathcal{T}) = b$ iff b is a cofinal branch through $\mathcal{T}$ such that either*

(i) *$\mathcal{Q} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$, or*
(ii) *b does not drop (so in particular the iteration embedding $i_b^{\mathcal{T}}$ exists), there exists an ordinal $\delta < N \cap \mathrm{Ord}$ such that $i_b^{\mathcal{T}}(\delta) = \delta(\mathcal{T})$ and*
$$N \vDash \text{“}\delta \text{ is Woodin”},$$
and there exists a $\tilde{\mathcal{Q}} \trianglerighteq N|\delta$ such that
$$\tilde{\mathcal{Q}} \vDash \text{“}\delta \text{ is Woodin”},$$
but δ is not definably Woodin over $\tilde{\mathcal{Q}}$ and if we lift the iteration tree $\mathcal{T}$ on $\tilde{\mathcal{Q}}$, call this iteration tree $\mathcal{T}^*$, then
$$i_b^{\mathcal{T}^*} : \tilde{\mathcal{Q}} \to \mathcal{Q}.$$

Definition 3.7.10. *Let N be a countable premouse. Then we say that N is* $\mathcal{Q}$-structure iterable *iff for every iteration tree $\mathcal{T}$ on N which is according to the iteration strategy Λ from Definition 3.7.9 the following holds.*

(i) *If $\mathcal{T}$ has limit length and there exists a $\mathcal{Q}$-structure $\mathcal{Q} \trianglerighteq \mathcal{M}(\mathcal{T})$ for $\mathcal{T}$ which is fully iterable above $\delta(\mathcal{T})$, then there exists a cofinal well-founded branch b through $\mathcal{T}$ such that $\Lambda(\mathcal{T}) = b$.*
(ii) *If $\mathcal{T}$ has a last model, then every putative iteration tree $\mathcal{U}$ extending $\mathcal{T}$ such that $\mathrm{lh}(\mathcal{U}) = \mathrm{lh}(\mathcal{T}) + 1$ has a well-founded last model.*

The premouse

$$(K^c)^{Lp^{2n-1}(z)}$$

is countably iterable in $Lp^{2n-1}(z)$ by the iterability proof of Chapter 9 in [St96] adapted as in Section 2 in [MSch04].

Subclaim 5. *In $Lp^{2n-1}(z)$,*

$$(K^c)^{Lp^{2n-1}(z)} \textit{ is } \mathcal{Q}\textit{-structure iterable.}$$

PROOF. Assume there exists an iteration tree $\mathcal{T}$ on $(K^c)^{Lp^{2n-1}(z)}$ which witnesses that $(K^c)^{Lp^{2n-1}(z)}$ is not $\mathcal{Q}$-structure iterable inside $Lp^{2n-1}(z)$. Since the other case is easier assume that $\mathcal{T}$ has limit length. That means in particular that there exists a $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{T}) \trianglerighteq \mathcal{M}(\mathcal{T})$ for $\mathcal{T}$ which is fully iterable above $\delta(\mathcal{T})$, but there is no cofinal well-founded branch b through $\mathcal{T}$ in $Lp^{2n-1}(z)$ such that $\Lambda(\mathcal{T}) = b$.

For some large enough natural number m let H be the Mostowski collapse of $Hull_m^{Lp^{2n-1}(z)}$ such that H is sound and $\rho_\omega(H) = \rho_{m+1}(H) = \omega$. Furthermore let

$$\pi : H \rightarrow Lp^{2n-1}(z)$$

be the uncollapse map such that $\mathcal{T}, \mathcal{Q}(\mathcal{T}) \in \operatorname{ran}(\pi)$. Moreover let $\bar{\mathcal{T}}, \bar{\mathcal{Q}} \in H$ be such that $\pi(\bar{\mathcal{T}}) = \mathcal{T}$, $\pi(\bar{\mathcal{Q}}) = \mathcal{Q}(\mathcal{T})$, and let $\bar{K} \subset H$ be such that $\pi(\bar{K}|\gamma) = (K^c)^{Lp^{2n-1}(z)}|\pi(\gamma)$ for any $\gamma < H \cap \mathrm{Ord}$. That means in particular that $\bar{\mathcal{T}}$ is an iteration tree on $\bar{K}$.

As argued above we have that $(K^c)^{Lp^{2n-1}(z)}$ is countably iterable in $Lp^{2n-1}(z)$. Therefore there exists a cofinal well-founded branch $\bar{b}$ through $\bar{\mathcal{T}}$ in $Lp^{2n-1}(z)$.

Now we consider two different cases.

Case 1. There is a drop along the branch $\bar{b}$.

In this case there exists a $\mathcal{Q}$-structure $\mathcal{Q}^* \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$ for $\bar{\mathcal{T}}$, because there is a drop along $\bar{b}$. A standard comparison argument shows that $\mathcal{Q}^* = \bar{\mathcal{Q}}$ and thus $\bar{\mathcal{Q}} \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$ is a $\mathcal{Q}$-structure for $\bar{\mathcal{T}}$.

Now consider the statement

$$\phi(\bar{\mathcal{T}}, \bar{\mathcal{Q}}) \equiv \text{ “there is a cofinal branch } b \text{ through } \bar{\mathcal{T}} \text{ such that } \bar{\mathcal{Q}} \trianglelefteq \mathcal{M}_b^{\bar{\mathcal{T}}}\text{”.}$$

This statement $\phi(\bar{\mathcal{T}}, \bar{\mathcal{Q}})$ is Σ^1_1-definable from the parameters $\bar{\mathcal{T}}$ and $\bar{\mathcal{Q}}$ and holds in the model $Lp^{2n-1}(z)$ as witnessed by the branch $\bar{b}$.

By $\boldsymbol{\Sigma}^1_1$-absoluteness the statement $\phi(\bar{\mathcal{T}}, \bar{\mathcal{Q}})$ also holds in the model $H^{\mathrm{Col}(\omega,\eta)}$ as witnessed by some branch b, where $\eta < H \cap \mathrm{Ord}$ is a large enough ordinal such that $\bar{\mathcal{T}}, \bar{\mathcal{Q}} \in H^{\mathrm{Col}(\omega,\eta)}$ are countable inside $H^{\mathrm{Col}(\omega,\eta)}$.

Since b is uniquely definable from $\bar{\mathcal{T}}$ and the $\mathcal{Q}$-structure $\bar{\mathcal{Q}}$, and $\bar{\mathcal{T}}, \bar{\mathcal{Q}} \in H$, we have by homogeneity of the forcing $\mathrm{Col}(\omega, \eta)$ that actually $b \in H$. This contradicts the fact that the iteration tree $\mathcal{T}$ witnesses that the premouse $(K^c)^{Lp^{2n-1}(z)}$ is not $\mathcal{Q}$-structure iterable inside $Lp^{2n-1}(z)$.

Case 2. There is no drop along the branch $\bar{b}$.

In this case we consider two subcases as follows.

Case 2.1. There is no Woodin cardinal $\bar{\delta}$ in $\bar{K}$ such that $i_{\bar{b}}^{\bar{\mathcal{T}}}(\bar{\delta}) = \delta(\bar{\mathcal{T}})$.

In this case we have that $\delta(\bar{\mathcal{T}})$ is not Woodin in $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$. Thus there exists an initial segment $\mathcal{Q}^* \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$ such that

$$\mathcal{Q}^* \vDash \text{“}\delta(\bar{\mathcal{T}}) \text{ is a Woodin cardinal”},$$

and definably over $\mathcal{Q}^*$ there exists a witness for the fact that $\delta(\bar{\mathcal{T}})$ is not Woodin (in the sense of Definition 2.2.4). As $(K^c)^{Lp^{2n-1}(z)}$ is countably iterable in $Lp^{2n-1}(z)$ we can successfully coiterate the premice $\mathcal{Q}^*$ and $\bar{\mathcal{Q}}$. Therefore we have that in fact $\mathcal{Q}^* = \bar{\mathcal{Q}}$ and thus as in Case 1 it follows that $\bar{\mathcal{Q}} \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$ is a $\mathcal{Q}$-structure for $\bar{\mathcal{T}}$ and we can derive a contradiction from that as above.

Case 2.2. There is a Woodin cardinal $\bar{\delta}$ in $\bar{K}$ such that $i_{\bar{b}}^{\bar{\mathcal{T}}}(\bar{\delta}) = \delta(\bar{\mathcal{T}})$.

Let $\delta = \pi(\bar{\delta})$ be the corresponding Woodin cardinal in $(K^c)^{Lp^{2n-1}(z)}$. Then the real z is generic over the model $(K^c)^{Lp^{2n-1}(z)}$ for the extender algebra at the Woodin cardinal δ since all extenders appearing in the K^c-construction in the sense of [MSch04] satisfy the axioms of the extender algebra. In fact by the same argument the model $Lp^{2n-1}(z)|\delta$ is generic over $(K^c)^{Lp^{2n-1}(z)}|(\delta + \omega)$ for the δ-version of the extender algebra $\mathbb{Q}_\delta$ (see the proof of Lemma 1.3 in [SchSt09] for a definition of the δ-version of the extender algebra).

Therefore we can perform a maximal $\mathcal{P}$-construction, which is defined as in [SchSt09], inside $Lp^{2n-1}(z)$ over $(K^c)^{Lp^{2n-1}(z)}|(\delta+\omega)$ to obtain a model $\mathcal{P}$. We have that $\mathcal{P} \vDash$ “δ is Woodin” by the definition of a maximal $\mathcal{P}$-construction.

Let $\bar{\mathcal{P}}$ be the corresponding result of a maximal $\mathcal{P}$-construction in H over $\bar{K}|(\bar{\delta}+\omega)$. Moreover let $\mathcal{T}^*$ be the iteration tree obtained by considering the tree $\bar{\mathcal{T}}$ based on $\bar{K}|\bar{\delta}$ as an iteration tree on $\bar{\mathcal{P}} \rhd \bar{K}|\bar{\delta}$. Let

$$i_{\bar{b}}^{\mathcal{T}^*} : \bar{\mathcal{P}} \rightarrow \mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$$

denote the corresponding iteration embedding, where the branch through $\mathcal{T}^*$ we consider is induced by the branch $\bar{b}$ through $\bar{\mathcal{T}}$ we fixed above, so we call them both $\bar{b}$. Then we have that

$$i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta}) = i_{\bar{b}}^{\bar{\mathcal{T}}}(\bar{\delta}) = \delta(\bar{\mathcal{T}}).$$

Case 2.2.1. We have that

$$\mathcal{P} \cap \mathrm{Ord} < Lp^{2n-1}(z) \cap \mathrm{Ord}\,.$$

Then we have in particular that $\bar{\mathcal{P}} \cap \mathrm{Ord} < H \cap \mathrm{Ord}$ and thus $\bar{\mathcal{P}} \in H$.

Consider the coiteration of $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ with $\bar{\mathcal{Q}}$ inside $Lp^{2n-1}(z)$. By the definition of a maximal $\mathcal{P}$-construction (see [SchSt09]) we have that $\bar{\delta}$ is not definably Woodin over $\bar{\mathcal{P}}$ since $\bar{\mathcal{P}} \cap \mathrm{Ord} < H \cap \mathrm{Ord}$. Since $\bar{b}$ is non-dropping this implies that $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$ is not definably Woodin over $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$. Furthermore we have that

$$\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \vDash \text{“}i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta}) \text{ is Woodin”}.$$

Concerning the other side of the coiteration we also have that $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta}) = \delta(\bar{\mathcal{T}})$ is a Woodin cardinal in $\bar{\mathcal{Q}}$ but it is not definably Woodin over $\bar{\mathcal{Q}}$.
Since the coiteration of $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ with $\bar{\mathcal{Q}}$ takes place above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta}) = \delta(\bar{\mathcal{T}})$ we have that it is successful inside $Lp^{2n-1}(z)$ using that $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ inherits the realization strategy for H above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$ and it follows that in fact

$$\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} = \bar{\mathcal{Q}}.$$

Consider the statement

$$\psi(\mathcal{T}^*, \bar{\mathcal{Q}}) \equiv \quad \text{“there is a cofinal branch } b \text{ through } \mathcal{T}^* \text{ such that } \bar{\mathcal{Q}} = \mathcal{M}_b^{\mathcal{T}^*}\text{”}.$$

This statement $\psi(\mathcal{T}^*, \bar{\mathcal{Q}})$ is Σ^1_1-definable from the parameters $\mathcal{T}^*$ and $\bar{\mathcal{Q}}$ and holds in the model $Lp^{2n-1}(z)$ as witnessed by the branch $\bar{b}$. We have that $\bar{\mathcal{T}}, \bar{\mathcal{P}} \in H$ and thus $\mathcal{T}^* \in H$.
Therefore an absoluteness argument exactly as in Case 1 yields that $\psi(\mathcal{T}^*, \bar{\mathcal{Q}})$ holds in $H^{\mathrm{Col}(\omega,\eta)}$, where $\eta < H \cap \mathrm{Ord}$ is an ordinal such that $\mathcal{T}^*, \bar{\mathcal{Q}} \in H^{\mathrm{Col}(\omega,\eta)}$ are countable inside the model $H^{\mathrm{Col}(\omega,\eta)}$. Thus it follows as before that $\bar{b} \in H$, which contradicts the fact that $\bar{\mathcal{T}}$ witnesses in H that $\bar{K}$ is not $\mathcal{Q}$-structure iterable.

Case 2.2.2. We have that

$$\mathcal{P} \cap \mathrm{Ord} = Lp^{2n-1}(z) \cap \mathrm{Ord}\,.$$

Then it follows that $\bar{\mathcal{P}} \cap \mathrm{Ord} = H \cap \mathrm{Ord}$ and Lemma 1.5 in [SchSt09] applied to the maximal $\mathcal{P}$-construction inside H yields that $\bar{\delta}$ is not definably Woodin over $\bar{\mathcal{P}}$ since we have that $\rho_\omega(H) = \omega$. As $\bar{b}$ is non-dropping, this implies that $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$ is not definably Woodin over $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$.
So as in Case 2.2.1 we can successfully coiterate $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ and $\bar{\mathcal{Q}}$ inside $Lp^{2n-1}(z)$ and obtain again that

$$\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} = \bar{\mathcal{Q}}.$$

Then it follows that

$$\bar{\mathcal{Q}} \cap \mathrm{Ord} < H \cap \mathrm{Ord} = \bar{\mathcal{P}} \cap \mathrm{Ord} \leq \mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \cap \mathrm{Ord},$$

where the first inequality holds true since $\bar{\mathcal{Q}} \in H$. This is a contradiction to the fact that $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} = \bar{\mathcal{Q}}$.
This finishes the proof of Subclaim 5. □

Working in $Lp^{2n-1}(z)$ we consider the coiteration of $K^{M_x}|\omega_2^{Lp^{2n-1}(z)}$ with $(K^c|\omega_2)^{Lp^{2n-1}(z)}$. From what we proved so far it easily follows that this coiteration is successful as shown in the next subclaim.

SUBCLAIM 6. *The coiteration of* $K^{M_x}|\omega_2^{Lp^{2n-1}(z)}$ *with* $(K^c|\omega_2)^{Lp^{2n-1}(z)}$ *in* $Lp^{2n-1}(z)$ *is successful.*

PROOF. Write $W = Lp^{2n-1}(z)$. Then we can successfully coiterate the premice $K^{M_x}|\omega_2^W$ and $K^c|\omega_2^W$ inside the model W since $K^{M_x}|\omega_2^W = K^{M_x}|(\gamma^+)^{M_x}$ is fully iterable in W by Subclaim 3 and $K^c|\omega_2^W$ is $\mathcal{Q}$-structure iterable in W by Subclaim 5. In particular the $K^{M_x}|\omega_2^W$-side of the coiteration provides $\mathcal{Q}$-structures for the $K^c|\omega_2^W$-side and therefore the coiteration is successful. □

In what follows we want to argue that the $(K^c|\omega_2)^{Lp^{2n-1}(z)}$-side cannot lose this coiteration. For that we want to use the following lemma, which we can prove similar as Theorem 3.8 in [MSch04]. As the version we aim to use is slightly stronger than what is shown in [MSch04], we will sketch a proof of this lemma.

LEMMA 3.7.11. *Let* $\kappa \geq \omega_2$ *be a regular cardinal such that* κ *is inaccessible in* K^c *(constructed in the sense of [MSch04]). Then* $K^c|\kappa$ *is universal with respect to every premouse* M *with* $M \cap \mathrm{Ord} = \kappa$ *to which it can be successfully coiterated.*

In [MSch04] the universality of the premouse $K^c|\kappa$ is only proved with respect to smaller premice, i.e. premice M such that $M \cap \mathrm{Ord} < \kappa$, to which $K^c|\kappa$ can be successfully coiterated. As shown below, their argument can easily be modified to yield Lemma 3.7.11.

PROOF. Let $\kappa \geq \omega_2$ be a regular cardinal such that κ is inaccessible in K^c and write $N = K^c|\kappa$.

Analogous to the notation in [MSch04] we say that M *iterates past* N iff M is a premouse with $M \cap \mathrm{Ord} = \kappa$ and there are iteration trees $\mathcal{T}$ on M and $\mathcal{U}$ on N of length $\lambda + 1$ arising from a successful comparsion such that there is no drop along $[0, \lambda]_U$ and $\mathcal{M}_\lambda^\mathcal{U} \lhd \mathcal{M}_\lambda^\mathcal{T}$ or $\mathcal{M}_\lambda^\mathcal{U} = \mathcal{M}_\lambda^\mathcal{T}$ and we in addition have that there is a drop along $[0, \lambda]_T$.

Assume toward a contradiction that there is a premouse M which iterates past N and let $\mathcal{T}$ and $\mathcal{U}$ be iteration trees of length $\lambda + 1$ on M and N respectively witnessing this. Let $i_{0\lambda}^\mathcal{U} : N \to \mathcal{M}_\lambda^\mathcal{U}$ denote the corresponding iteration embedding on the N-side, which exists as there is no drop on the main branch through $\mathcal{U}$.

We distinguish the following cases.

Case 1. We have that for some $\xi < \kappa$,

$$i_{0\lambda}^{\mathcal{U}}\text{"}\kappa \subset \kappa \text{ and } i_{\beta\lambda}^{\mathcal{T}}(\xi) \geq \kappa$$

for some ordinal $\beta < \lambda$ such that the iteration embedding $i_{\beta\lambda}^{\mathcal{T}}$ is defined.

In this case we can derive a contradiction as in Section 3 in [MSch04] because we can prove that the consequences of Lemma 3.5 in [MSch04] also hold in this setting.

We assume that the reader is familiar with the argument for Lemma 3.5 in [MSch04] and we use the notation from there. So we let

$$X \prec H_{\kappa^+}$$

be such that $|X| < \kappa$, $X \cap \kappa \in \kappa$, $\{M, N, \mathcal{T}, \mathcal{U}, \beta, \xi\} \subset X$, and $X \cap \kappa \in (0, \kappa)_T \cap (0, \kappa)_U$, as in this case $\lambda = \kappa$. We write $\alpha = X \cap \kappa$. Let

$$\pi : \bar{H} \cong X \prec H_{\kappa^+}$$

be such that $\bar{H}$ is transitive. Then we have that $\alpha = \text{crit}(\pi)$ and $\pi(\alpha) = \kappa$. Let $\bar{\mathcal{T}}, \bar{\mathcal{U}} \in \bar{H}$ be such that $\pi(\bar{\mathcal{T}}, \bar{\mathcal{U}}) = (\mathcal{T}, \mathcal{U})$.

As in the proof of Lemma 3.5 in [MSch04] we aim to show that $\mathcal{P}(\alpha) \cap N \subset \bar{H}$. Exactly as in [MSch04] we get that

$$\mathcal{M}_\alpha^{\mathcal{T}} || (\alpha^+)^{\mathcal{M}_\alpha^{\mathcal{T}}} = \mathcal{M}_\alpha^{\mathcal{U}} || (\alpha^+)^{\mathcal{M}_\alpha^{\mathcal{U}}}.$$

The case assumption that $i_{\beta\kappa}^{\mathcal{T}}(\xi) \geq \kappa$ for ordinals $\beta, \xi < \kappa$ implies that there are ordinals $\bar{\beta}, \bar{\xi} < \alpha$ such that $i_{\bar{\beta}\alpha}^{\bar{\mathcal{T}}}(\bar{\xi}) \geq \alpha$. As $i_{\bar{\beta}\alpha}^{\bar{\mathcal{T}}} \upharpoonright \alpha = i_{\bar{\beta}\alpha}^{\mathcal{T}} \upharpoonright \alpha$ and $\mathcal{M}_\alpha^{\bar{\mathcal{T}}} | \alpha = \mathcal{M}_\alpha^{\mathcal{T}} | \alpha$, this yields that $\mathcal{P}(\alpha) \cap \mathcal{M}_\alpha^{\mathcal{T}} = \mathcal{P}(\alpha) \cap \mathcal{M}_\alpha^{\bar{\mathcal{T}}}$. Therefore we have that

$$\mathcal{P}(\alpha) \cap \mathcal{M}_\alpha^{\mathcal{U}} = \mathcal{P}(\alpha) \cap \mathcal{M}_\alpha^{\mathcal{T}} = \mathcal{P}(\alpha) \cap \mathcal{M}_\alpha^{\bar{\mathcal{T}}} \in \bar{H}.$$

Moreover we are assuming that $i_{0\kappa}^{\mathcal{U}}\text{"}\kappa \subset \kappa$ and we have

$$i_{0\alpha}^{\mathcal{U}} \upharpoonright \alpha = i_{0\alpha}^{\bar{\mathcal{U}}} \upharpoonright \alpha \in \bar{H}.$$

Therefore we can again argue exactly as in the proof of Lemma 3.5 in [MSch04] to get that $\mathcal{P}(\alpha) \cap N \subset \bar{H}$. Following [MSch04] this now yields a contradiction to the assumption that M iterates past N.

Case 2. We have that

$$i_{0\lambda}^{\mathcal{U}}\text{"}\kappa \subset \kappa \text{ and there are no } \xi, \beta \text{ such that } i_{\beta\lambda}^{\mathcal{T}}(\xi) \geq \kappa,$$

where $\xi < \kappa$ and $\beta < \lambda$.

By assumption we have that κ is inaccessible in N. Assume first that M has a largest cardinal η. In this case we have that there are no cardinals between the image of η under the iteration embedding and κ in $\mathcal{M}_\lambda^{\mathcal{T}}$ but there are cardinals between the image of η under the iteration embedding

and κ in $\mathcal{M}_\lambda^\mathcal{U}$. This contradicts the fact that $\mathcal{T}$ and $\mathcal{U}$ were obtained by a successful comparison of M and N with $\mathcal{M}_\lambda^\mathcal{U} \trianglelefteq \mathcal{M}_\lambda^\mathcal{T}$.
Now assume that $M \vDash$ ZFC. Then the case assumption implies that in particular $\mathcal{M}_\lambda^\mathcal{T} \cap \mathrm{Ord} \leq \kappa$. This contradicts the assumption that M iterates past N.

Case 3. We have that for some $\zeta < \kappa$,

$$i_{0\lambda}^\mathcal{U}(\zeta) \geq \kappa \text{ and there are no } \xi, \beta \text{ such that } i_{\beta\lambda}^\mathcal{T}(\xi) \geq \kappa,$$

where $\xi < \kappa$ and $\beta < \lambda$.

This again easily contradicts the assumption that M iterates past N.

Case 4. We have that for some $\zeta < \kappa$ and for some $\xi < \kappa$,

$$i_{0\lambda}^\mathcal{U}(\zeta) \geq \kappa \text{ and } i_{\beta\lambda}^\mathcal{T}(\xi) \geq \kappa$$

for some ordinal $\beta < \lambda$ such that the iteration embedding $i_{\beta\lambda}^\mathcal{T}$ is defined.

In this case we have that $\lambda = \kappa$. Let

$$X \prec H_\theta$$

for some large enough ordinal θ be such that $|X| = \kappa$ and $\{M, N, \mathcal{T}, \mathcal{U}\} \subset X$. Let

$$\pi : \bar{H} \cong X \prec H_\theta$$

be such that $\bar{H}$ is transitive.
Then a reflection argument as in the proof of the Comparison Lemma (see Theorem 3.11 in [St10]) yields that there is an ordinal $\gamma < \kappa$ such that the embeddings $i_{\gamma\kappa}^\mathcal{T}$ and $i_{\gamma\kappa}^\mathcal{U}$ agree. This implies that there are extenders $E_\alpha^\mathcal{T}$ used in $\mathcal{T}$ at stage α and $E_{\alpha'}^\mathcal{U}$ used in $\mathcal{U}$ at stage α' which are compatible. Again as in the proof of Theorem 3.11 in [St10] this yields a contradiction.
This finishes the proof of Lemma 3.7.11. □

Now we can use Lemma 3.7.11 to prove the following subclaim.

Subclaim 7. *$\omega_2^{Lp^{2n-1}(z)}$ is a successor cardinal in $(K^c)^{Lp^{2n-1}(z)}$.*

Proof. Work in $W = Lp^{2n-1}(z)$ and assume the converse. That means we are assuming that ω_2^W is inaccessible in K^c.
As above consider the successful coiteration of $K^{M_x}|\omega_2^W$ with $K^c|\omega_2^W$ and let $\mathcal{T}$ and $\mathcal{U}$ be the resulting trees on $K^{M_x}|\omega_2^W$ and $K^c|\omega_2^W$ respectively of length $\lambda + 1$ for some ordinal λ. Since we assume that ω_2^W is inaccessible in K^c, it follows by Lemma 3.7.11 that $K^c|\omega_2^W$ is universal in W for the coiteration with premice of height $\leq \omega_2^W$. Therefore the $K^c|\omega_2^W$-side has to win the comparison. That means we have that $\mathcal{M}_\lambda^\mathcal{T} \trianglelefteq \mathcal{M}_\lambda^\mathcal{U}$ and there is no drop on the main branch through $\mathcal{M}_\lambda^\mathcal{T}$. In particular the iteration

embedding
$$i^{\mathcal{T}}_{0\lambda} : K^{M_x}|\omega_2^W \rightarrow \mathcal{M}^{\mathcal{T}}_{\lambda}$$
exists. Now we distinguish the following cases.

Case 1. We have that
$$i^{\mathcal{T}}_{0\lambda}\text{"}\omega_2^W \subset \omega_2^W.$$

This means in particular that $(\gamma^+)^{K^{M_x}} = (\gamma^+)^{M_x} = \omega_2^W$ stays a successor cardinal in the model $\mathcal{M}^{\mathcal{T}}_{\lambda}$. So say that we have $(\eta^+)^{\mathcal{M}^{\mathcal{T}}_{\lambda}} = \omega_2^W$ for some cardinal $\eta < \omega_2^W$ in $\mathcal{M}^{\mathcal{T}}_{\lambda}$. In particular this means that there are no cardinals between η and ω_2^W in $\mathcal{M}^{\mathcal{T}}_{\lambda}$. But by assumption we have that ω_2^W is inaccessible in K^c and thus also in $\mathcal{M}^{\mathcal{U}}_{\lambda}$. In particular there are cardinals between η and ω_2^W in $\mathcal{M}^{\mathcal{U}}_{\lambda}$. This contradicts the fact that $\mathcal{M}^{\mathcal{T}}_{\lambda}$ and $\mathcal{M}^{\mathcal{U}}_{\lambda}$ were obtained by a successful comparison with $\mathcal{M}^{\mathcal{T}}_{\lambda} \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\lambda}$.

Case 2. We have that
$$i^{\mathcal{T}}_{0\lambda}\text{"}\omega_2^W \not\subset \omega_2^W.$$

In this case we distinguish two subcases as follows.

Case 2.1. We have that
$$\sup i^{\mathcal{T}}_{0\lambda}\text{"}\gamma < \omega_2^W.$$

In this case we have that
$$(i^{\mathcal{T}}_{0\lambda}(\gamma)^+)^{\mathcal{M}^{\mathcal{T}}_{\lambda}} = \omega_2^W,$$
because we are also assuming that $i^{\mathcal{T}}_{0\lambda}\text{"}\omega_2^W \not\subset \omega_2^W$.
So in particular we again have that ω_2^W is a successor cardinal in the model $\mathcal{M}^{\mathcal{T}}_{\lambda}$ and so there are no cardinals between $i^{\mathcal{T}}_{0\lambda}(\gamma) < \omega_2^W$ and ω_2^W in $\mathcal{M}^{\mathcal{T}}_{\lambda}$. From this we can derive the same contradiction as in Case 1, because ω_2^W is inaccessible in K^c.

Case 2.2. We have that
$$\exists \eta < \omega_2^W \text{ such that } i^{\mathcal{T}}_{0\lambda}(\eta) \geq \omega_2^W.$$

Let $\alpha < \lambda$ be the least ordinal such that the iteration embedding $i^{\mathcal{U}}_{\alpha\lambda}$ is defined. That means the last drop on the main branch in $\mathcal{U}$ is at stage α. Since $K^{M_x}|\omega_2^W$ has height ω_2^W we have by universality of $K^c|\omega_2^W$ in W (see Lemma 3.7.11) that the case assumption implies that there exists an ordinal $\nu < \omega_2^W$ such that
$$i^{\mathcal{U}}_{\alpha\lambda}(\nu) \geq \omega_2^W.$$
Moreover in this case we have in fact $\lambda = \omega_2^W$.

Let $X \prec W|\theta$ for some large ordinal θ be such that $\mathcal{T}, \mathcal{U} \in X$ and $|X| = \omega_2^W$. Moreover let H be the Mostowski collapse of X and let $\pi : H \to W|\theta$ be the uncollapse map. Then a reflection argument as in the proof of the Comparison Lemma (see Theorem 3.11 in [St10]) yields that there is an ordinal $\xi < \lambda$ such that the embeddings $i^{\mathcal{T}}_{\xi\lambda}$ and $i^{\mathcal{U}}_{\xi\lambda}$ agree. This implies that there are extenders $E^{\mathcal{T}}_{\beta}$ used in $\mathcal{T}$ at stage β and $E^{\mathcal{U}}_{\beta'}$ used in $\mathcal{U}$ at stage β' which are compatible. Again as in the proof of Theorem 3.11 in [St10] this yields a contradiction.

This finishes the proof of Subclaim 7. □

Recall that we have

$$\mathrm{HOD}^{Lp^{2n-1}(z)} \vDash \text{“}\omega_2^{Lp^{2n-1}(z)} \text{ is inaccessible”}$$

by Theorem 3.7.7 as $z \geq_T x$. Since we have that

$$(K^c)^{Lp^{2n-1}(z)} \subset \mathrm{HOD}^{Lp^{2n-1}(z)}$$

by Subclaim 4, this contradicts Subclaim 7 and thereby finishes the proof of Claim 3. □

Work in V now and let $x \in {}^\omega\omega$ be arbitrary in the cone of reals from Theorem 3.7.7. Then by Claim 3 we have that

$$M_x \vDash \text{“}(M_{2n}^\#)^{M_x} \text{ is } \omega_1\text{-iterable”}.$$

Hence

$$M_x \vDash \text{“}(M_{2n}^\#)^{M_x} \text{ is } \Pi^1_{2n+2}\text{-iterable”}.$$

Since M_x is $\boldsymbol{\Sigma}^1_{2n+2}$-correct in V we have that

$$V \vDash \text{“}(M_{2n}^\#)^{M_x} \text{ is } \Pi^1_{2n+2}\text{-iterable”}.$$

By $\boldsymbol{\Sigma}^1_{2n+2}$-correctness in V again we have for every real $y \geq_T x$ such that in particular $(M_{2n}^\#)^{M_x} \in M_y$ that

$$M_y \vDash \text{“}(M_{2n}^\#)^{M_x} \text{ is } \Pi^1_{2n+2}\text{-iterable”}.$$

Consider the comparison of the premice $(M_{2n}^\#)^{M_x}$ and $(M_{2n}^\#)^{M_y}$ inside the model M_y. This comparison is successful by Lemma 2.2.9 for all reals $y \geq_T x$ as above, since $(M_{2n}^\#)^{M_y}$ is ω_1-iterable in M_y and $(M_{2n}^\#)^{M_x}$ is Π^1_{2n+2}-iterable in M_y. Moreover both premice are ω-sound and we have that $\rho_\omega((M_{2n}^\#)^{M_x}) = \rho_\omega((M_{2n}^\#)^{M_y}) = \omega$. Thus the premice $(M_{2n}^\#)^{M_x}$ and $(M_{2n}^\#)^{M_y}$ are in fact equal.

Therefore we have that all premice $(M_{2n}^\#)^{M_x}$ for $x \in {}^\omega\omega$ in the cone of reals from Theorem 3.7.7 are equal in V. Call this unique premouse $M_{2n}^\#$.

We now finally show that this premouse $M_{2n}^\#$ is ω_1-iterable in V via the $\mathcal{Q}$-structure iteration strategy (see Definition 2.2.2). So let $\mathcal{T}$ be an iteration tree on $M_{2n}^\#$ in V of limit length $< \omega_1^V$ according to the $\mathcal{Q}$-structure iteration strategy. Pick $z \in {}^\omega\omega$ such that $M_{2n}^\#$ and $\mathcal{T}$ are in M_z and $\mathrm{lh}(\mathcal{T}) < \omega_1^{M_z}$.

Since M_z is $\mathbf{\Sigma}^1_{2n+2}$-correct in V we have that $\mathcal{T}$ is according to the $\mathcal{Q}$-structure iteration strategy in M_z, because $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ is $2n$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$ for all limit ordinals $\lambda < \mathrm{lh}(\mathcal{T})$ and therefore Π^1_{2n+1}-iterability above $\delta(\mathcal{T} \upharpoonright \lambda)$ is enough for $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ to determine a unique cofinal well-founded branch b through $\mathcal{T}$ by Lemma 2.2.10. Moreover we have that $(M_{2n}^{\#})^{M_z} = M_{2n}^{\#}$ and therefore

$$M_z \vDash \text{“}M_{2n}^{\#} \text{ is } \omega_1\text{-iterable”}.$$

So in M_z there exists a cofinal well-founded branch b through $\mathcal{T}$, which is determined by $\mathcal{Q}$-structures $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ which are ω_1-iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$ and therefore also Π^1_{2n+1}-iterable above $\delta(\mathcal{T} \upharpoonright \lambda)$ in M_z for all limit ordinals $\lambda \leq \mathrm{lh}(\mathcal{T})$. That means we in particular have that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$. Since M_z is $\mathbf{\Sigma}^1_{2n+2}$-correct in V, it follows as above that b is also the unique cofinal well-founded branch in V which is determined by the same $\mathcal{Q}$-structures as in M_z. Therefore

$$V \vDash \text{“}M_{2n}^{\#} \text{ exists and is } \omega_1\text{-iterable”}$$

and we finished the proof of Theorem 3.7.1. □

CHAPTER 4

Conclusion

By the results proved in Sections 3.6 and 3.7 together with some arguments mentioned in Section 3.4 we have that the following theorem which is due to Itay Neeman and W. Hugh Woodin and was announced in Section 3.3 holds true.

THEOREM 3.3.1. *Let $n \geq 1$ and assume there is no $\boldsymbol{\Sigma}^1_{n+2}$-definable ω_1-sequence of pairwise distinct reals. Then the following are equivalent.*

(1) $\boldsymbol{\Pi}^1_n$ determinacy and Π^1_{n+1} determinacy,
(2) for all $x \in {}^\omega\omega$, $M^\#_{n-1}(x)$ exists and is ω_1-iterable, and $M^\#_n$ exists and is ω_1-iterable,
(3) $M^\#_n$ exists and is ω_1-iterable.

PROOF. This follows from Theorems 3.6.1 and 3.7.1 together with Theorem 2.14 in [Ne02] as in the proof of Theorem 3.4.1. □

Moreover Theorems 3.6.1 and 3.7.1 together with Lemma 3.5.1 immediately imply the following main theorem due to W. Hugh Woodin.

THEOREM 2.1.1. *Let $n \geq 1$ and assume $\boldsymbol{\Pi}^1_{n+1}$ determinacy holds. Then $M^\#_n(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$.*

4.1. Applications

From these results we can now obtain a boldface version of the Determinacy Transfer Theorem as in Theorem 3.6.2 for all projective levels $\boldsymbol{\Pi}^1_{n+1}$ of determinacy. The lightface version of the Determinacy Transfer Theorem for even levels of projective determinacy Π^1_{2n} (see Theorem 3.6.2) is used in the proof of Theorem 3.6.1, and therefore in the proof of Theorem 2.1.1 for odd levels n.

COROLLARY 4.1.1 (Determinacy Transfer Theorem). *Let $n \geq 1$. Then $\boldsymbol{\Pi}^1_{n+1}$ determinacy is equivalent to $\Game^{(n)}(< \omega^2 - \boldsymbol{\Pi}^1_1)$ determinacy.*

PROOF. By Theorem 1.10 in [KW08] we have for the even levels that

$$\mathrm{Det}(\boldsymbol{\Pi}^1_{2n}) \leftrightarrow \mathrm{Det}(\boldsymbol{\Delta}^1_{2n}) \leftrightarrow \mathrm{Det}(\Game^{(2n-1)}(< \omega^2 - \boldsymbol{\Pi}^1_1)).$$

Here the first equivalence is due to Martin (see [Ma73]) and proven in Theorem 5.1 in [KS85]. The second equivalence due to Kechris and Woodin can be proven using purely descriptive set theoretic methods (see Theorem 1.10 in [KW08]).

The results in this part of this thesis (using this version of the Determinacy Transfer Theorem for even levels) together with results due to Itay Neeman yield the Determinacy Transfer Theorem for all levels n as follows.

By basic facts about the game quantifier "$\Game$" (see the proof of Lemma 3.4.2) we have that

$$\mathrm{Det}(\Game^{(n)}(< \omega^2 - \mathbf{\Pi}^1_1))$$

implies $\mathbf{\Pi}^1_{n+1}$ determinacy.

For the other direction assume that $\mathbf{\Pi}^1_{n+1}$ determinacy holds. Then Theorem 2.1.1 yields that the premouse $M_n^\#(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$. This implies that

$$\mathrm{Det}(\Game^{(n)}(< \omega^2 - \mathbf{\Pi}^1_1))$$

holds true by Theorem 2.5 in [Ne95]. □

4.2. Open problems

We close Part 1 of this thesis with the following open problem, which is the lightface version of Theorem 2.1.1.

Conjecture. *Let $n > 1$ and assume that $\mathbf{\Pi}^1_n$ determinacy and Π^1_{n+1} determinacy hold. Then $M_n^\#$ exists and is ω_1-iterable.*

This conjecture holds true for $n = 0$ which is due to L. Harrington (see [Ha78]) and for $n = 1$ which is due to W. H. Woodin (see Corollary 4.17 in [StW16]), but it is open for $n > 1$.

Part 2

Beyond Projective

Overview

In this second part of the thesis we want to prove a generalization of results from Part 1 to certain sets of reals in $L(\mathbb{R})$. The results we are going to prove will in fact be more general, because they hold for an arbitrary adequate pointclass Γ which is $\mathbb{R}$-parametrized and has the scale property and a premouse $\mathcal{N}$ which has an iteration strategy $\Sigma \in \forall^{\mathbb{N}}\Gamma$ which condenses well. We consider sets $A_i \in \mathcal{P}(\mathbb{R})$ for $i < \omega$ such that the pair $(\mathcal{N}, \Sigma)$ captures every individual A_i and show that the following theorem holds for Γ and Σ as above, where the pointclass $\mathbf{\Pi}^1_{2k+1}\Gamma$ is as defined in Definition 6.1.2.

THEOREM 7.3.1. *Let $k < \omega$ and assume that every $\mathbf{\Sigma}^1_{2k+2}\Gamma$-definable set of reals is determined. Moreover assume that there is no $\mathbf{\Sigma}^1_{2k+4}\Gamma$-definable ω_1-sequence of pairwise distinct reals. Then the Σ-premouse $M_k^{\Sigma,\#}$ exists and is ω_1-iterable.*

From this we will obtain the following corollary.

COROLLARY 7.3.2. *Let $k < \omega$ and assume that every $\mathbf{\Pi}^1_{2k+3}\Gamma$-definable set of reals is determined. Then the Σ-premouse $M_k^{\Sigma,\#}(x)$ exists and is ω_1-iterable for all reals x.*

We want to apply these results to the following setting.
Let A be a set of reals such that $A \in \mathbf{\Sigma}_n(J_\beta(\mathbb{R}))$, where $[\alpha, \beta]$ is a weak Σ_1-gap and $n < \omega$ is least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$. So we have that $A = \bigcup_{i<\omega} A_i$ for sets $A_i \in J_\beta(\mathbb{R})$ by [St08].
Then, by results of Chapter 5 in [SchSt] due to W. H. Woodin, we obtain a premouse $\mathcal{N}$ and an iteration strategy Σ for $\mathcal{N}$ to which we can apply Theorem 7.3.1. This will yield the following result in the $L(\mathbb{R})$-hierarchy.

THEOREM 8.3.2. *Let $\alpha < \beta$ be ordinals such that $[\alpha, \beta]$ is a weak Σ_1-gap, let $k \geq 0$, and let*

$$A \in \Gamma = \mathbf{\Sigma}_n(J_\beta(\mathbb{R})) \cap \mathcal{P}(\mathbb{R}),$$

where $n < \omega$ is the least natural number such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$. Moreover assume that every $\mathbf{\Pi}^1_{2k+5}\Gamma$-definable set of reals is determined. Then there exists an ω_1-iterable hybrid Σ-premouse $\mathcal{N}$ which captures every set of reals in the pointclass $\Sigma^1_k(A)$ or $\Pi^1_k(A)$.

Here $\Sigma^1_n(A)$ and $\Pi^1_n(A)$ for $n < \omega$ and a set of reals A denote the set of all reals which are definable over the model $(V_{\omega+1}, \in, A)$ by a Σ_n- or a Π_n-formula respectively.

Outline. This second part of the thesis is organized as follows. Chapter 5 contains a short introduction to hybrid mice. In particular we introduce the hybrid premouse $M_k^{\Sigma,\#}$ and state some related results. In Section 5.3 we introduce capturing and show how hybrid mice can be used to capture certain sets of reals.

The main result of this part is proven in Chapters 6 and 7. We will introduce the necessary properties a pointclass needs to have for our argument in Section 6.1 and then prove the main result (see Theorem 7.3.1) for such an abstract pointclass Γ.

Chapter 6 is devoted to the construction of a model with Woodin cardinals from determinacy hypotheses. The model we will construct there will be used in Chapter 7 to construct the hybrid premouse $M_k^{\Sigma,\#}$ from determinacy hypotheses.

The main difference between Chapter 6 and Chapter 2 in the first part of this thesis is the following: Recall that in Chapter 2 we proved (under a suitable determinacy hypothesis) that OD-determinacy holds in the premouse $M_{n-1}(x)|\delta_x$. Here we will first construct a non fine-structural model which we call k-rich in Section 6.3 and prove in Section 6.5 (under a suitable determinacy hypothesis) that OD-determinacy holds in this k-rich model. From this we will afterwards provide the basis to construct a hybrid premouse with one Woodin cardinal inside this k-rich model in Section 6.6.

In Chapter 7 we will then prove that the hybrid premouse $M_k^{\Sigma,\#}$ exists and is ω_1-iterable from a suitable determinacy hypothesis. For this purpose we will first introduce canonical hybrid premice in Section 7.1 which we call (A, k)-suitable. Then we will use the results from Chapter 6 to show in Section 7.2 that such canonical hybrid premice exist under certain determinacy hypotheses. In Section 7.3 we will finally conclude, analogous to Section 3.4 in the first part of this thesis, that $M_k^{\Sigma,\#}$ exists and is ω_1-iterable under a suitable determinacy hypothesis.

In Chapter 8 we will outline applications of our result to the setting of the core model induction technique. In Section 8.1 we first give an introduction to the $L(\mathbb{R})$-hierarchy and the relevant results concerning scales in $L(\mathbb{R})$ from [St08]. In Section 8.2 we will outline how the results in Chapter 5 in [SchSt] due to W. H. Woodin can be connected to our setting. Finally we join everything together in Section 8.3.

We close this thesis with mentioning some related open problems concerning the connection between sets of reals in the $L(\mathbb{R})$-hierarchy and inner models in Section 8.4.

CHAPTER 5

Hybrid Mice

This chapter is devoted to an introduction to hybrid mice and an outline of their basic properties. In the projective hierarchy it was enough to consider countable mice with finitely many Woodin cardinals, but at certain levels of the $L(\mathbb{R})$ hierarchy we will look at more complicated mice, namely hybrid mice, to be able to capture more complicated sets of reals.

We will start with an introduction to hybrid mice and outline that a lot of constructions using mice generalize to this hybrid context. Then we will sketch how hybrid mice can be used to capture sets of reals.

5.1. Introduction

In this section we will introduce the concept of hybrid mice. The whole presentation will be very sketchy and we will not provide any proofs, because these details will mostly not be relevant in the following chapters. Most of the details can for example be found in [SchSt] which our presentation will follow or in the more recent revised write-up in [SchT].

Informally hybrid mice are like ordinary mice, but equipped with an iteration strategy Σ for a countable premouse $\mathcal{N}$ inside them. To ensure that these hybrid mice behave nicely for example in terms of fine structure, we have to consider iteration strategies Σ which condense well. The following definitions are reformulations of Definitions 5.3.6 and 5.3.7 in [SchSt].

Definition 5.1.1. *Let $\mathcal{T}$ and $\mathcal{U}$ be iteration trees on a premouse $\mathcal{N}$. We say that $\mathcal{U}$* is a hull of $\mathcal{T}$ *iff the following holds true. Let*

$$\sigma : \mathrm{lh}(\mathcal{U}) \to \mathrm{lh}(\mathcal{T})$$

be an order preserving map such that $\mathrm{ran}(\sigma)$ *is support-closed. Then we have that $\mathcal{U}$ is the unique iteration tree on $\mathcal{N}$ such that for all $\gamma < \mathrm{lh}(\mathcal{U})$ there are maps*

$$\pi_\gamma : \mathcal{M}_\gamma^{\mathcal{U}} \to \mathcal{M}_{\sigma(\gamma)}^{\mathcal{T}},$$

which are commuting with the tree embeddings such that $\pi_0 = \mathrm{id}$ *and for all $\gamma < \mathrm{lh}(\mathcal{U})$, we have that*

$$\pi_\gamma(E_\gamma^{\mathcal{U}}) = E_{\sigma(\gamma)}^{\mathcal{T}},$$

and the map $\pi_{\gamma+1}$ is determined by the shift lemma.

Definition 5.1.2. *Let Σ be an iteration strategy for a premouse $\mathcal{N}$. Then we say that Σ* condenses well *iff for every iteration tree $\mathcal{T}$ on $\mathcal{N}$ according to Σ, we have that whenever $\mathcal{U}$ is a hull of $\mathcal{T}$ then $\mathcal{U}$ is also according to the iteration strategy Σ.*

Let $\mathcal{N}$ be a countable premouse and assume that Σ is a possibly partial iteration strategy for $\mathcal{N}$ which condenses well. We just informally define that a *(hybrid) Σ-premouse* is a premouse M which is constructed above $\mathcal{N}$ and while extenders are added to the extender sequence of M we are closing the model additionally under the iteration strategy Σ for $\mathcal{N}$ in a way that preserves the usual fine structural properties for premice. The exact way this can be done is not relevant for our purposes so we refer the interested reader to Section 5.6 in [SchSt] and Section 3 in [SchT].

In informal discussions and whenever it is clear from the context we might omit the reference to the iteration strategy Σ when referring to hybrid Σ-premice if this does not lead to any confusions. Moreover we will tacitly assume that Σ is an iteration strategy which condenses well whenever we consider hybrid Σ-premice.

Premice which are like the ones we considered in Part 1 of this thesis (so which are in particular not hybrid) we call *pure* or *ordinary* premice.

An important reason why we restrict ourselves to iteration strategies Σ which condense well is the following lemma, which is Lemma 5.6.5 in [SchSt].

Lemma 5.1.3. *Let $\mathcal{N}$ be a countable premouse and let Σ be an iteration strategy for $\mathcal{N}$ which condenses well. Moreover let M be a Σ-premouse. Suppose that*

$$\pi : \bar{M} \to M$$

is a sufficiently elementary embedding such that $\pi \upharpoonright (\mathcal{N} \cup \{\mathcal{N}\}) = id$ and $\bar{M}$ is transitive. Then $\bar{M}$ is a Σ-premouse.

A proof of this lemma can be found in [SchSt].

5.2. $K^{c,\Sigma}$ and $M_k^{\Sigma,\#}$

We will use this section to generalize standard concepts like small premice and core models to the context of hybrid mice, again without giving too many details.

Definition 5.2.1. *Let Σ be an iteration strategy for a countable premouse $\mathcal{N}$ and let M be a hybrid Σ-premouse. Then for $k < \omega$ we say that M* is (k, Σ)-small *iff whenever κ is the critical point of an extender on the M-sequence, then*

$$M|\kappa \nvDash \text{“there are } k \text{ Woodin cardinals”}.$$

Analogous to the projective case we can now define the Σ-premice $M_k^{\Sigma,\#}$ and M_k^{Σ}.

Whenever we say that a Σ-premouse M is ω_1-iterable, we mean that there exists an iteration strategy for trees of length $< \omega_1$ on M such that every iterate according to this iteration strategy is again a Σ-premouse.

DEFINITION 5.2.2. *Let $k \geq 1$ and let Σ be an iteration strategy for a countable premouse $\mathcal{N}$. Then we let $M_k^{\Sigma,\#}$ denote the unique countable, sound, ω_1-iterable Σ-premouse (constructed above the premouse $\mathcal{N}$) which is not (k,Σ)-small, but all of whose proper initial segments are (k,Σ)-small, if it exists.*

DEFINITION 5.2.3. *Let $k \geq 1$ and assume that the Σ-premouse $M_k^{\Sigma,\#}$ exists for an iteration strategy Σ for a countable premouse $\mathcal{N}$ as above. Then M_k^{Σ} is the unique premouse which is obtained from $M_k^{\Sigma,\#}$ by iterating its top measure out of the universe.*

We sometimes write $M_k^{\Sigma,\#}(\mathcal{N})$ and $M_k^{\Sigma}(\mathcal{N})$ for $M_k^{\Sigma,\#}$ and M_k^{Σ} to visualize the premouse $\mathcal{N}$ for which Σ is an iteration strategy.

For a real x the (Σ, x)-premice $M_k^{\Sigma,\#}(x)$ and $M_k^{\Sigma}(x)$ relativized to x are defined similarly. Sometimes we also write $M_k^{\Sigma,\#}(\mathcal{N}, x)$ and $M_k^{\Sigma}(\mathcal{N}, x)$ for them to again visualize the fixed premouse $\mathcal{N}$ for which Σ is an iteration strategy.

Let Σ be an iteration strategy for a countable premouse $\mathcal{N}$ and let x be a real. Then we can define a $K^{c,\Sigma}(x)$*-construction* above x and $\mathcal{N}$ as in Definition 1.3.10 in [SchSt] as a generalization of the standard K^c-construction from [St96]. Moreover we remark without further proof that the results of Jensen and Steel in [JS13] concerning the construction of a core model generalize to the context of hybrid mice. Thus the following theorem and in particular the subsequent remark follow from Theorem 1.3.20 in [SchSt] and Theorem 1.1 in [JS13].

THEOREM 5.2.4 (Core Model Existence Dichotomy). *Let $k \geq 1$. Let Σ be an iteration strategy for a countable premouse $\mathcal{N}$ and suppose that for all $x \in {}^\omega\omega$ the (Σ, x)-premouse $M_k^{\Sigma,\#}(x)$ exists and is (ω_1+1)-iterable. Then exactly one of the following holds.*

(1) *For all $x \in {}^\omega\omega$ the (Σ, x)-premouse $M_{k+1}^{\Sigma,\#}(x)$ exists and is (ω_1+1)-iterable, or*
(2) *for some $x \in {}^\omega\omega$, $K^{c,\Sigma}(x)$ is (k,Σ)-small, has no Woodin cardinals, and is (ω_1+1)-iterable.*

REMARK. In alternative (2) of the Core Model Existence Dichotomy in fact the hybrid core model $K^{\Sigma}(x)$ exists, where $K^{\Sigma}(x)$ is a generalization of the core model $K(x)$ at the level of finitely many Woodin cardinals in the fashion of [St96] and [JS13] to the hybrid context. This follows from the appropriate generalization of Theorem 1.1 in [JS13].

DEFINITION 5.2.5. *We define an $L^{\Sigma}(x)$*-construction *as a special case of a $K^{c,\Sigma}(x)$-construction where no extenders are added to the sequence.*

Moreover we let $M_0^\Sigma(x) = L^\Sigma(x)$ *and we let* $M_0^{\Sigma,\#}(x)$ *denote the least active* (Σ, x)*-premouse. In particular we say that a* Σ*-premouse is* $(0, \Sigma)$*-small iff it is an initial segment of the model* L^Σ*.*

5.3. Capturing Sets of Reals with Hybrid Mice

Our main interest lies in hybrid mice which capture certain sets of reals. We first define what we mean by "capturing" analogous to Section 1.4 in [SchSt], starting with some preliminary definitions.

DEFINITION 5.3.1. *Let* $\mathcal{N}$ *be an ordinary or a hybrid premouse and let* Σ *be an iteration strategy for* $\mathcal{N}$*. Moreover let* δ *be a cardinal in* $\mathcal{N}$*. Then we say that the pair* $(\mathcal{N}, \Sigma)$ absorbs reals at δ *iff for every ordinal* $\eta < \delta$ *and for every real* x*, whenever* $\mathcal{T}$ *is an iteration tree based on* $\mathcal{N}|\eta$ *by the iteration strategy* Σ *which does not drop on the main branch and* $i : \mathcal{N} \to \mathcal{N}^*$ *is the corresponding iteration embedding, then there exists an iteration tree* $\mathcal{U}$ *based on* $\mathcal{N}^*|i(\delta)$ *such that*

(1) *all critical points of extenders used in* $\mathcal{U}$ *are strictly above* $i(\eta)$*,*
(2) $\mathcal{U}$ *gives rise to an iteration map* $j : \mathcal{N}^* \to \mathcal{N}^{**}$*, so in particular* $\mathcal{U}$ *does not drop on the main branch, and*
(3) $x \in \mathcal{N}^{**}[g]$*, for some* $\mathrm{Col}(\omega, j(i(\delta)))$*-generic* g *over* $\mathcal{N}^{**}$*.*

We have that premice do absorb reals at some cardinal δ if δ is a Woodin cardinal and they are sufficiently iterable, using Woodin's or Neeman's genericity iteration (see Theorem 7.14 in [St10] or Corollary 1.8 in [Ne95]).

DEFINITION 5.3.2. *Let* A *be a set of reals and let* $\mathcal{N}$ *be an ordinary or a hybrid premouse with an iteration strategy* Σ *for* $\mathcal{N}$*. Moreover let* δ *be a cardinal in* $\mathcal{N}$*.*

(1) *Let* $\tau \in \mathcal{N}$ *be a term. Then we say that* τ *is an* $(\mathcal{N}, \Sigma)$-term for A at δ *iff whenever* $\mathcal{T}$ *is an iteration tree based on* $\mathcal{N}|\delta$ *by the iteration strategy* Σ *which does not drop on the main branch,* $i : \mathcal{N} \to \mathcal{N}^*$ *is the corresponding iteration embedding and* g *is* $\mathrm{Col}(\omega, i(\delta))$*-generic over* $\mathcal{N}^*$*, then*
$$i(\tau)^g = A \cap \mathcal{N}^*[g].$$
(2) *We say that the pair* $(\mathcal{N}, \Sigma)$ understands A at δ *iff there exists an* $(\mathcal{N}, \Sigma)$*-term* τ *for* A *at* δ*.*

Now we can finally define when a pair $(\mathcal{N}, \Sigma)$ captures a set of reals A.

DEFINITION 5.3.3. *Let* A *be a set of reals and let* $\mathcal{N}$ *be an ordinary or a hybrid premouse with an iteration strategy* Σ *for* $\mathcal{N}$*. Moreover let* δ *be a cardinal in* $\mathcal{N}$*. Then we say that the pair* $(\mathcal{N}, \Sigma)$ captures A at δ *iff there is an* $(\mathcal{N}, \Sigma)$*-term* τ *for* A *at* δ *and* $(\mathcal{N}, \Sigma)$ *absorbs reals at* δ*.*

REMARK. In case there exists a unique iteration strategy Σ for a premouse $\mathcal{N}$ or if we already fixed an iteration strategy for $\mathcal{N}$, we might omit the reference to the iteration strategy Σ in the definition of capturing as long as this does not lead to any confusion, i.e. we will just say that $\mathcal{N}$ captures A at δ.

The following lemma and its corollary now suggest why we are considering the Σ-premouse $M_k^{\Sigma,\#}$ if we are searching for mice capturing certain sets of reals.

LEMMA 5.3.4. *Let $A \subset \mathbb{R}\times\mathbb{R}$ and let $\mathcal{N}$ be an ordinary or a hybrid premouse with an iteration strategy Σ for $\mathcal{N}$ and cardinals $\eta < \delta$ in $\mathcal{N}$. Suppose that the pair $(\mathcal{N}, \Sigma)$ captures the set A at δ. Then $(\mathcal{N}, \Sigma)$ understands $\exists^{\mathbb{R}} A$ and $\forall^{\mathbb{R}} A$ at η.*

This lemma is proven in [SchSt] as Lemma 1.4.18 and it yields the following corollary.

COROLLARY 5.3.5. *Let $n \geq 0$ and let $A = \bigcup_{k\in\omega} A_k$ be a set of reals. Moreover let $\mathcal{N}$ be an ordinary or a hybrid countable premouse with an iteration strategy Σ for $\mathcal{N}$ (which condenses well) and a cardinal δ such that for each $k < \omega$ the pair $(\mathcal{N}, \Sigma)$ captures A_k at δ. Then the hybrid Σ-premouse $M_{n+1}^{\Sigma,\#}(\mathcal{N})$ constructed above $\mathcal{N}$ captures every set of reals which is in the pointclass $\Sigma^1_n(A)$ or $\Pi^1_n(A)$ at its bottom Woodin cardinal.*

Here we denote by $\Sigma^1_n(A)$ and $\Pi^1_n(A)$ for $n < \omega$ and a set of reals A the set of all reals which are definable over the model $(V_{\omega+1}, \in, A)$ by a Σ_n- or a Π_n-formula respectively.

PROOF OF COROLLARY 5.3.5. The argument for Claim 2 in the proof of Lemma 5.6.8 in [SchSt] shows that $M_{n+1}^{\Sigma,\#}(\mathcal{N})$ captures A at its top Woodin cardinal. So Corollary 5.3.5 follows from Lemma 5.3.4. $\square$

We will outline in Section 8.2 how such premice $\mathcal{N}$ as in the statement of Corollary 5.3.5 can be obtained in the setting of the core model induction technique, i.e. at the end of Σ_1-gaps in the $L(\mathbb{R})$-hierarchy, from results in Chapter 5 in [SchSt] due to W. Hugh Woodin.

CHAPTER 6

A Model with Woodin Cardinals from Determinacy Hypotheses

In this chapter we will provide the basis to construct hybrid models with finitely many Woodin cardinals from determinacy assumptions. The amount of determinacy we are going to assume varies from section to section, so we will always point out how much determinacy we need to assume to prove a particular result.

6.1. Introduction

We aim to prove the results in this and the following chapter in a general context such that they can be used for different applications. So we will isolate properties for a pointclass Γ under which we are able to prove the results. The reader can always imagine this pointclass to be for example $\Gamma = \boldsymbol{\Sigma}_n(J_\beta(\mathbb{R})) \cap \mathcal{P}(\mathbb{R})$ as defined in [St08] for a weak Σ_1-gap $[\alpha, \beta]$ and $n < \omega$ least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$, which is one of the pointclasses used in the core model induction technique. In what follows we might confuse $\boldsymbol{\Sigma}_n(J_\beta(\mathbb{R})) \cap \mathcal{P}(\mathbb{R})$ with $\boldsymbol{\Sigma}_n(J_\beta(\mathbb{R}))$ for example whenever we refer to "the pointclass $\boldsymbol{\Sigma}_n(J_\beta(\mathbb{R}))$". We will explain the connection of our results to these pointclasses and the core model induction technique in more detail in Chapter 8.

We start with some preliminary definitions which introduce properties we want the pointclass Γ to satisfy.

Definition 6.1.1. *A pointclass Λ is called* $\mathbb{R}$-parametrized *iff there exists some set $U \subset \mathbb{R} \times \mathbb{R}$, such that $U \in \Lambda$ and U is universal for Λ. That means we have $U \in \Lambda$ and U satisfies that for all sets $A \subseteq \mathbb{R}$,*

$$A \in \Lambda \textit{ iff } \exists y \in \mathbb{R}(A = U_y),$$

where $U_y = \{x \in \mathbb{R} \mid (y, x) \in U\}$.

Definition 6.1.2. *Let Γ be an $\mathbb{R}$-parametrized pointclass and let $n \geq 1$. Then we say a set of reals A is in the pointclass $\Pi^1_n\Gamma$ iff there is a set of reals B in Γ and a Π^1_n-formula φ such that for all $x \in {}^\omega\omega$,*

$$x \in A \textit{ iff } \varphi(x, B, {}^\omega\omega \setminus B).$$

Here we mean by "$\varphi(x, B, {}^\omega\omega \setminus B)$" that the parameter B is allowed to occur positively and negatively in the formula φ.

Furthermore we say a set of reals A is in the pointclass $\boldsymbol{\Pi}^1_n\Gamma$ iff there is a set of reals B in Γ, a real y and a Π^1_n-formula φ such that for all $x \in {}^\omega\omega$,

$$x \in A \textit{ iff } \varphi(x, y, B, {}^\omega\omega \setminus B).$$

The pointclasses $\Sigma^1_n\Gamma$, $\boldsymbol{\Sigma}^1_n\Gamma$, $\Delta^1_n\Gamma$ and $\boldsymbol{\Delta}^1_n\Gamma$ are defined in the same way.

We now define the notion of *scale* which was introduced by Moschovakis in [Mo71]. A more detailed presentation of scales and their basic properties can be found in Section 3 of [KM08].

DEFINITION 6.1.3. *A* norm *on a set of reals A is a function $\varphi : A \twoheadrightarrow \gamma$ from A onto some ordinal γ, which we call the* length of φ.

DEFINITION 6.1.4. *A* scale *on a set of reals A is a sequence of norms $(\varphi_n \mid n < \omega)$ on A such that the following holds. For every sequence of reals $(x_i \mid i < \omega)$ such that for all $i < \omega$, $x_i \in A$ and*

$$\lim_{i<\omega} x_i = x,$$

and such that for each $n < \omega$ there exists an ordinal γ_n and a natural number i_0 such that for all $i \geq i_0$,

$$\varphi_n(x_i) = \gamma_n,$$

we have $x \in A$ and for each n, $\varphi_n(x) \leq \gamma_n$.

DEFINITION 6.1.5. *Let Λ be a pointclass. We say a scale $(\varphi_n \mid n < \omega)$ on a set of reals A is a* Λ-scale *iff there are relations R and R' in Λ and $\neg\Lambda$ respectively such that for every $a \in A$ we have that for every real x and every $n < \omega$,*

$$[x \in A \wedge \varphi_n(x) \leq \varphi_n(a)] \textit{ iff } R(n, x, a) \textit{ iff } R'(n, x, a).$$

DEFINITION 6.1.6. *We say a pointclass Λ has the* scale property *iff every set A in Λ admits a Λ-scale.*

We will join all properties we want a typical pointclass Γ we consider to satisfy in the following definition.

DEFINITION 6.1.7. *Let $\mathcal{N}$ be a (possibly hybrid) countable premouse, which is ω_1-iterable as witnessed by an iteration strategy Σ which can be coded by a set of reals. Futhermore assume that Σ condenses well. Then we say that a pointclass Γ is* $(\mathcal{N}, \Sigma)$-apt *iff it satisfies the following properties.*

(*i*) $\Sigma \in \forall^{\mathbb{N}}\Gamma$,
(*ii*) *Γ is $\mathbb{R}$-parametrized,*
(*iii*) *Γ is adequate in the sense of [Mo09],*
(*iv*) *Γ has the scale property.*

REMARK. Assume that the pointclass Γ is $\mathbb{R}$-parametrized. This implies that the pointclasses $\Pi^1_{2k+1}\Gamma$, $\boldsymbol{\Pi}^1_{2k+1}\Gamma$, $\Sigma^1_{2k+2}\Gamma$ and $\boldsymbol{\Sigma}^1_{2k+2}\Gamma$ for $k < \omega$ are all $\mathbb{R}$-parametrized (see Theorem 1D.2 in [Mo09]). Further note that this implies the existence of a universal $\Pi^1_{2k+1}\Gamma$-definable set (or $\Sigma^1_{2k+2}\Gamma$-definable set) for the pointclass $\Pi^1_{2k+1}\Gamma(\mathbb{R})$ (or $\Sigma^1_{2k+2}\Gamma(\mathbb{R})$, respectively), where we additionally allow real parameters.

Remark. Properties (*iii*) and (*iv*) in Definition 6.1.7 already imply that for all $k < \omega$ assuming that every $\boldsymbol{\Delta}^1_{2k}\Gamma$-definable set of reals is determined, the pointclasses $\Pi^1_{2k+1}\Gamma$, $\boldsymbol{\Pi}^1_{2k+1}\Gamma$, $\Sigma^1_{2k+2}\Gamma$ and $\boldsymbol{\Sigma}^1_{2k+2}\Gamma$ all have the uniformization property by Moschovakis' Second Periodicity Theorem (see Theorem $6C.3$ in [Mo09]).

Remark. We will show in Theorem 8.2.2 that for appropriate sets A we can construct $\mathcal{N}$ and Σ as above with the property that there is a pointclass Γ such that Γ is $(\mathcal{N}, \Sigma)$-apt and the pair $(\mathcal{N}, \Sigma)$ captures the set A using results in Chapter 5 in [SchSt] due to W. Hugh Woodin.

For the rest of this chapter fix for all $i < \omega$ sets $A_i \in \mathcal{P}(\mathbb{R})$, a pointclass Γ, a (possibly hybrid) premouse $\mathcal{N}$ and an ω_1-iteration strategy Σ for $\mathcal{N}$ which condenses well and witnesses that $(\mathcal{N}, \Sigma)$ captures every set A_i at the same cardinal δ, such that the pointclass Γ is $(\mathcal{N}, \Sigma)$-apt.

The proof of Theorem 7.3.1 (see overview at the beginning of this part of this thesis or Section 7.3) which we present in this and the following chapter of this thesis will be organized inductively. Therefore we will assume throughout the rest of this and the following chapter that Theorem 7.3.1 holds for $k - 1$.

6.2. A Consequence of Determinacy

In what follows we will need the following consequence of determinacy which is an analogue of Lemma 3.5.1.

Theorem 6.2.1. *Let $k < \omega$ and let $\mathcal{N}$ be a countable premouse and Σ an ω_1-iteration strategy for $\mathcal{N}$ which condenses well. Moreover let Γ be a pointclass which is $(\mathcal{N}, \Sigma)$-apt. Assume that every $\boldsymbol{\Pi}^1_{2k+3}\Gamma$-definable set of reals is determined. Then there exists no uncountable $\boldsymbol{\Sigma}^1_{2k+4}\Gamma$-definable sequence of pairwise distinct reals.*

Remark. The statement "there exists no uncountable $\boldsymbol{\Sigma}^1_{2k+4}\Gamma$-definable sequence of pairwise distinct reals" is defined analogous to Part 1 (see remark after the statement of Theorem 3.3.2).

Proof of Theorem 6.2.1. The proof of this theorem is analogous to the descriptive set theoretic argument for the odd levels of the projective hierarchy in the proof of Lemma 3.5.1. We will present it here to convince the reader, that it also works in this setting with some minor adjustments. Note that the inner model theoretic argument we gave for the even levels of the projective hierarchy in the proof of Lemma 3.5.1 does not work here, because at the moment we are not able to construct hybrid premice which are suitable enough to apply this argument from the amount of determinacy we are assuming.

Assume toward a contradiction that there exists a $\boldsymbol{\Sigma}^1_{2k+4}\Gamma$-definable ω_1-sequence of pairwise distinct reals. That means in particular that there exists a well-order $\leq^*$ of ordertype ω_1 for reals such that if we let $X_{\leq^*} = field(\leq^*)$, that means if we have for all $y \in {}^\omega\omega$,

$$y \in X_{\leq^*} \;\Leftrightarrow\; \exists x\, (x \leq^* y \vee y \leq^* x),$$

then there exists a $\boldsymbol{\Sigma}^1_{2k+4}\Gamma$-definable relation R such that we have for all $x, y \in {}^\omega\omega$,

$$R(x,y) \;\Leftrightarrow\; x, y \in X_{\leq^*} \;\wedge\; x \leq^* y.$$

Let A be a $\boldsymbol{\Pi}^1_{2k+3}\Gamma$-definable relation such that we have for all $x, y \in {}^\omega\omega$,

$$R(x,y) \;\Leftrightarrow\; \exists z\, A(z,x,y).$$

Moreover consider the relation A_2 such that for all $u \in ({}^\omega\omega)^2$ and for all $y \in {}^\omega\omega$,

$$A_2(y,u) \;\Leftrightarrow\; A((u)_0,(u)_1,y).$$

Recall that we fixed Γ such that the pointclass $\boldsymbol{\Pi}^1_{2k+3}\Gamma$ has the uniformization property. Therefore there exists a $\boldsymbol{\Pi}^1_{2k+3}\Gamma$-definable function F which uniformizes the set A_2, that means we have

$$(y, F(y)) \in A_2$$

for all $y \in \mathrm{dom}(F)$. We have that $X_{\leq^*} \subseteq \mathrm{dom}(F)$. So the relation A^* defined by

$$A^*(z,x,y,u) \;\Leftrightarrow\; A(z,x,y) \wedge z = (u)_0 \wedge x = (u)_1 \wedge u = F(y)$$

is $\boldsymbol{\Pi}^1_{2k+3}\Gamma$-definable. Let A^* denote the corresponding set of all tuples (z,x,y,u) such that $A^*(z,x,y,u)$ holds.

As before we now consider the following game $G^p(A^*)$ which is due to M. Davis (see [Da64] or Theorem 12.11 in [Sch14]). We again identify the elements of the set A^* with elements of the Cantor space ${}^\omega 2$.

I	s_0		s_1		$\dots$	
II		n_0		n_1		$\dots$

Player I plays finite $0-1$-sequences $s_i \in {}^{<\omega}2$ (allowing $s_i = \emptyset$), player II responds with $n_i \in \{0,1\}$ and the game lasts ω steps. We say player I wins the game $G^p(A^*)$ iff

$$s_0{}^\frown n_0{}^\frown s_1{}^\frown n_1{}^\frown \dots \in A^*.$$

Otherwise player II wins. As a consequence of our determinacy hypothesis we have that the game $G^p(A^*)$ is determined, because we may code $G^p(A^*)$ into a Gale-Stewart game $G(A')$ for some set of reals A' which is $\boldsymbol{\Pi}^1_{2k+3}\Gamma$-definable.

Then we can argue again as in the proof Theorem 12.11 in [Sch14] to obtain that the set A^* has a perfect subset, because A^* is uncountable. As in the proof of Lemma 3.5.1 this yields that we have the following claim.

Claim 1. *The set*

$$B = \{y \mid \exists z \exists x \exists u \; A^*(z,x,y,u)\}$$

has a perfect subset.

Therefore there exists a continuous function $f : \mathbb{R} \to B$ and we can consider the following order $\leq$ on the reals. We say for two reals x and y that

$$x \leq y \Leftrightarrow f(x) \leq^* f(y).$$

Then we again have the following claim.

Claim 2. *The order $\leq$ is a $\boldsymbol{\Sigma}^1_{2k+4}\Gamma$-definable well-order of the reals.*

We can now use this claim to prove the following claim exactly as in the proof of Lemma 3.5.1 with Bernstein's argument, so we omit the proof here.

Claim 3. *There are two disjoint $\boldsymbol{\Sigma}^1_{2k+4}\Gamma$-definable sets D and D' of size $\aleph_1$ without perfect subsets.*

Let E be a $\boldsymbol{\Pi}^1_{2k+3}\Gamma$-definable set such that

$$x \in D \;\Leftrightarrow\; \exists y \; (x,y) \in E.$$

Since we assumed that $\boldsymbol{\Pi}^1_{2k+3}\Gamma$-uniformization holds, there exists a $\boldsymbol{\Pi}^1_{2k+3}\Gamma$-definable partial function $C : \mathbb{R} \to \mathbb{R}$ uniformizing E, that means we have that

$$\exists y \, (x,y) \in E \Leftrightarrow (x, C(x)) \in E.$$

The set C' which is defined as

$$C' = \{(x, C(x)) \mid x \in D\} = C \cap E$$

is $\boldsymbol{\Pi}^1_{2k+3}\Gamma$-definable and consider the Davis game $G^p(C')$, which is again determined by assumption.

As above this yields that C' has a perfect subset $P \subset C'$, because D and thus C' is uncountable. Since P is a perfect set and D does not contain a perfect subset, there are reals x, y and y' such that $y \neq y'$ and $(x,y) \in P$ and $(x,y') \in P$. But we have $y = C(x) = y'$. This is a contradiction. □

6.3. The Construction of a k-rich Model

In this section we are going to construct a model $P^k(x;\Sigma)$ for which we can prove a version of Lemma 2.1.3. We will construct this model in a way that makes it suitable for our later analysis. In particular we will be able to obtain a hybrid premouse from it which is short tree iterable in an adapted sense. Moreover the construction will enable us to construct canonical hybrid premice with Woodin cardinals.

Fix an arbitrary real x and recall that we fixed a premouse $\mathcal{N}$ together with the pointclass Γ. Fix a natural number $k \geq 1$ and suppose in the following construction of the model $P^k(x;\Sigma)$ that every $\boldsymbol{\Delta}^1_{2k}\Gamma$-definable set of reals is determined.

The model $P^k(x;\Sigma)$ is build level-by-level in a construction of length ω_1^V. That means we build a chain of models $(P^*_\alpha \mid \alpha \leq \omega_1^V)$ and let

$$P^k(x;\Sigma) = P^*_{\omega_1^V} = \bigcup_{\alpha<\omega_1^V} P^*_\alpha.$$

We are starting from $P^*_0 = \{x, \mathcal{N}\}$ and are taking unions at limit steps of the construction. That means for a limit ordinal $\lambda \leq \omega_1^V$ we let

$$P^*_\lambda = \bigcup_{\alpha<\lambda} P^*_\alpha.$$

For the construction of the successor steps we first introduce the following notation.

DEFINITION 6.3.1. *Let $n \geq 1$. Then we say that a model P is* $\mathbf{\Sigma}^\Sigma_n$-correct in *V and write $P \prec_{\mathbf{\Sigma}^\Sigma_n} V$, if Σ is a predicate over P and for all Σ^1_n-formulas ψ and for all $y \in P \cap {}^\omega\omega$,*

$$P \vDash \psi(y, \Sigma, {}^\omega\omega \setminus \Sigma) \textit{ iff } V \vDash \psi(y, \Sigma, {}^\omega\omega \setminus \Sigma).$$

By this notation we mean that we identify the parameter Σ with a set of reals and allow Σ to occur positively and negatively in the formula ψ.

For the successor steps $\alpha + n$ in the construction of $P^k(x;\Sigma)$ we consider different cases for a limit ordinal α (or $\alpha = 0$) and each natural number n. At the first successor level $\alpha+1$ we are going to ensure that $P^k(x;\Sigma)$ will be closed under the iteration strategy Σ for $\mathcal{N}$. In the next successor step $\alpha+2$ we uniformly close $P^*_{\alpha+2}$ under Skolem functions to obtain that $P^k(x;\Sigma)$ is $\mathbf{\Sigma}^\Sigma_{2k+2}$-correct in V. Then we use the successor levels of the form $\alpha + 3$ to add witnesses for the fact that $P^k(x;\Sigma)$ will be $\mathbf{\Sigma}^\Sigma_{2k+2}$-absolute. That means we ensure that $P^k(x;\Sigma)[g]$ will be $\mathbf{\Sigma}^\Sigma_{2k+2}$-correct in V for comeager many $\mathrm{Col}(\omega,\eta)$-generic $g \in V$ over $P^k(x;\Sigma)$ for any ordinal $\eta < \omega_1^V$.

Moreover we want to obtain the additional property that for a comeager set of reals $y \in V$ which are $\mathrm{Col}(\omega,\eta)$-generic over the model $P^k(x;\Sigma)$ for an ordinal $\eta < \omega_1^V$, we have that

$$P^k(x;\Sigma)[y] = P^k(x \oplus y;\Sigma).$$

Here we mean by this equality that the two models have the same universe (and not necessarily the same hierarchies).

To ensure this property we will use the other successor steps $\alpha+n$ for $n \geq 4$ to recursively close $P^k(x;\Sigma)$ under "names for names" in a uniform way. This will be done for names for reals which are added to levels like $P^*_{\alpha+2}$ and for names for branches through iteration trees on $\mathcal{N}$ which are added to levels like $P^*_{\alpha+1}$.

We will also recursively define the order of construction for $P^k(x;\Sigma)$ along the way. At limit steps of the construction we define the order of construction analogous to the order of construction for L.

Furthermore we are, analogously to the constructions in Chapter 3, going to ensure during the construction that each level of the model $P^k(x;\Sigma)$ is $\Sigma^1_{2k+2}\Gamma$-definable uniformly in codes for $\mathcal{N}$ and the real x. This will enable us to prove in Lemma 6.3.4 that $P^k(x;\Sigma)$ is a model of ZFC from a determinacy hypothesis.

We have that $P^k(x;\Sigma)$ is a non fine structural model, but we define $P^k(x;\Sigma)$ such that it will be closed under the iteration strategy Σ for the premouse $\mathcal{N} \in P^k(x;\Sigma)$.

Remark. The successor levels $(\alpha+1)$ and $(\alpha+2)$ in our construction are not necessary, because they are also included in the level $(\alpha+3)$ if we consider a trivial forcing. We added them to the construction as a warm up for the construction at level $(\alpha+3)$ to illustrate what happens in this special case.

Before we are going to describe the construction at the successor levels in more detail we fix a $\Pi^1_{2k+1}\Gamma$-definable set U which is universal for $\mathbf{\Pi}^1_{2k+1}\Gamma$-definable sets of reals in V. We can pick this set U such that we have

$$U_{\ulcorner\varphi\urcorner\frown(a\oplus b)} = A^\Gamma_{\varphi,a,b}$$

for every Π^1_{2k+1}-formula φ and every $a,b \in {}^\omega\omega$, where $\ulcorner\varphi\urcorner$ denotes the Gödel number of the formula φ,

$$A^\Gamma_{\varphi,a,b} = \{x \mid \varphi(x,a,U^\Gamma_b, {}^\omega\omega \setminus U^\Gamma_b)\},$$

where as above this means that U^Γ_b is allowed to occur positively and negatively in φ, and $U^\Gamma \in \Gamma$ is a universal set for the pointclass Γ which exists since we assume that Γ is $\mathbb{R}$-parametrized.

Moreover we fix a $\Pi^1_{2k+1}\Gamma$-definable uniformizing function F for U. That means for all $z \in \operatorname{dom}(F)$ we have that

$$(z, F(z)) \in U,$$

where $\operatorname{dom}(F) = \{z \mid \exists y\,(z,y) \in U\}$. We have that U and F as above exist by our assumptions on Γ (see second remark after Definition 6.1.7).

We start the construction with letting $P^*_0 = \{x, \mathcal{N}\}$ and are now going to describe the construction at the successor levels in more detail.

Level $\alpha+1$**:** Assume inductively that we already constructed the model P^*_α for a limit ordinal $\alpha < \omega_1^V$ or $\alpha = 0$. Then we first construct the level $(\alpha+1)$ of $P^k(x;\Sigma)$, in which we close under the iteration strategy Σ for $\mathcal{N}$. Recall that we have $\mathcal{N} \in P^*_\alpha$ for all $\alpha < \omega_1^V$ since we picked $P^*_0 = \{x, \mathcal{N}\}$.

Assume now that there is an iteration tree $\mathcal{T}$ on $\mathcal{N}$ of limit length $< \omega_1^V$ such that $\mathcal{T} \in P^*_\alpha$, but $\Sigma(\mathcal{T})$ is not in P^*_α. If there is no such tree we just let $P^*_{\alpha+1} = rud(P^*_\alpha)$. If there exists such a tree $\mathcal{T}$, we additionally add the pair $(\mathcal{T}, \Sigma(\mathcal{T}))$ to the model $P^*_{\alpha+1}$ for all such trees $\mathcal{T} \in P^*_\alpha$ before we close under rudimentary functions.

More precisely we add the set

$$\begin{aligned}(\mathcal{T}, \{(\xi, h) \mid h \in \{0,1\}, \xi < \mathrm{lh}(\mathcal{T}), \text{ and} \\ h = 1 \text{ and } \xi \in \Sigma(\mathcal{T}) \text{ or} \\ h = 0 \text{ and } \xi \notin \Sigma(\mathcal{T})\})\end{aligned}$$

for all iteration trees $\mathcal{T}$ on $\mathcal{N}$ in P^*_α of limit length to the model $P^*_{\alpha+1}$, where we define $\Sigma(\mathcal{T}) = \emptyset$ if $\mathcal{T}$ is not an iteration tree on $\mathcal{N}$ of limit length according to the iteration strategy Σ, before we close under rudimentary functions.

Order of construction: For two iteration trees $\mathcal{T}$ and $\mathcal{U}$ on $\mathcal{N}$ of limit length $< \omega_1^V$ such that $\mathcal{T}, \mathcal{U} \in P^*_\alpha$ and $(\mathcal{T}, \Sigma(\mathcal{T})) \neq (\mathcal{U}, \Sigma(\mathcal{U}))$ are added in the construction of $P^*_{\alpha+1}$, we say that $(\mathcal{T}, \Sigma(\mathcal{T}))$ is constructed before $(\mathcal{U}, \Sigma(\mathcal{U}))$ iff $\mathcal{T}$ is constructed before $\mathcal{U}$ in the order of construction for elements of P^*_α. For elements which are added during the closure under rudimentary functions we define the order of construction analogous to the order of construction for L.

Level $\alpha + 2$: Assume now that we already constructed the model $P^*_{\alpha+1}$. Then we close $P^*_{\alpha+1}$ under the uniformizing function F we fixed above. That means we let

$$P^*_{\alpha+2} = rud(P^*_{\alpha+1} \cup \{y \in {}^\omega\omega \mid \exists z \in P^*_{\alpha+1} \cap {}^\omega\omega \; \varphi_F(z, y, U^\Gamma_b, {}^\omega\omega \setminus U^\Gamma_b)\}),$$

where φ_F is a Π^1_{2k+1}-formula and b is a fixed real such that for all $z, y \in {}^\omega\omega$

$$F(z) = y \text{ iff } \varphi_F(z, y, U^\Gamma_b, {}^\omega\omega \setminus U^\Gamma_b).$$

This will add witnesses to ensure that $P^k(x;\Sigma)$ is $\mathbf{\Sigma}^\Sigma_{2k+2}$-correct in V as $\Sigma \in \forall^{\mathbb{N}}\Gamma$.

Order of construction: First we say that $F(z)$ is constructed before $F(z')$ for $F(z) \neq F(z')$ with $z, z' \in \mathrm{dom}(F) \cap P^*_{\alpha+1}$ if z is constructed before z' in the order of construction for elements of $P^*_{\alpha+1}$ where z and z' are the minimal (according to the order of construction in $P^*_{\alpha+1}$) reals y and y' in $P^*_{\alpha+1}$ such that $F(y) = F(z)$ and $F(y') = F(z')$. Then we define the order of construction for elements added by the closure under rudimentary functions analogous to the order of construction for L.

Level $\alpha+3$: Assume that we already constructed $P^*_{\alpha+2}$. Let η be a countable ordinal in V and let τ be a $\mathrm{Col}(\omega, \eta)$-name for a real in $P^*_{\alpha+2}$. Then we define the $\mathrm{Col}(\omega, \eta)$-name $\sigma^\eta_1(\tau)$ as follows.

$$\begin{aligned}\sigma^\eta_1(\tau) = \{((n, \check{m}), p) \mid & \text{ for comeager many } g \text{ which are} \\ & \mathrm{Col}(\omega, \eta)\text{-generic over } P^*_{\alpha+2} \text{ such that } p \in g \\ & \text{we have } \tau^g \in {}^\omega\omega \text{ and } F(\tau^g)(n) = m\}.\end{aligned}$$

We now add $\sigma^\eta_1(\tau)$ for all ordinals $\eta \in P^*_{\alpha+2}$ and for all such names τ to the model $P^*_{\alpha+3}$.

Moreover we uniformly add names for branches through iteration trees on $\mathcal{N}$ to $P^*_{\alpha+3}$ as follows.

Recall that Σ is an iteration strategy for the premouse $\mathcal{N}$ and define as above $\Sigma(\mathcal{T}) = \emptyset$ if $\mathcal{T}$ is not an iteration tree on $\mathcal{N}$ of limit length according to Σ.

Now let η again be a countable ordinal in V and let $\dot{\mathcal{T}}$ be a $\mathrm{Col}(\omega,\eta)$-name for an iteration tree $\mathcal{T}$ of limit length on $\mathcal{N}$ in $P^*_{\alpha+2}$. Then we define the $\mathrm{Col}(\omega,\eta)$-name $\sigma_1^{\eta,*}(\dot{\mathcal{T}})$ for a branch through $\mathcal{T}$ as follows.

$$\begin{aligned}\sigma_1^{\eta,*}(\dot{\mathcal{T}}) = \{((\check{\xi,h}),p) \mid \; & \text{for comeager many } g \text{ which are} \\ & \mathrm{Col}(\omega,\eta)\text{-generic over } P^*_{\alpha+2} \text{ such that } p \in g \\ & \text{we have that } h = 1 \text{ and } \xi \in \Sigma(\dot{\mathcal{T}}^g) \text{ or} \\ & \text{we have that } h = 0 \text{ and } \xi \notin \Sigma(\dot{\mathcal{T}}^g)\}.\end{aligned}$$

We now also add $\sigma_1^{\eta,*}(\dot{\mathcal{T}})$ for all ordinals $\eta \in P^*_{\alpha+2}$ and all such names $\dot{\mathcal{T}}$ for iteration trees $\mathcal{T}$ on $\mathcal{N}$ of limit length to the model $P^*_{\alpha+3}$. That means we let

$$\begin{aligned}P^*_{\alpha+3} = rud(P^*_{\alpha+2} \cup \{\sigma_1^{\eta}(\tau) \mid \tau \text{ is a } \mathrm{Col}(\omega,\eta)\text{-name for a real as above}\} \cup \\ \{\sigma_1^{\eta,*}(\dot{\mathcal{T}}) \mid \dot{\mathcal{T}} \text{ is a } \mathrm{Col}(\omega,\eta)\text{-name for an iteration tree } \mathcal{T} \text{ as above}\}).\end{aligned}$$

Order of construction: Let σ and σ' be two elements added to the model $P^*_{\alpha+3}$ during the construction. Then we say that σ is constructed before σ' if $\sigma \neq \sigma'$ and

(i) there is a $\mathrm{Col}(\omega,\eta)$-name $\tau \in P^*_{\alpha+2}$ for a real and a $\mathrm{Col}(\omega,\eta')$-name $\dot{\mathcal{T}} \in P^*_{\alpha+2}$ for an iteration tree on $\mathcal{N}$ for the largest such ordinals η, η' in $P^*_{\alpha+2}$ such that we have $\sigma = \sigma_1^{\eta}(\tau)$ and $\sigma' = \sigma_1^{\eta',*}(\dot{\mathcal{T}})$ for minimal such τ and $\dot{\mathcal{T}}$, or

(ii) there are $\mathrm{Col}(\omega,\eta)$- and $\mathrm{Col}(\omega,\eta')$-names τ and τ' in $P^*_{\alpha+2}$ for reals for the largest such ordinals η, η' in $P^*_{\alpha+2}$ such that $\sigma = \sigma_1^{\eta}(\tau)$ and $\sigma' = \sigma_1^{\eta'}(\tau')$ and $\eta < \eta'$ or $\eta = \eta'$ and τ is constructed before τ' in the order of construction for elements of $P^*_{\alpha+2}$ for minimal such τ and τ', or

(iii) there are $\mathrm{Col}(\omega,\eta)$- and $\mathrm{Col}(\omega,\eta')$-names $\dot{\mathcal{T}}$ and $\dot{\mathcal{U}}$ in $P^*_{\alpha+2}$ for iteration trees on $\mathcal{N}$ of limit length for the largest such ordinals η, η' in $P^*_{\alpha+2}$ such that $\sigma = \sigma_1^{\eta,*}(\dot{\mathcal{T}})$ and $\sigma' = \sigma_1^{\eta',*}(\dot{\mathcal{U}})$ and $\eta < \eta'$ or $\eta = \eta'$ and $\dot{\mathcal{T}}$ is constructed before $\dot{\mathcal{U}}$ in the order of construction for elements of $P^*_{\alpha+2}$ for minimal such $\dot{\mathcal{T}}$ and $\dot{\mathcal{U}}$.

For elements added to the model $P^*_{\alpha+3}$ by the closure under rudimentary functions we define the order of construction analogous to the order of construction for L.

Level $\alpha + n$: We now construct the other successor levels $P^*_{\alpha+n}$ for $n > 3$ as follows. To perform this construction in a uniform way we first make the following definition.

DEFINITION 6.3.2. *We define n-names for $n < \omega$ recursively as follows.*

(*i*) *We say that τ is a* 0-name *iff τ is a branch b through an iteration tree $\mathcal{T}$ on $\mathcal{N}$ as for example added to $P^*_{\alpha+1}$ during the construction or τ is a real $F(z)$ as for example added to $P^*_{\alpha+2}$ during the construction.*

(*ii*) *We say that τ is a* 1-name *iff τ is a* $\mathrm{Col}(\omega, \eta)$*-name for a real or a* $\mathrm{Col}(\omega, \eta)$*-name for a branch for some ordinal $\eta < \omega_1^V$ as for example added to $P^*_{\alpha+3}$ during the construction. That means*

$$\tau = \sigma_1^\eta(\tau')$$

for some $\mathrm{Col}(\omega, \eta)$*-name τ' for a real or*

$$\tau = \sigma_1^{\eta,*}(\dot{\mathcal{T}})$$

for some $\mathrm{Col}(\omega, \eta)$*-name $\dot{\mathcal{T}}$ for an iteration tree $\mathcal{T}$ on $\mathcal{N}$.*

(*iii*) *We recursively define that τ is an* $(n + 1)$-name *for some $n \geq 1$ iff τ is a* $\mathrm{Col}(\omega, \eta)$*-name for an n-name for some ordinal $\eta < \omega_1^V$ as for example added to $P^*_{\alpha+n+3}$ during the construction. That means*

$$\tau = \sigma_{n+1}^\eta(\sigma)$$

for some $\mathrm{Col}(\omega, \eta)$*-name σ for an n-name, where the operation $\sigma \mapsto \sigma_{n+1}^\eta(\sigma)$ is as defined below.*

Assume now inductively that we already defined the model $P^*_{\alpha+n+2}$ for some $n \geq 1$ and construct the next model $P^*_{\alpha+n+3}$ as follows.

Let η be a countable ordinal in V and let τ be an n-name such that for non-meager many $g \in V \cap {}^\omega\omega$ which are $\mathrm{Col}(\omega, \eta)$-generic over $P^*_{\alpha+n+2}$ we have that $\tau \in P^*_{\alpha+n+2}(x \oplus g)$, where $P^*_{\alpha+n+2}(x \oplus g)$ is the model which is obtained by the same construction as $P^*_{\alpha+n+2}$, but starting with $P^*_0(x \oplus g) = \{x \oplus g, \mathcal{N}\}$ instead of $P^*_0 = \{x, \mathcal{N}\}$. Let σ be a $\mathrm{Col}(\omega, \eta)$-name for τ in $P^*_{\alpha+n+2}(x \oplus g)$. Then we define $\sigma_{n+1}^\eta(\sigma)$ to be a canonical $\mathrm{Col}(\omega, \eta)$-name for τ, that means

$$\begin{aligned}\sigma_{n+1}^\eta(\sigma) = \{(\check{u}, p) \mid\ & \text{for comeager many } g \text{ which are} \\ & \mathrm{Col}(\omega, \eta)\text{-generic over } P^*_{\alpha+n+2} \text{ such that } p \in g \\ & \text{we have that } u \in \sigma^g\}.\end{aligned}$$

We add all such $\mathrm{Col}(\omega, \eta)$-names $\sigma_{n+1}^\eta(\sigma)$ for ordinals η in $P^*_{\alpha+n+2}$ to the model $P^*_{\alpha+n+3}$. That means we define

$$\begin{aligned}P^*_{\alpha+n+3} = rud(P^*_{\alpha+n+2} \cup \{\sigma_{n+1}^\eta(\sigma) \mid \sigma \text{ is a } & \mathrm{Col}(\omega, \eta)\text{-name} \\ & \text{for an } n\text{-name as above}\}).\end{aligned}$$

Order of construction: We say that $\sigma_{n+1}^\eta(\sigma)$ is constructed before $\sigma_{n+1}^{\eta'}(\sigma')$ for $\sigma_{n+1}^\eta(\sigma) \neq \sigma_{n+1}^{\eta'}(\sigma')$ where σ and σ' are $\mathrm{Col}(\omega, \eta)$- and $\mathrm{Col}(\omega, \eta')$-names for n-names τ and τ' as above for the largest such ordinals η, η' in $P^*_{\alpha+n+2}$ iff $\eta < \eta'$ or $\eta = \eta'$ and τ is constructed before τ' in the order of construction for elements of the model $P^*_{\alpha+n+2}(x \oplus g)$ for non-meager many $g \in V \cap {}^\omega\omega$ which are $\mathrm{Col}(\omega, \eta)$-generic over $P^*_{\alpha+n+2}$ such that $\tau, \tau' \in P^*_{\alpha+n+2}(x \oplus g)$ for minimal such τ, τ'. We define the order of construction for elements added by the closure under rudimentary functions analogous to the order of construction for L.

Limit steps: As mentioned before we take unions at limit steps of the construction, so we let

$$P^*_\lambda = \bigcup_{\gamma<\lambda} P^*_\gamma$$

for every limit ordinal $\lambda \leq \omega_1^V$ and finally

$$P^k(x;\Sigma) = P^*_{\omega_1^V} = \bigcup_{\alpha<\omega_1^V} P^*_\alpha.$$

Order of construction: The order of construction at the limit steps is defined analogous to the order of construction for L at limit steps.

So we can finally make the following definition.

Definition 6.3.3. *Let $k \geq 1$, let x be a real and let $P^k(x;\Sigma)$ be the model of height ω_1^V constructed above. Then we say that $P^k(x;\Sigma)$ is the k-*rich *$L(x;\Sigma)$*-model above *x.*

These models are called *k-rich $L(x;\Sigma)$-models* since they are constructed similar to the construction of L but we enrich them by additionally closing them under Σ, under witnesses for Skolem functions and under names for these objects at certain levels of the construction. The superscript k indicates that $P^k(x;\Sigma)$ is the model we want to use for the construction of the hybrid premouse $M_k^{\Sigma,\#}$ later. More precisely we are going to use the k-rich $L(x;\Sigma)$-model $P^k(x;\Sigma)$ for the construction of an (A,k)-suitable premouse (see Section 7.2).

To prove that $P^k(x;\Sigma)$ is a model of ZFC we need a stronger determinacy hypothesis as in the construction above, because we want to use Theorem 6.2.1. So in fact we only need to assume that there is no uncountable $\boldsymbol{\Sigma}^1_{2k+2}\Gamma$-definable sequence of pairwise distinct reals in addition to $\boldsymbol{\Pi}^1_{2k}\Gamma$ determinacy to obtain the following lemma. For simplicity we state the lemma under the slightly stonger assumption that every $\boldsymbol{\Pi}^1_{2k+1}\Gamma$-definable set of reals is determined as we need to assume this much determinacy later on anyway.

Lemma 6.3.4. *Let $k \geq 1$ and let x be a real. Assume that every $\boldsymbol{\Pi}^1_{2k+1}\Gamma$-definable set of reals is determined and let $P^k(x;\Sigma)$ be the k-rich $L(x;\Sigma)$-model as constructed above. Then*

$$P^k(x;\Sigma) \models \mathrm{ZFC}.$$

Proof. Assume this is not the case. Then the power set axiom has to fail. So let γ be a countable ordinal in V such that

$$\mathcal{P}(\gamma) \cap P^k(x;\Sigma) \notin P^k(x;\Sigma).$$

This implies that the set $\mathcal{P}(\gamma) \cap P^k(x;\Sigma)$ has size $\aleph_1$.
Let $P^*_\gamma = P^k(x;\Sigma)|\gamma$ be the γ-th level in the construction of $P^k(x;\Sigma)$. Then we can fix a real a in V which codes the countable set P^*_γ.
If it exists, we let A_ξ for $\gamma < \xi < \omega_1^V$ be the smallest subset of γ in

$$P^k(x;\Sigma)\,|\,(\xi+1) \;\setminus\; P^k(x;\Sigma)\,|\,\xi$$

according to the order of construction we defined above. Moreover we let X be the set of all ordinals ξ with $\gamma < \xi < \omega_1^V$ such that A_ξ exists. Then X is cofinal in ω_1^V.
Finally we let a_ξ be a real coding the set $A_\xi \subseteq \gamma$ relative to the code a we fixed for P^*_γ. For $\xi \in X$ we have that $A_\xi \in \mathcal{P}(\gamma) \cap P^k(x;\Sigma)$ and thus $A_\xi \subseteq P^*_\gamma$, so the canonical code a_ξ for A_ξ relative to a exists.
Now consider the following ω_1^V-sequence of reals

$$A = (a_\xi \in {}^\omega\omega \mid \xi \in X).$$

Analogous to the proof of Claim 3 in the proof of Theorem 3.4.1 we have that a real y codes an element of the model $P^k(x;\Sigma)\,|\,(\xi+1) \;\setminus\; P^k(x;\Sigma)\,|\,\xi$ for some $\xi < \omega_1^V$ iff there is a sequence of countable models $(P_\beta \mid \beta \leq \xi+1)$ and an element $Y \in P_{\xi+1}$ such that

(1) $P_0 = \{x, \mathcal{N}\}$,
(2) $P_{\beta+1}$ is constructed from P_β as described in the construction above for all $\beta \leq \xi$,
(3) $P_\lambda = \bigcup_{\beta<\lambda} P_\beta$ for all limit ordinals $\lambda \leq \xi$,
(4) y does not code an element of P_ξ, and
(5) y codes Y.

This shows that the levels $P^k(x;\Sigma)\,|\,(\xi+1) \;\setminus\; P^k(x;\Sigma)\,|\,\xi$ of the model $P^k(x;\Sigma)$ can be constructed in a $\Sigma^1_{2k+2}\Gamma$-definable way from the real x and a code for the countable premouse $\mathcal{N}$ and the countable ordinal ξ by the following argument.
Let $\alpha < \omega_1^V$ be a limit ordinal or let $\alpha = 0$. For the successor levels of the form $\alpha+1$ we can compute $\Sigma(\mathcal{T})$ from $\mathcal{T}$ in a $\forall^{\mathbb{N}}\Gamma$-definable way by our choice of the pointclass Γ.
For the successor levels of the form $\alpha+2$ recall that U is a $\Pi^1_{2k+1}\Gamma$-definable set and F is a $\Pi^1_{2k+1}\Gamma$-definable function uniformizing U. So we have that the elements of the model $P^*_{\alpha+2}$ can be computed in a $\Pi^1_{2k+1}\Gamma$-definable way from $P^*_{\alpha+1}$.

For the $P^*_{\alpha+3}$ levels of the construction, we have that for an ordinal $\eta \in P^*_{\alpha+2}$ and a $\mathrm{Col}(\omega,\eta)$-name τ for a real, the $\mathrm{Col}(\omega,\eta)$-name

$$\begin{aligned}\sigma_1^\eta(\tau) = \{((n, \check{m}), p) \mid \; & \text{for comeager many } g \text{ which are} \\ & \mathrm{Col}(\omega,\eta)\text{-generic over } P^*_{\alpha+2} \text{ such that } p \in g \\ & \text{we have } \tau^g \in {}^\omega\omega \text{ and } F(\tau^g)(n) = m\}\end{aligned}$$

is $\Pi^1_{2k+1}\Gamma$-definable from τ and $P^*_{\alpha+2}$ for the following reason:

First recall that we chose F such that it is $\Pi^1_{2k+1}\Gamma$-definable. Furthermore we can replace every comeager set by a subset which is comeager and G_δ, so in particular Borel. Hence $\sigma_1^\eta(\tau)$ is $\Pi^1_{2k+1}\Gamma$-definable from τ and $P^*_{\alpha+2}$.

For a similar reason we have that the $\mathrm{Col}(\omega,\eta)$-name $\sigma_1^{\eta,*}(\dot{\mathcal{T}})$ for a branch through $\mathcal{T}$,

$$\begin{aligned}\sigma_1^{\eta,*}(\dot{\mathcal{T}}) = \{((\xi, \check{h}), p) \mid \; & \text{for comeager many } g \text{ which are} \\ & \mathrm{Col}(\omega,\eta)\text{-generic over } P^*_{\alpha+2} \text{ such that } p \in g \\ & \text{we have that } h = 1 \text{ and } \xi \in \Sigma(\dot{\mathcal{T}}^g) \text{ or} \\ & \text{we have that } h = 0 \text{ and } \xi \notin \Sigma(\dot{\mathcal{T}}^g)\},\end{aligned}$$

is $\Pi^1_{2k+1}\Gamma$-definable from $\dot{\mathcal{T}}$ and $P^*_{\alpha+2}$. It is in fact even $\Pi^1_1\Gamma$-definable from $\dot{\mathcal{T}}$ and $P^*_{\alpha+2}$, because as already mentioned above the branch $\Sigma(\mathcal{T})$ is $\forall^{\mathbb{N}}\Gamma$-definable from $\mathcal{T}$ by our choice of the pointclass Γ.

Therefore we have that the elements of the $(\alpha+3)$-levels of $P^k(x;\Sigma)$ can be constructed in a $\Pi^1_{2k+1}\Gamma$-definable way from the previous level.

Moreover we can define the level $P^*_{\alpha+n+3}$ of $P^k(x;\Sigma)$ for a fixed $n<\omega$ in a $\Sigma^1_{2k+2}\Gamma$-definable way from $P^*_{\alpha+n+2}$ by the following argument:

Let $P^*_{\alpha+n+2}(x\oplus g)$ denote the level $\alpha+n+2$ in the construction of the model $P^k(x\oplus g;\Sigma)$ which is constructed as above but starting from $P^*_0(x\oplus g) = \{x\oplus g, \mathcal{N}\}$. What we already showed gives in particular that this model is $\Delta^1_{2k+2}\Gamma$-definable from the real $x\oplus g$ and a code for $\mathcal{N}$ as the construction of the model $P^k(x\oplus g;\Sigma)$ is defined in a unique way.

Now a name σ is added to the model $P^*_{\alpha+n+3}$ for some $n<\omega$ if for non-meager many $g \in V \cap {}^\omega\omega$ which are $\mathrm{Col}(\omega,\eta)$-generic over $P^*_{\alpha+n+2}$ for some ordinal $\eta \in P^*_{\alpha+n+2}$, there is a finite sequence of names $(\tau_i \mid 0 \leq i \leq n)$ and a finite sequence of names $(\sigma_i \mid 0 \leq i \leq n)$ such that τ_0 is a 0-name, $\tau_1 = \sigma_1^{\eta_1}(\sigma_0)$ or $\tau_1 = \sigma_1^{\eta_1,*}(\sigma_0)$ for some ordinal $\eta_1 \in P^*_{\alpha+2}$ and a $\mathrm{Col}(\omega,\eta_1)$-name σ_0, and for all $1 < i \leq n$ we have that $\tau_i \in P^*_{\alpha+i+2}(x\oplus g)$, σ_i is a $\mathrm{Col}(\omega,\eta_i)$-name for τ_i in $P^*_{\alpha+i+2}(x\oplus g)$ for some ordinal $\eta_i \in P^*_{\alpha+i+1}(x\oplus g)\cap \mathrm{Ord}$,

$$\tau_i = \sigma_i^{\eta_i}(\sigma_{i-1}),$$

and

$$\sigma = \sigma_{n+1}^\eta(\sigma_n).$$

Therefore we have as above that $P^*_{\alpha+n+3}$ is constructed in a $\Delta^1_{2k+2}\Gamma$-definable way from the previous levels.

After all it follows that the sequence A as defined above is $\Sigma^1_{2k+2}\Gamma$-definable in the parameters a, x and a code for $\mathcal{N}$. Moreover A has size $\aleph_1$ because the set $\mathcal{P}(\gamma) \cap P^k(x;\Sigma)$ has size $\aleph_1$. Hence A contradicts Theorem 6.2.1 because A is an uncountable $\boldsymbol{\Sigma}^1_{2k+2}\Gamma$-definable sequence of pairwise distinct reals. □

The following lemma summarizes some properties of the k-rich $L(x;\Sigma)$-model $P^k(x;\Sigma)$ which follow from the construction.

LEMMA 6.3.5. *Let $k \geq 1$ and let x be a real. Assume that every $\boldsymbol{\Pi}^1_{2k+1}\Gamma$-definable set of reals is determined and let $P^k(x;\Sigma)$ be the k-rich model above x as constructed above. Then $P^k(x;\Sigma)$ satisfies the following properties.*

- (*i*) *$P^k(x;\Sigma)$ is $\boldsymbol{\Sigma}^\Sigma_{2k+2}$-correct in V for real parameters from $P^k(x;\Sigma)$, that means*
$$P^k(x;\Sigma) \prec_{\boldsymbol{\Sigma}^\Sigma_{2k+2}} V,$$
- (*ii*) *$P^k(x;\Sigma)[g]$ is $\boldsymbol{\Sigma}^\Sigma_{2k+2}$-correct in V for real parameters from $P^k(x;\Sigma)[g]$ for comeager many $\mathrm{Col}(\omega,\eta)$-generic $g \in V$ over $P^k(x;\Sigma)$ for any ordinal $\eta < \omega_1^V$, that means*
$$P^k(x;\Sigma)[g] \prec_{\boldsymbol{\Sigma}^\Sigma_{2k+2}} V,$$
- (*iii*) *for comeager many reals y in V which are $\mathrm{Col}(\omega,\eta)$-generic over $P^k(x;\Sigma)$ for an ordinal $\eta < \omega_1^V$, we have that*
$$P^k(x;\Sigma)[y] = P^k(x \oplus y;\Sigma),$$
and
- (*iv*) *$P^k(x;\Sigma)$ is $\Sigma^1_{2k+2}\Gamma$-definable in the codes from x and $\mathcal{N}$.*

Moreover if $P^k(x;\Sigma)$ and $P^k(y;\Sigma)$ are both k-rich models for reals $x \leq_T y$ such that $y \in P^k(x;\Sigma)$, then $P^k(x;\Sigma) = P^k(y;\Sigma)$.

REMARK. Again by the equality
$$P^k(x;\Sigma)[y] = P^k(x \oplus y;\Sigma)$$
in property (*iii*) in Lemma 6.3.5 we do *not* mean that the hierarchies of the models $P^k(x;\Sigma)[y]$ and $P^k(x\oplus y;\Sigma)$ are equal. We only want to express that the two models have the same universes, which follows from our construction since we inductively close the models $P^*_{\alpha+n+2}$ under names for n-names for every $n < \omega$. This will suffice for our application in the proof of Theorem 6.6.2 later.

PROOF OF LEMMA 6.3.5. Property (*i*) follows from the construction exactly as in the proof of property (3) in Claim 2 in the proof of Theorem 3.7.1. Moreover property (*ii*) can be obtained from the construction of $P^k(x;\Sigma)$ at the successor levels $\alpha + 3$ by a similar argument, as follows.

First of all notice that every $g \in V$ which is $\mathrm{Col}(\omega, \eta)$-generic over $P^k(x;\Sigma)$ for an ordinal $\eta < \omega_1^V$, is also $\mathrm{Col}(\omega, \eta)$-generic over some level $P^*_{\alpha+2}$ in the construction of $P^k(x;\Sigma)$. Now the names of the form $\sigma_1^\eta(\tau)$ which are added to the model $P^*_{\alpha+3}$ provide witnesses for comeager many g which are $\mathrm{Col}(\omega, \eta)$-generic over $P^*_{\alpha+2}$. Property (ii) follows from this using that the intersection of countably many comeager sets is comeager again.

Property (iii) follows from the construction of $P^k(x;\Sigma)$ we described above, because we added names for n-names at the levels $\alpha+n$ of the construction for $n \geq 4$. We easily have that $P^k(x;\Sigma)[y] \subseteq P^k(x \oplus y;\Sigma)$. The other implication also holds true by the following argument.

For a comeager set of reals y we have that whenever for example a name τ for a real is added to the model $P^k(x \oplus y;\Sigma)$ during the construction (at some level $\alpha+2$), then in the construction of $P^k(x;\Sigma)$ a name for τ is added at some other level. Therefore it follows that $\tau \in P^k(x;\Sigma)[y]$. The same argument now applies to names for names for reals and so on, which are added to the model $P^k(x \oplus y;\Sigma)$ during the construction.

Moreover we added branches $\Sigma(\mathcal{T})$ for iteration trees $\mathcal{T}$ on $\mathcal{N}$ at levels $\alpha+1$ of the construction and names for branches $\Sigma(\dot{\mathcal{T}})$ for a name $\dot{\mathcal{T}}$ for an iteration tree at levels $\alpha+3$. As above we also closed under names for names for these branches and so on during the construction of the model $P^k(x;\Sigma)$ at the levels $\alpha+n$ for $n \geq 4$.

Therefore it follows after all that the models $P^k(x;\Sigma)[y]$ and $P^k(x \oplus y;\Sigma)$ have the same universes, as claimed in property (iii).

Finally we have that property (iv) follows immediately from the proof of Lemma 6.3.4. □

6.4. Definable Iterability and a Comparison Lemma

In this section we will prove that for certain (k,Σ)-small premice M the statement "M is ω_1-iterable" is Π^Σ_{2k+1}-definable uniformly in any code for M. This will heavily be used in correctness arguments later. Moreover we will show a version of the Comparison Lemma for certain ω_1-iterable (k,Σ)-small hybrid premice we can prove using some techniques from the previous section. At the same time we prove that $P^k(x;\Sigma)$ is closed under the operation $a \mapsto M^{\Sigma,\#}_{k-1}(a)$.

First we define the pointclasses Π^Σ_{2k+1} and $\boldsymbol{\Pi}^\Sigma_{2k+1}$ for an ω_1-iteration strategy Σ for a countable premouse $\mathcal{N}$ as fixed above.

Definition 6.4.1. *Let $k \geq 1$. Then we say a set of reals A is in the pointclass Π^Σ_k iff there is a Π^1_k-formula φ such that for all $x \in {}^\omega\omega$,*

$$x \in A \text{ iff } \varphi(x, \Sigma, {}^\omega\omega \setminus \Sigma).$$

Here we identify the ω_1-iteration strategy Σ for the countable premouse $\mathcal{N}$ with a set of reals. Moreover we as before mean by the notation "$\varphi(x, \Sigma, {}^\omega\omega \setminus$

Σ)" that the parameter Σ is allowed to occur positively and negatively in the formula φ.
Moreover we say a set of reals A is in the pointclass $\boldsymbol{\Pi}^{\Sigma}_{k}$ iff there is a real y and a Π^1_k-formula φ such that for all $x \in {}^\omega\omega$,

$$x \in A \text{ iff } \varphi(x, y, \Sigma, {}^\omega\omega \setminus \Sigma).$$

The pointclasses Σ^{Σ}_{k} and $\boldsymbol{\Sigma}^{\Sigma}_{k}$ are defined in a similar fashion.

As in the projective hierarchy we have that Σ^{Σ}_{k} and $\boldsymbol{\Sigma}^{\Sigma}_{k}$ are the dual pointclasses of Π^{Σ}_{k} and $\boldsymbol{\Pi}^{\Sigma}_{k}$ respectively. We analogously define the pointclasses Δ^{Σ}_{k} and $\boldsymbol{\Delta}^{\Sigma}_{k}$.
Now we can prove the following lemma.

LEMMA 6.4.2. *Let $k \geq 0$ and assume that every $\boldsymbol{\Pi}^1_{2k+1}\Gamma$-definable set of reals is determined.*

- (i) *Let M be a countable, $\mathcal{N}$-sound, (k, Σ)-small Σ-premouse such that $\rho_\omega(M) = \mathcal{N}$, where $\mathcal{N}$ is the fixed countable premouse such that Σ is an iteration strategy for $\mathcal{N}$ and M is constructed above $\mathcal{N}$. Then the statement "M is ω_1-iterable" is Π^{Σ}_{2k+1}-definable uniformly in any code for M (relative to Σ).*
- (ii) *For $k \geq 1$ and $x \in {}^\omega\omega$, the k-rich $L(x; \Sigma)$-model $P^k(x; \Sigma)$ is closed under the operation*
$$a \mapsto M^{\Sigma,\#}_{k-1}(a).$$
- (iii) *Let M and N be countable Σ-premice which are ω_1-iterable such that every proper initial segment of M or N is (k, Σ)-small. Moreover assume that M and N are $\mathcal{N}$-sound and that $\rho_\omega(M) = \rho_\omega(N) = \mathcal{N}$, where $\mathcal{N}$ is the fixed countable premouse such that Σ is an iteration strategy for $\mathcal{N}$ and the Σ-premice M and N are constructed above $\mathcal{N}$. Then we have that*
$$M \trianglelefteq N \text{ or } N \trianglelefteq M.$$

REMARK. Lemma 6.4.2 (i) generalizes to countable (k, Σ)-small Σ-premice M such that M is γ-sound and $\rho_\omega(M) = \gamma$ for some ordinal γ which is a cutpoint of M, if we only consider ω_1-iterability for M above γ, by the same proof as the one we will give for the case that M is $\mathcal{N}$-sound and $\rho_\omega(M) = \mathcal{N}$. A similar generalization holds for Lemma 6.4.2 (iii).

PROOF OF LEMMA 6.4.2. We proof (i), (ii) and (iii) simultaneously by an inductive argument using Lemma 6.3.5 (ii) and the inductive proof of Lemma 2.2.8 in the first part of this thesis.
For $k = 0$ we easily get that (i) holds as then the statement "M is ω_1-iterable" is Π^{Σ}_{1}-definable uniformly in any code for M (relative to Σ), because in the case that M is $(0, \Sigma)$-small, we have that in fact M is ω_1-iterable iff M is a Σ-premouse and M is well-founded. Moreover we have that (ii) is empty for $k = 0$ and (iii) also easily holds true as in this case every proper

initial segment of M or N is $(0, \Sigma)$-small and thus by definition an initial segment of L^Σ.

So let $k \geq 1$ be arbitrary and assume inductively that in particular (iii) holds for countable Σ-premice M and N such that every proper initial segment of M or N is $(k-1, \Sigma)$-small. We aim to show that (i) holds. So let M be a countable, $\mathcal{N}$-sound, (k, Σ)-small Σ-premouse such that $\rho_\omega(M) = \mathcal{N}$.

We iterate the Σ-premouse M via the $\mathcal{Q}$-structure iteration strategy, which is the straightforward generalization to the hybrid context of the iteration strategy defined in Definition 2.2.2.

It now follows inductively that the statement "M is ω_1-iterable" is Π^Σ_{2k+1}-definable uniformly in any code for M (relative to Σ), because it can be defined as follows. We first consider trees of limit length.

$\forall \mathcal{T}$ iteration tree on M of limit length $\text{lh}(\mathcal{T}) < \omega_1$ such that

$\forall \lambda < \text{lh}(\mathcal{T})$ limit, $\exists \mathcal{Q} \trianglelefteq \mathcal{M}^{\mathcal{T}}_\lambda$ such that

$\mathcal{Q}$ is $(k-1, \Sigma)$-small above $\delta(\mathcal{T} \restriction \lambda), \omega_1$-iterable above $\delta(\mathcal{T} \restriction \lambda)$,

and is a $\mathcal{Q}$-structure for $\mathcal{T} \restriction \lambda$,

$\exists b$ branch through $\mathcal{T}$ such that

$\mathcal{Q}(b, \mathcal{T})$ exists, is a $\mathcal{Q}$-structure for $\mathcal{T}$, is $(k-1, \Sigma)$-small

above $\delta(\mathcal{T})$ and ω_1-iterable above $\delta(\mathcal{T})$.

For trees of successor length we get a similar statement as follows.

$\forall \mathcal{T}$ putative iteration tree on M of successor length $\text{lh}(\mathcal{T}) < \omega_1$ such that

$\forall \lambda < \text{lh}(\mathcal{T})$ limit, $\exists \mathcal{Q} \trianglelefteq \mathcal{M}^{\mathcal{T}}_\lambda$ such that

$\mathcal{Q}$ is $(k-1, \Sigma)$-small above $\delta(\mathcal{T} \restriction \lambda), \omega_1$-iterable above $\delta(\mathcal{T} \restriction \lambda)$,

and is a $\mathcal{Q}$-structure for $\mathcal{T} \restriction \lambda$,

the last model of $\mathcal{T}$ is a well-founded Σ-premouse.

Since for $\lambda \leq \text{lh}(\mathcal{T})$ the relevant $\mathcal{Q}$-structures for $\mathcal{T} \restriction \lambda$ here are $(k - 1, \Sigma)$-small above $\delta(\mathcal{T} \restriction \lambda)$, ω_1-iterability above $\delta(\mathcal{T} \restriction \lambda)$ for them is a Π^Σ_{2k-1}-definable statement uniformly in any code by our inductive hypothesis for (i). (Here we are using the slight generalization of Lemma 6.4.2 (i) mentioned in the remark above.) Moreover these $\mathcal{Q}$-structures identify a unique cofinal well-founded branch through $\mathcal{T}$ using the inductive hypothesis for (iii) and the standard arguments in the non-hybrid setting (see Section 2.2). Therefore ω_1-iterability for M is Π^Σ_{2k+1}-definable in any code for M (relative to Σ).

Note that this argument in fact shows that for (i) it suffices to assume $\boldsymbol{\Pi}^1_{2k-1}\Gamma$ determinacy as we are only using (iii) at the level $k-1$.

Now we aim to prove (ii). So let $k \geq 1$ be arbitrary and assume inductively that in particular (iii) holds for countable Σ-premice M and N such that every proper initial segment of M or N is $(k-1, \Sigma)$-small and (i) holds for

(k, Σ)-small countable Σ-premice M. We aim to show that this implies that the model $P^k(x;\Sigma)$ is closed under the operation

$$a \mapsto M_{k-1}^{\Sigma,\#}(a).$$

Recall that we inductively assume that Theorem 7.3.1 holds at the level $k-1$ and therefore we have from $\mathbf{\Pi}^1_{2k+1}\Gamma$ determinacy that the Σ-premouse $M_{k-1}^{\Sigma,\#}(a)$ exists in V as a is countable in V. Moreover we have that ω_1-iterability for the (k,Σ)-small premouse $M_{k-1}^{\Sigma,\#}(a)$ is Π^Σ_{2k+1}-definable uniformly in any code for $M_{k-1}^{\Sigma,\#}(a)$ (relative to Σ) by (i). So we can consider the following Σ^Σ_{2k+2}-formula $\varphi(a)$.

$$\begin{aligned}\varphi(a) \;\equiv\; & \exists N \text{ countable } (\Sigma,a)\text{-premouse such that} \\ & N \text{ is } (k,\Sigma)\text{-small, but not } (k-1,\Sigma)\text{-small,} \\ & \text{every proper initial segment of } N \text{ is } (k-1,\Sigma)\text{-small,} \\ & N \text{ is } a\text{-sound}, \rho_\omega(N) = a, \text{ and} \\ & N \text{ is } \omega_1\text{-iterable.}\end{aligned}$$

As argued above we have that $\varphi(a)$ holds in V as witnessed by the premouse $M_{k-1}^{\Sigma,\#}(a)$. Let $\eta < \omega_1^V$ be a large enough ordinal such that a is countable in $P^k(x;\Sigma)^{\mathrm{Col}(\omega,\eta)}$. Moreover let $g \in V$ be $\mathrm{Col}(\omega,\eta)$-generic over $P^k(x;\Sigma)$ such that $P^k(x;\Sigma)[g]$ is $\mathbf{\Sigma}^\Sigma_{2k+2}$-correct in V for parameters in $P^k(x;\Sigma)[g] \cap {}^\omega\omega$, as in property (ii) in Lemma 6.3.5.

Then we have that $\varphi(a)$ holds in $P^k(x;\Sigma)[g]$, say witnessed by a (Σ,a)-premouse N. In this case N is the unique (Σ,a)-premouse in $P^k(x;\Sigma)[g]$ which witnesses that $\varphi(a)$ holds by the following argument. Assume that M is another (Σ,a)-premouse in $P^k(x;\Sigma)[g]$ which witnesses $\varphi(a)$. By our inductive hypothesis for (iii) we can successfully compare N and M as both only have proper initial segments which are $(k-1,\Sigma)$-small. This yields that in fact $N = M$.

Since N is uniquely definable from $a \in P^k(x;\Sigma)$ it follows by homogeneity of the forcing $\mathrm{Col}(\omega,\eta)$ that $N \in P^k(x;\Sigma)$. We in fact have that N is ω_1^V-iterable in $P^k(x;\Sigma)$ by the following argument. Let $\mathcal{T}$ be an iteration tree on N in $P^k(x;\Sigma)$ of limit length λ. Let g be $\mathrm{Col}(\omega,\lambda)$-generic over $P^k(x;\Sigma)$ such that $P^k(x;\Sigma)[g]$ is $\mathbf{\Sigma}^\Sigma_{2k+2}$-correct in V. Then we have in particular that N is ω_1-iterable in $P^k(x;\Sigma)[g]$ and therefore there exists a cofinal well-founded branch b through $\mathcal{T}$ in $P^k(x;\Sigma)[g]$. If we now argue as in the proof of Lemma 2.2.8, using that (ii) holds for $a \mapsto M_{k-2}^{\Sigma,\#}(a)$ inductively, it follows that b is given by a $\mathcal{Q}$-structure in $P^k(x;\Sigma)$ and therefore $b \in P^k(x;\Sigma)$ by homogeneity of the forcing $\mathrm{Col}(\omega,\lambda)$, as already argued several times before. This implies that $P^k(x;\Sigma)$ is closed under the operation

$$a \mapsto M_{k-1}^{\Sigma,\#}(a).$$

Now we prove (*iii*) for Σ-premice M and N such that every proper inital segment of M or N is (k, Σ)-small.

Let z be a real coding the Σ-premice M and N relative to Σ. Then the inductive proof of Lemma 2.2.8 in the first part of this thesis shows that we can successfully coiterate M and N inside the k-rich $L(z;\Sigma)$-model $P^k(z;\Sigma)$ if we make the following changes.

We have that the model $P^k(z;\Sigma)$ is closed under the operation $a \mapsto M_{k-1}^{\Sigma,\#}(a)$ and we can in fact show that the operation $a \mapsto M_{k-1}^{\Sigma,\#}(a)$ is contained in the model $P^k(x;\Sigma)$. So the analogue of Lemma 2.2.8 (1) holds true in this setting.

Then the proof of Lemma 2.2.8 (2) can be performed inside the model $P^k(z;\Sigma)$ as before, but using Lemma 6.3.5 (*ii*) for the absoluteness arguments as follows.

The following argument is the same for M and N, so we only present it for M. If $\mathcal{T}$ is an iteration tree on M of length $\lambda+1$ in V for some limit ordinal $\lambda < \omega_1^V$ such that

$$\mathcal{T} \restriction \lambda \in P^k(z;\Sigma),$$

then the proof of Lemma 2.2.8 (2) gives that there exists a $\mathcal{Q}$-structure $\mathcal{Q}$ for $\mathcal{T} \restriction \lambda$ inside $P^k(z;\Sigma)$. Consider the statement

$$\phi(\mathcal{T} \restriction \lambda, \mathcal{Q}) \equiv \text{ "there is a cofinal branch } b \text{ through } \mathcal{T} \restriction \lambda \text{ such that } \mathcal{Q} \trianglelefteq \mathcal{M}_b^{\mathcal{T}} \text{ and } \mathcal{M}_b^{\mathcal{T}} \text{ is a } \Sigma\text{-premouse".}$$

This statement $\phi(\mathcal{T} \restriction \lambda, \mathcal{Q})$ is Σ_1^Σ-definable uniformly in any code for the parameters $\mathcal{T} \restriction \lambda$ and $\mathcal{Q}$ and holds true in V since we assumed that M is ω_1-iterable in V.

Let $\eta < \omega_1^V$ be an ordinal such that $\mathcal{T} \restriction \lambda$ and $\mathcal{Q}$ are countable inside $P^k(z;\Sigma)^{\mathrm{Col}(\omega,\eta)}$. Since the model $P^k(z;\Sigma)$ was constructed such that it is $\boldsymbol{\Sigma}_{2k+2}^\Sigma$-absolute (see Lemma 6.3.5 (*ii*)) it follows that $\phi(\mathcal{T} \restriction \lambda, \mathcal{Q})$ holds in $P^k(z;\Sigma)[g]$ for comeager many $g \in V$ which are $\mathrm{Col}(\omega,\eta)$-generic over $P^k(z;\Sigma)$.

Say $\phi(\mathcal{T} \restriction \lambda, \mathcal{Q})$ is witnessed by a branch b in $P^k(z;\Sigma)[g]$ for such a generic g. Since this branch b is uniquely definable from the parameters $\mathcal{T} \restriction \lambda$ and $\mathcal{Q}$ and moreover we have that $\mathcal{T} \restriction \lambda, \mathcal{Q} \in P^k(z;\Sigma)$, it follows by homogeneity of the forcing $\mathrm{Col}(\omega,\eta)$ that in fact $b \in P^k(z;\Sigma)$.

Therefore we have that the analogue of Lemma 2.2.8 (2) for M (and by the same argument also for N) holds in this setting and in particular M and N are iterable inside $P^k(z;\Sigma)$ with respect to iteration trees in $P^k(z;\Sigma)$.

Moreover we have that $P^k(z;\Sigma) \vDash \mathrm{ZFC}$ by Lemma 6.3.4 and

$$\omega_1^{P^k(z;\Sigma)} < \omega_1^V.$$

Therefore the coiteration of M and N inside the model $P^k(z;\Sigma)$ terminates successfully and we have as in the proof of Lemma 2.2.8 (3) and (4) that

$$M \trianglelefteq N \text{ or } N \trianglelefteq M.$$

□

6.5. OD-Determinacy for a k-rich Model

In this section we will use the results from the previous sections to prove a version of Lemma 2.1.3 for k-rich $L(x;\Sigma)$-models. The argument will again be a generalization of the original proof due to Kechris and Solovay in [KS85].

LEMMA 6.5.1. *Let $k \geq 1$ and assume that every $\mathbf{\Sigma}^1_{2k+2}\Gamma$-definable set of reals is determined. For all $x \in {}^\omega\omega$ let $P^k(x;\Sigma)$ be the k-rich $L(x;\Sigma)$-model as constructed in Section 6.3 above. Then there exists a real x_0 such that for all reals $x \geq_T x_0$,*

$$P^k(x;\Sigma) \models \mathrm{OD}_\Sigma\text{-}determinacy.$$

REMARK. We say that a set of reals A is in OD_Σ iff it is definable from some finite sequence of ordinals and Σ as an additional parameter.

PROOF OF LEMMA 6.5.1. Assume this is not the case. Then, for cofinally many reals x, we have that

$$P^k(x;\Sigma) \vDash \neg\mathrm{OD}_\Sigma\text{-determinacy}.$$

Write $P^*_\eta(x;\Sigma) = P^k(x;\Sigma)|\eta$ for an ordinal $\eta < \omega_1^V$ for the η-th level in the construction of $P^k(x;\Sigma)$. Then we have that for all reals z there exists a real $y \geq_T z$ and an ordinal $\eta < \omega_1^V$ such that

$$P^*_\eta(y;\Sigma) \vDash \text{“}\neg\mathrm{OD}_\Sigma\text{-determinacy} + \mathrm{ZFC}^*\text{”},$$

where ZFC^* denotes a large enough segment of ZFC to carry out the arguments which follow. Furthermore we have that the set

$$\{y \in {}^\omega\omega \mid \exists\eta < \omega_1^V (P^*_\eta(y;\Sigma) \vDash \text{“}\neg\mathrm{OD}_\Sigma\text{-determinacy} + \mathrm{ZFC}^*\text{”})\}$$

is Turing-invariant and $\mathbf{\Sigma}^1_{2k+2}\Gamma$-definable by Lemma 6.3.5 (iv). Thus $\mathbf{\Sigma}^1_{2k+2}\Gamma$ Turing-determinacy yields that we can fix a real z_0 such that

$$\forall y \geq z_0 \exists\eta < \omega_1^V (P^*_\eta(y;\Sigma) \vDash \text{“}\neg\mathrm{OD}_\Sigma\text{-determinacy} + \mathrm{ZFC}^*\text{”}).$$

Let $\eta(y)$ denote the least ordinal η such that

$$P^*_\eta(y;\Sigma) \vDash \text{“}\neg\mathrm{OD}_\Sigma\text{-determinacy} + \mathrm{ZFC}^*\text{”},$$

if it exists. We have that

$$\forall z \exists z' \geq_T z \forall y \geq_T z' (\eta(y) \geq \eta(z')),$$

because otherwise there exists an infinite descending chain of ordinals. In particular this real $z' \geq_T z$ is such that the ordinal $\eta(y)$ is defined for all reals $y \geq_T z'$.

Now consider the game G which is defined as follows.

$$\begin{array}{c|l} \mathrm{I} & x \oplus a \\ \hline \mathrm{II} & \qquad\quad y \oplus b \end{array} \qquad \text{for } x, a, y, b \in {}^\omega\omega.$$

As usual the players I and II alternate playing natural numbers and the game lasts ω steps. Say player I produces a real $x \oplus a$ and player II produces a real $y \oplus b$. Then player I wins the game G iff the ordinal $\eta(x \oplus y)$ is defined and

$$a \oplus b \in A_{P^*_{\eta(x\oplus y)}(x \oplus y; \Sigma)},$$

where $A_{P^*_{\eta(x\oplus y)}(x \oplus y; \Sigma)}$ denotes the least non-determined OD_Σ-set in the model $P^*_{\eta(x\oplus y)}(x \oplus y; \Sigma)$.

The winning condition for this game G is $\boldsymbol{\Sigma}^1_{2k+2}\Gamma$-definable, since the model $P^k(x \oplus y; \Sigma)$ is $\Sigma^1_{2k+2}\Gamma$-definable from the parameter $x \oplus y$ and any real coding the countable premouse $\mathcal{N}$ by Lemma 6.3.5. Therefore it follows from the hypothesis that the game G is determined. Assume first that player I has a winning strategy in G and call this winning strategy τ. Let x_τ be a real coding this strategy τ.

Pick a real $z \geq_T x_\tau$ such that for all reals $x \geq_T z$, we have that the ordinal $\eta(x)$ is defined and $\eta(x) \geq \eta(z)$. We aim to prove that

$$P^*_{\eta(z)}(z; \Sigma) \vDash \text{“} A_{P^*_{\eta(z)}(z;\Sigma)} \text{ is determined”}$$

to derive a contradiction.

For this purpose we work in the model $P^*_{\eta(z)}(z; \Sigma)$ and consider the following run of the game G defined above.

$$\begin{array}{c|l} \mathrm{I} & x \oplus a = \tau((z \oplus b) \oplus b) \\ \hline \mathrm{II} & \qquad\qquad\qquad\qquad\quad (z \oplus b) \oplus b \end{array}$$

Assume player II plays the real $(z \oplus b) \oplus b$ and player I responds with $x \oplus a$ according to his winning strategy τ. That means $x \oplus a = \tau((z \oplus b) \oplus b)$. Recall that $z \geq_T x_\tau$, so we have that $\tau, z \in P^*_{\eta(z)}(z; \Sigma)$.

We define a strategy τ^* for player I such that in a run of the Gale-Stewart game $G(A_{P^*_{\eta(z)}(z;\Sigma)})$ inside the model $P^*_{\eta(z)}(z; \Sigma)$ according to the strategy τ^*, player I has to respond to the real b with producing the real a, where a and b are as above. So we have that $\tau^*(b) = a$.

$$\begin{array}{c|l} \mathrm{I} & a = \tau^*(b) \\ \hline \mathrm{II} & \qquad\quad b \end{array}$$

Then, since τ is a winning strategy for player I in the original game G, we have that the ordinal $\eta(x \oplus (z \oplus b))$ is defined and that

$$a \oplus b \in A_{P^*_{\eta(x\oplus(z\oplus b))}(x \oplus (z \oplus b); \Sigma)}.$$

Now it is enough to show that

$$A_{P^*_{\eta(z)}(z;\Sigma)} = A_{P^*_{\eta(x\oplus(z\oplus b))}(x\oplus(z\oplus b);\Sigma)}$$

to derive a contradiction, because then τ^* is a winning strategy for player I in the Gale-Stewart game with payoff set $A_{P^*_{\eta(z)}(z;\Sigma)}$ played in $P^*_{\eta(z)}(z;\Sigma)$. This implies that the set $A_{P^*_{\eta(z)}(z;\Sigma)}$ is determined in $P^*_{\eta(z)}(z;\Sigma)$, which is a contradiction since $A_{P^*_{\eta(z)}(z;\Sigma)}$ was supposed to be the least non-determined OD_Σ-set of reals in $P^*_{\eta(z)}(z;\Sigma)$.

Since $\tau, z \in P^*_{\eta(z)}(z;\Sigma)$ and as we worked inside the model $P^*_{\eta(z)}(z;\Sigma)$, we have that

$$x \oplus (z \oplus b) \in P^*_{\eta(z)}(z;\Sigma).$$

Moreover $\eta(z)$ was defined such that $P^*_{\eta(z)}(z;\Sigma) \models \mathrm{ZFC}^*$, so $\eta(z)$ is in particular a limit step in the construction of $P^k(z;\Sigma)$. Therefore we have by our construction of k-rich $L(z;\Sigma)$-models that

$$P^*_{\eta(z)}(x \oplus (z \oplus b);\Sigma) \sim_\Sigma P^*_{\eta(z)}(z;\Sigma),$$

where $P \sim_\Sigma Q$ abbreviates that P and Q have the same sets of reals and the same OD_Σ-sets of reals in the same order. This equality holds true because the k-rich $L(x\oplus(z\oplus b);\Sigma)$-model up to $\eta(z)$, $P^*_{\eta(z)}(x\oplus(z\oplus b);\Sigma)$, constructed inside the model $P^*_{\eta(z)}(z;\Sigma)$ is the same as the k-rich $L(x \oplus (z \oplus b);\Sigma)$-model up to $\eta(z)$ constructed in V, since during the construction of the model $P^*_{\eta(z)}(z;\Sigma)$ we close under the predicate Σ, under appropriate Skolem functions and under certain names, $\eta(z)$ is a limit step in the construction such that $P^*_{\eta(z)}(z;\Sigma) \models \mathrm{ZFC}^*$, and we defined the construction in a unique way.

Moreover we have that $\eta(x \oplus (z \oplus b)) \geq \eta(z)$ by our choice of z since $z \leq_T x\oplus(z\oplus b)$. This implies by the definition of the ordinal $\eta(z)$ that we already have that

$$\eta(x \oplus (z \oplus b)) = \eta(z)$$

since the models $P^*_{\eta(z)}(x \oplus (z \oplus b);\Sigma)$ and $P^*_{\eta(z)}(z;\Sigma)$ agree on their sets of reals and OD_Σ-sets of reals and on the order of their OD_Σ-sets of reals. Therefore we have that

$$P^*_{\eta(x\oplus(z\oplus b))}(x \oplus (z \oplus b);\Sigma) \sim_\Sigma P^*_{\eta(z)}(z;\Sigma).$$

This finally gives

$$A_{P^*_{\eta(x\oplus(z\oplus b))}(x\oplus(z\oplus b);\Sigma)} = A_{P^*_{\eta(z)}(z;\Sigma)},$$

as desired.

Now suppose that player II has a winning strategy σ in the game G introduced above and let x_σ be a real coding this strategy σ. Recall that z_0 is a base of a cone of reals z such that

$$\exists \eta < \omega_1^V (P^*_\eta(z;\Sigma) \models \neg\mathrm{OD}_\Sigma\text{-determinacy} + \mathrm{ZFC}^*).$$

Then as above we can pick a real $z \geq_T x_\sigma \oplus z_0$ such that for all reals $x \geq_T z$, we have that the ordinal $\eta(x)$ is defined and $\eta(x) \geq \eta(z)$. We again want to prove that

$$P^*_{\eta(z)}(z;\Sigma) \vDash \text{“}A_{P^*_{\eta(z)}(z;\Sigma)} \text{ is determined”}$$

to derive a contradiction.

For this purpose we also work in the model $P^*_{\eta(z)}(z;\Sigma)$. Analogous to the case when player I has a winning strategy in G we consider the following run of the game G defined above.

I	$(z \oplus a) \oplus a$	
II		$y \oplus b = \sigma((z \oplus a) \oplus a)$

Assume player I plays the real $(z \oplus a) \oplus a$ and player II responds with the real $y \oplus b$ according to his winning strategy σ. That means we have that $y \oplus b = \sigma((z \oplus a) \oplus a)$. So as above we have that $\sigma, z \in P^*_{\eta(z)}(z;\Sigma)$.

We define a strategy σ^* for player II such that in a run of the Gale-Stewart game $G(A_{P^*_{\eta(z)}(z;\Sigma)})$ inside the model $P^*_{\eta(z)}(z;\Sigma)$ according to the strategy σ^*, player II has to respond to the real a with producing the real b, where a and b are as above. So we have that $\sigma^*(a) = b$.

I	a	
II		$b = \sigma^*(a)$

We have by our choice of the real z that the ordinal $\eta((z \oplus a) \oplus y)$ is defined because $(z \oplus a) \oplus y \geq_T z$. Then, since σ is a winning strategy for player II in the original game G, it follows that

$$a \oplus b \notin A_{P^*_{\eta((z\oplus a)\oplus y)}((z\oplus a)\oplus y;\Sigma)}.$$

Now it is again enough to show that

$$A_{P^*_{\eta(z)}(z;\Sigma)} = A_{P^*_{\eta((z\oplus a)\oplus y)}((z\oplus a)\oplus y;\Sigma)}$$

to derive a contradiction, because then σ^* is a winning strategy for player II in the Gale-Stewart game with payoff set $A_{P^*_{\eta(z)}(z;\Sigma)}$ played in the model $P^*_{\eta(z)}(z;\Sigma)$. This implies that $A_{P^*_{\eta(z)}(z;\Sigma)}$ is determined inside $P^*_{\eta(z)}(z;\Sigma)$, which is a contradiction since $A_{P^*_{\eta(z)}(z;\Sigma)}$ was supposed to be the least non-determined OD_Σ-set of reals inside $P^*_{\eta(z)}(z;\Sigma)$.

This now follows exactly as in the case that player I has a winning strategy in the game G by the following argument. Since we worked inside the model $P^*_{\eta(z)}(z;\Sigma)$ we have that $(z \oplus a) \oplus y \in P^*_{\eta(z)}(z;\Sigma)$ and therefore it follows as above by our construction of k-rich $L(z;\Sigma)$-models that

$$P^*_{\eta(z)}((z \oplus a) \oplus y;\Sigma) \sim_\Sigma P^*_{\eta(z)}(z;\Sigma).$$

Moreover we have that $\eta((z \oplus a) \oplus y) \geq \eta(z)$ by our choice of the real z since $z \leq_T (z \oplus a) \oplus y$. This implies as before by the definition of the ordinal $\eta(z)$ that we already have that $\eta((z \oplus a) \oplus y) = \eta(z)$. Therefore it follows that

$$P^*_{\eta((z \oplus a) \oplus y)}((z \oplus a) \oplus y; \Sigma) \sim_\Sigma P^*_{\eta(z)}(z; \Sigma)$$

and this finally gives that

$$A_{P^*_{\eta((z \oplus a) \oplus y)}((z \oplus a) \oplus y; \Sigma)} = A_{P^*_{\eta(z)}(z;\Sigma)},$$

as desired. □

6.6. A Hybrid Model with One Woodin Cardinal in a k-rich Model

In this section we will show how to use Lemma 6.5.1 to provide the basis for constructing a hybrid inner model with one Woodin cardinal, which is "strong enough" to survive putting $(k-1, \Sigma)$-small Σ-mice on top. This is a generalization of Theorem 7.7 in [St96] analogous to Theorem 2.5.1.

As in Section 2.4 we first need the following consequence of Lemma 6.5.1 which is essentially due to Solovay. Since the proof is a straightforward generalization to the hybrid case of the proof of Corollary 2.4.1 we omit it here.

Corollary 6.6.1. *Let $k \geq 1$ and assume that every $\boldsymbol{\Sigma}^1_{2k+2}\Gamma$-definable set of reals is determined. For all $x \in {}^\omega\omega$ let $P^k(x; \Sigma)$ be the k-rich $L(x; \Sigma)$-model above x as constructed in Section 6.3. Then*

$$\omega_1^{P^k(x;\Sigma)} \text{ is measurable in } \mathrm{HOD}_\Sigma^{P^k(x;\Sigma)}$$

for a cone of reals x.

Remark. We denote by HOD_Σ the class of all hereditarily OD_Σ-definable sets. That means

$$\mathrm{HOD}_\Sigma = \{x \mid \mathrm{TC}(\{x\}) \subset \mathrm{OD}_\Sigma\},$$

where $\mathrm{TC}(\{x\})$ denotes the transitive closure of the set $\{x\}$.

Now we are ready to prove the main theorem of this section, which will be the analogue of Theorem 2.5.1 for k-rich models.

Theorem 6.6.2. *Let $k \geq 1$ and assume that every $\boldsymbol{\Sigma}^1_{2k+2}\Gamma$-definable set of reals is determined. For all $x \in {}^\omega\omega$ let $P^k(x; \Sigma)$ be the k-rich $L(x; \Sigma)$-model above x as constructed in Section 6.3. Then there is a cone of reals x such that*

$$(K^{c,\Sigma})^{P^k(x;\Sigma)}$$

has a Woodin cardinal or is not fully iterable via the $\mathcal{Q}$-structure iteration strategy.

REMARK. By $(K^{c,\Sigma})^{P^k(x;\Sigma)}$ we denote the K^c-construction of Chapter 2 in [St96] generalized to the hybrid context as in Section 5.2 and furthermore generalized to (k,Σ)-small Σ-premice instead of $(1,\Sigma)$-small Σ-premice. As the notation suggests the construction is performed inside the model $P^k(x;\Sigma)$. The $\mathcal{Q}$-structure iteration strategy for the premouse $(K^{c,\Sigma})^{P^k(x;\Sigma)}$ is the straightforward generalization of the $\mathcal{Q}$-structure iteration strategy in the projective case as defined in Definition 2.2.2.

PROOF OF THEOREM 6.6.2. By Lemma 6.5.1 there exists a real x such that we have for all reals $y \geq_T x$ that

$$P^k(y;\Sigma) \vDash \text{OD}_\Sigma\text{-determinacy}.$$

Fix such a real x and assume toward a contradiction that the Σ-premouse $(K^{c,\Sigma})^{P^k(x;\Sigma)}$ does not have Woodin cardinals and is fully iterable via the $\mathcal{Q}$-structure iteration strategy. Then we can isolate the core model $(K^\Sigma)^{P^k(x;\Sigma)}$ from $(K^{c,\Sigma})^{P^k(x;\Sigma)}$ by a generalization to the hybrid case of Theorem 1.1 in [JS13].

In particular $(K^\Sigma)^{P^k(x;\Sigma)}$ is absolute for set sized forcings over $P^k(x;\Sigma)$ and by a generalization of [MSch95] it follows that $(K^\Sigma)^{P^k(x;\Sigma)}$ satisfies weak covering, that means we have

$$P^k(x;\Sigma) \vDash (\alpha^+)^{K^\Sigma} = \alpha^+,$$

for all singular cardinals α (see Theorem 1.1 in [JS13]).

Let $\alpha = \aleph_\omega^{P^k(x;\Sigma)} < \omega_1^V$, so α is a singular cardinal inside $P^k(x;\Sigma)$. Therefore we have that in particular

$$P^k(x;\Sigma) \vDash (\alpha^+)^{K^\Sigma} = \alpha^+.$$

Pick a real z in V which is generic over the model $P^k(x;\Sigma)$ for $\text{Col}(\omega,\alpha)$ such that we have [1]

$$P^k(x;\Sigma)[z] = P^k(x \oplus z;\Sigma),$$

using Lemma 6.3.5 (iii).

Recall that we chose the real x such that we have

$$P^k(x \oplus z;\Sigma) \vDash \text{OD}_\Sigma\text{-determinacy}.$$

This implies that

$$P^k(x;\Sigma)[z] \vDash \text{OD}_\Sigma\text{-determinacy}.$$

Furthermore we have by Corollary 6.6.1 that $\omega_1^{P^k(x\oplus z;\Sigma)}$ is measurable in $\text{HOD}_\Sigma^{P^k(x\oplus z;\Sigma)}$ since by the proof of Corollary 6.6.1 (see the proof of Corollary 2.4.1 in Part 1) it follows that x is also a base for a cone of reals as in Corollary 6.6.1.

[1] Again by this equality we mean that the models $P^k(x;\Sigma)[z]$ and $P^k(x \oplus z;\Sigma)$ have the same universe (and not necessarily the same hierarchies).

Work in the model $P^k(x;\Sigma)[z]$ from now on. Then we also have that ω_1 is measurable in HOD_Σ. Since $K^\Sigma \subseteq \mathrm{HOD}_\Sigma$ and $\omega_1 = (\alpha^+)^{K^\Sigma}$ it follows that $\omega_1 = (\alpha^+)^{\mathrm{HOD}_\Sigma}$. This is a contradiction, because all measurable cardinals in HOD_Σ are inaccessible as $\mathrm{HOD}_\Sigma \vDash \mathrm{ZFC}$. $\square$

With the help of Theorem 6.6.2 we will construct a hybrid premouse with a Woodin cardinal in Section 7.2, such that the Woodin cardinal "survives", if we put a $(k-1,\Sigma)$-small Σ-mouse on top. Such premice will be called (A,k)-suitable. This property will be ensured by the fact, that we build the premouse inside the k-rich model $P^k(x;\Sigma)$.

CHAPTER 7

Proving Iterability

In this chapter we will prove that we can in fact construct iterable hybrid models with Woodin cardinals. For that purpose we will show that the important concepts of Sections 3.1 and 3.4 generalize to our context beyond the projective hierarchy.

7.1. Canonical Hybrid Mice

As in Chapter 6 we fix for $i < \omega$ sets $A_i \in \mathcal{P}(\mathbb{R})$, a pointclass Γ, a (possibly hybrid) premouse $\mathcal{N}$ and an ω_1-iteration strategy Σ for $\mathcal{N}$ which condenses well and witnesses that $\mathcal{N}$ captures every set A_i at some cardinal δ, such that the pointclass Γ is $(\mathcal{N}, \Sigma)$-apt (see Definition 6.1.7). Moreover we let $A = \bigcup_{i<\omega} A_i$.
In what follows we define canonical hybrid premice at the level of k-rich $L(x;\Sigma)$-models. We call these premice pre-(A,k)-suitable and tacitly assume, whenever we are mentioning pre-(A,k)-suitable Σ-premice, that the (Σ, x)-premouse $M_{k-1}^{\Sigma,\#}(x)$ exists for all $x \in {}^\omega\omega$, which follows from $\mathbf{\Pi}^1_{2k+1}\Gamma$ determinacy as we inductively assume that Theorem 7.3.1 holds at the level $k-1$.

Definition 7.1.1. *Let $k \geq 1$. Then we say that a countable (hybrid) Σ-premouse M is* pre-(A,k)-suitable *iff there exists an ordinal $\delta < \omega_1^V$ such that*

(1) $M \vDash$ "ZFC$^-$ + δ *is the largest cardinal*",
(2) $\mathcal{N} \in M$,

$$M = M_{k-1}^\Sigma(\mathcal{N}, M|\delta)|(\delta^+)^{M_{k-1}^\Sigma(\mathcal{N}, M|\delta)},$$

and for all $\gamma < \delta$

$$M_{k-1}^\Sigma(\mathcal{N}, M|\gamma)|(\gamma^+)^{M_{k-1}^\Sigma(\mathcal{N}, M|\gamma)} \lhd M,$$

(3) $M_{k-1}^\Sigma(\mathcal{N}, M|\delta)$ *is a proper class model and*

$$M_{k-1}^\Sigma(\mathcal{N}, M|\delta) \vDash \text{“}\delta \text{ is Woodin”},$$

and
(4) *for every $\gamma < \delta$, $M_{k-1}^\Sigma(\mathcal{N}, M|\gamma)$ is a set, or*

$$M_{k-1}^\Sigma(\mathcal{N}, M|\gamma) \nvDash \text{“}\gamma \text{ is Woodin”}.$$

Remark. For $k = 0$ we let $M_k^\Sigma = L^\Sigma$. Moreover we say that a premouse is pre-A-suitable iff it is pre-$(A,1)$-suitable.

Remark. Here the hybrid premouse M_{k-1}^Σ is defined as in Section 5.2. As above we write $M_{k-1}^\Sigma(\mathcal{N}, M|\delta)$ to remind ourselves that Σ is an iteration strategy for the premouse $\mathcal{N}$ we fixed earlier.

We use $M_{k-1}^{\Sigma,\#}(\mathcal{N}, M)$ to denote the following model constructed above a Σ-premouse M such that $\mathcal{N} \in M$. We construct the Σ-premouse $M_{k-1}^{\Sigma,\#}$ on top of M including the extenders of M on the sequence and analogous to Definition 2.2.6 we stop the construction if it reaches a model which is not fully sound. (Note that constructing further in this case would mean that the resulting model is no longer a Σ-premouse.) In particular we have that the model $M_{k-1}^{\Sigma,\#}(\mathcal{N}, M)$ is not $(k-1,\Sigma)$-small above M if it does not reach a model which is not fully sound.

In this case we let $M_{k-1}^\Sigma(\mathcal{N}, M)$ denote the proper class model obtained from $M_{k-1}^{\Sigma,\#}(\mathcal{N}, M)$ by iterating the top measure out of the universe. If $M_{k-1}^{\Sigma,\#}(\mathcal{N}, M)$ is not fully sound, we just let $M_{k-1}^\Sigma(\mathcal{N}, M) = M_{k-1}^{\Sigma,\#}(\mathcal{N}, M)$.

The Σ-premice in Definition 7.1.1 are called (A,k)-suitable, because they are typical candidates for premice capturing the set A as introduced above. The only thing that holds them back from providing a premouse that captures A is their possible lack of iterability. We will take care of this issue during this chapter by using (A,k)-suitable premice to prove the existence of a particular Σ-mouse, namely $M_k^{\Sigma,\#}$, which will capture the set A.

We can define a notion of short tree iterability for pre-(A,k)-suitable Σ-premice analogous to the definition of short tree iterability for pre-n-suitable premice, see Definition 3.1.5, as follows. Again we tacitly assume that $M_{k-1}^{\Sigma,\#}(x)$ exists for all $x \in {}^\omega\omega$.

Definition 7.1.2. *Let $k \geq 1$ and let $\mathcal{T}$ be an iteration tree of length $< \omega_1^V$ on a pre-(A,k)-suitable premouse M. We say that $\mathcal{T}$ is* short *iff for all limit ordinals $\lambda < \mathrm{lh}(\mathcal{T})$, the $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{T} \upharpoonright \lambda)$ exists, is $(k-1,\Sigma)$-small above $\delta(\mathcal{T} \upharpoonright \lambda)$ and we have that,*

$$\mathcal{Q}(\mathcal{T} \upharpoonright \lambda) \trianglelefteq \mathcal{M}_\lambda^\mathcal{T},$$

and if $\mathcal{T}$ has limit length we in addition have that $\mathcal{Q}(\mathcal{T})$ exists and

$$\mathcal{Q}(\mathcal{T}) \trianglelefteq M_{k-1}^\Sigma(\mathcal{N}, \mathcal{M}(\mathcal{T})).$$

As usual we say that $\mathcal{T}$ is maximal *iff $\mathcal{T}$ is not short.*

This now yields a notion of short tree iterability.

Definition 7.1.3. *Let $k \geq 1$ and let M be a pre-(A,k)-suitable Σ-premouse. We say that M is* short tree iterable *iff whenever $\mathcal{T}$ is a short tree on M,* (*i*) *if* $\mathcal{T}$ *has a last model, then every putative iteration tree* $\mathcal{U}$ *extending* $\mathcal{T}$ *with* $\mathrm{lh}(\mathcal{U}) = \mathrm{lh}(\mathcal{T}) + 1$ *has a well-founded last model and the last model of* $\mathcal{U}$ *is a* Σ*-premouse, and*

(*ii*) *if* $\mathcal{T}$ *has limit length, then there exists a cofinal well-founded branch* b *through* $\mathcal{T}$ *such that*

$$\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$$

and the model $\mathcal{M}_b^{\mathcal{T}}$ *is a* Σ*-premouse.*

Now we can finally define when a hybrid premouse is (A, k)-suitable.

Definition 7.1.4. *Let* $k \geq 1$*. Then we say that a (hybrid)* Σ*-premouse* M *is* (A, k)-suitable *iff*

(*i*) M *is pre-*(A, k)*-suitable,*

(*ii*) M *is short tree iterable in the sense of Definition 7.1.3, and*

(*iii*) M *is fullness preserving for non-dropping short trees, that means whenever* $\mathcal{T}$ *is a short tree on* M *of length* $\lambda + 1$ *for some ordinal* $\lambda < \omega_1^V$*, which is non-dropping on the main branch* $[0, \lambda]_T$*, then* $\mathcal{M}_\lambda^{\mathcal{T}}$ *is again pre-*(A, k)*-suitable.*

Remark. In contrast to the situation in the projective hierarchy it is necessary to demand here that M is short tree iterable. At the projective levels we included a weak form of iterability in the definition of pre-n-suitability (see Definition 3.1.1) and were able to prove that every such pre-n-suitable premouse is in fact short tree iterable. So there it was not necessary to include short tree iterability explicitly in the definition of n-suitability. We do not see how to prove a similar thing in this setting, because we have to take care of the iteration strategy Σ. Nevertheless we are able to prove in Section 7.2 that an (A, k)-suitable premouse, which in particular is short tree iterable, exists from our determinacy assumption.

Remark. Again in contrast to the situation in the projective hierarchy we did not demand any fullness preservation for maximal trees in the definition of (A, k)-suitability. It should be possible to prove this analogous to the projective case and as in the proof of Claim 2 in the proof of Theorem 7.2.1, but we will not need it in what follows.

7.2. A Canonical Premouse with Finitely Many Woodin Cardinals

Fix $k \geq 1$ throughout this section. We will construct an (A, k)-suitable Σ-premouse using Theorem 6.6.2. Recall that similar to the projective case in Part 1 of this thesis we will assume inductively that if every $\mathbf{\Pi}^1_{2k+1}\Gamma$-definable set of reals is determined, then the (Σ, x)-premouse $M_{k-1}^{\Sigma,\#}(x)$ exists and is ω_1-iterable for every real x. That means we assume inductively that Theorem 7.3.1 and Corollary 7.3.2 (see overview at the beginning of this part of this thesis) hold.

THEOREM 7.2.1. *Let $k \geq 1$ and assume that every $\mathbf{\Sigma}^1_{2k+2}\Gamma$-definable set of reals is determined. Then there exists an (A,k)-suitable Σ-premouse.*

PROOF. The proof of this theorem divides into two parts. We first construct a pre-(A,k)-suitable Σ-premouse using Theorem 6.6.2, afterwards we prove in the second part that this premouse is in fact already (A,k)-suitable.

We fix a real x such that x is contained in a cone of reals as in Theorem 6.6.2. Recall that $P^k(x;\Sigma)$ denotes the k-rich $L(x;\Sigma)$-model as constructed in Section 6.3. Then we have that

$$(K^{c,\Sigma})^{P^k(x;\Sigma)}$$

as defined in Theorem 6.6.2 has a Woodin cardinal or is not fully iterable via the $\mathcal{Q}$-structure iteration strategy.

We now construct the Σ-premouse M as follows.

Case 1. Assume that $(K^{c,\Sigma})^{P^k(x;\Sigma)}$ has a Woodin cardinal.

Then $(K^{c,\Sigma})^{P^k(x;\Sigma)}$ has a largest Woodin cardinal and is (k,Σ)-small, because otherwise we already have that $M_k^{\Sigma,\#}$ exists and is ω_1-iterable for trivial reasons.

So let δ denote the largest Woodin cardinal in $(K^{c,\Sigma})^{P^k(x;\Sigma)}$. Then we let

$$M = M_{k-1}^{\Sigma,\#}(\mathcal{N}, (K^{c,\Sigma})^{P^k(x;\Sigma)}|\delta) \,|\, \delta^+,$$

where δ^+ denotes $(\delta^+)^{M_{k-1}^{\Sigma,\#}(\mathcal{N},(K^{c,\Sigma})^{P^k(x;\Sigma)}|\delta)}$ and we have that the Σ-premouse $M_{k-1}^{\Sigma,\#}(\mathcal{N}, (K^{c,\Sigma})^{P^k(x;\Sigma)}|\delta)$ exists in $P^k(x;\Sigma)$ by Lemma 6.4.2 (ii).

Case 2. Assume that $(K^{c,\Sigma})^{P^k(x;\Sigma)}$ is not fully iterable via the $\mathcal{Q}$-structure iteration strategy and does not have a Woodin cardinal.

Then there exists an iteration tree $\mathcal{T}$ of limit length on $(K^{c,\Sigma})^{P^k(x;\Sigma)}$ such that there is no $\mathcal{Q}$-structure for $\mathcal{T}$ inside the model $P^k(x;\Sigma)$. We have that the ω_1-iterable Σ-premouse $M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{T}))$ exists inside the model $P^k(x;\Sigma)$ using Lemma 6.4.2 (ii).

Moreover we have that the construction of $M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{T}))$ inside $P^k(x;\Sigma)$ does not break down, that means $M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{T}))$ is not $(k-1,\Sigma)$-small above $\delta(\mathcal{T})$, because otherwise we would already have a $\mathcal{Q}$-structure for $\mathcal{T}$ inside $P^k(x;\Sigma)$. In particular we have that

$$M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{T})) \vDash \text{“}\delta(\mathcal{T}) \text{ is Woodin”}.$$

We let $\delta = \delta(\mathcal{T})$ and

$$M = M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{T})) \,|\, \delta^+,$$

where similar as above δ^+ denotes $(\delta^+)^{M_{k-1}^{\Sigma,\#}(\mathcal{N},\mathcal{M}(\mathcal{T}))}$.
We have the following claim by our choice of M.

Claim 1. *There is a Σ-premouse M which is pre-(A,k)-suitable.*

We show that there is a Σ-premouse M which is pre-(A,k)-suitable in V (and not only in $P^k(x;\Sigma)$) using the absoluteness properties of the model $P^k(x;\Sigma)$ (see Lemma 6.3.5 and Lemma 6.4.2 (ii)).

Proof of Claim 1. We again distinguish two cases as above.

Case 1. Assume that we have $M = (K^{c,\Sigma})^{P^k(x;\Sigma)} \mid \delta^+$ as in the first case above.

Then properties (1) and (2) in Definiton 7.1.1 hold by our choice of M above.
To show that property (3) in Definition 7.1.1 holds, we assume toward a contradiction that the construction of the Σ-premouse $M_{k-1}^{\Sigma,\#}(\mathcal{N}, M|\delta)$ inside $P^k(x;\Sigma)$ reaches a model $N \trianglelefteq M_{k-1}^{\Sigma,\#}(\mathcal{N}, M|\delta)$ which is $(k-1,\Sigma)$-small above δ such that δ is not definably Woodin over N.
We have that

$$N \trianglerighteq (K^{c,\Sigma})^{P^k(x;\Sigma)} \mid \delta$$

and $\rho_\omega(N) \leq \delta$. Therefore we can consider the coiteration of the Σ-premice N and $(K^{c,\Sigma})^{P^k(x;\Sigma)}$ inside the model $P^k(x;\Sigma)$.
We have that $(K^{c,\Sigma})^{P^k(x;\Sigma)}$ is countably iterable above δ inside $P^k(x;\Sigma)$ by a generalization of Chapter 9 in [St96] to our context. Moreover N is ω_1-iterable above δ by construction. Since the coiteration takes place above δ this is enough to show that the coiteration is successful by a straightforward generalization to the hybrid setting of Claim 1 in the proof of Theorem 2.5.1 using the proof of Lemma 6.4.2 (iii).
We can perform the coiteration inside the model $P^k(x;\Sigma)$, so by universality of the Σ-premouse $(K^{c,\Sigma})^{P^k(x;\Sigma)}$ (which follows from a generalization of Corollary 3.6 in [St96]) it follows that the $(K^{c,\Sigma})^{P^k(x;\Sigma)}$-side has to win the comparison. That means there is an iterate K^* of $(K^{c,\Sigma})^{P^k(x;\Sigma)}$ and a non-dropping iterate N^* of N such that

$$N^* \lhd K^*.$$

This is a contradiction because we assumed that δ is not definably Woodin over N, but at the same time we have that

$$(K^{c,\Sigma})^{P^k(x;\Sigma)} \vDash \text{“}\delta \text{ is a Woodin cardinal”}.$$

Therefore it follows that $M = (K^{c,\Sigma})^{P^k(x;\Sigma)} \mid \delta^+$ satisfies property (3) in Definition 7.1.1. By minimizing M it follows that there is a pre-(A,k)-suitable premouse, which in particular satisfies property (4) in Definition 7.1.1.

Case 2. Assume that we have $M = M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{T})) \,|\, \delta^+$ as in the second case above.

Then again properties (1) and (2) in Definiton 7.1.1 hold by our choice of M above.
As argued in the second case above we have that $M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{T}))$ is not $(k-1, \Sigma)$-small above $\delta(\mathcal{T})$, so $M_{k-1}^{\Sigma}(\mathcal{N}, \mathcal{M}(\mathcal{T}))$ is by definition a proper class model and moreover we have that

$$M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{T})) \vDash \text{“}\delta(\mathcal{T}) \text{ is Woodin”}.$$

Therefore property (3) in Definition 7.1.1 holds true for M. Furthermore we again have that it follows by minimizing that there exists a pre-(A,k)-suitable premouse, which in particular satisfies property (4) in Definition 7.1.1. □

Let M be the pre-(A,k)-suitable premouse as constructed in Claim 1. The second step will be to show that M is in fact already (A,k)-suitable. For that we need to prove that M is short tree iterable and fullness preserving for short trees.

Claim 2. *M is (A,k)-suitable.*

Proof. Assume toward a contradiction that this is not the case and say this is witnessed by some short iteration tree $\mathcal{T}$ on M. Let δ be the ordinal from the definition of M.

Case 1. $\mathcal{T}$ witnesses that M is not short tree iterable.

For simplicity assume in this case that $\mathcal{T}$ has limit length since the other case is easier. Then $\mathcal{T}$ witnesses that the following statement $\phi_1(M)$ holds true in V.

$$\begin{aligned}\phi_1(M) \equiv\ & \exists \mathcal{T} \text{ tree on } M \text{ of length } \lambda \text{ for some limit ordinal } \lambda < \omega_1^V \\ & \exists\, (\mathcal{Q}_\gamma \mid \gamma \leq \lambda \text{ limit ordinal}), \text{ such that for all limit ordinals } \gamma \leq \lambda, \\ & \quad \mathcal{Q}_\gamma \text{ is } \omega_1\text{-iterable above } \delta(\mathcal{T} \upharpoonright \gamma), (k-1, \Sigma)\text{-small} \\ & \quad \text{above } \delta(\mathcal{T} \upharpoonright \gamma), \text{ and a } \mathcal{Q}\text{-structure for } \mathcal{T} \upharpoonright \gamma, \text{ and} \\ & \text{for all limit ordinals } \gamma < \lambda \text{ we have } \mathcal{Q}_\gamma \trianglelefteq \mathcal{M}_\gamma^{\mathcal{T}}, \text{ but} \\ & \text{there exists no cofinal branch } b \text{ through } \mathcal{T} \text{ such that} \\ & \quad \mathcal{M}_b^{\mathcal{T}} \text{ is a } \Sigma\text{-premouse and } \mathcal{Q}_\lambda \trianglelefteq \mathcal{M}_b^{\mathcal{T}}.\end{aligned}$$

We have that the statement $\phi_1(M)$ is Σ_{2k}^{Σ}-definable uniformly in any code for M (relative to Σ), because by Lemma 6.4.2 (i) the statement “P is ω_1-iterable” for some countable Σ-premouse P which is $(k-1, \Sigma)$-small is Π_{2k-1}^{Σ}-definable uniformly in any real coding the premouse P relative to Σ.

Case 2. $\mathcal{T}$ is a short tree on M of length $\lambda + 1$ for some ordinal $\lambda < \omega_1^V$ which is non-dropping on the main branch such that the final model $\mathcal{M}_\lambda^\mathcal{T}$ is not pre-(k, Σ)-suitable.

Let

$$i_{0\lambda}^\mathcal{T} : M \rightarrow \mathcal{M}_\lambda^\mathcal{T}$$

be the corresponding iteration embedding, which exists because the iteration tree $\mathcal{T}$ is non-dropping on the main branch. Assume that

$$M_{k-1}^\Sigma(\mathcal{N}, \mathcal{M}_\lambda^\mathcal{T} | i_{0\lambda}^\mathcal{T}(\delta)) \nvDash \text{“} i_{0\lambda}^\mathcal{T}(\delta) \text{ is Woodin”},$$

where δ is the largest cardinal in $\mathcal{M}_\lambda^\mathcal{T}$.
Let $\phi_2(M)$ be the following statement:

$$\begin{aligned}
\phi_2(M) \equiv\ & \exists \mathcal{T} \text{ tree on } M \text{ of length } \lambda + 1 \text{ for some } \lambda < \omega_1^V \text{ such that} \\
& \mathcal{T} \text{ is non-dropping on the main branch and} \\
& \forall \gamma < \mathrm{lh}(\mathcal{T}) \text{ limit } \exists \mathcal{Q} \trianglelefteq \mathcal{M}_\gamma^\mathcal{T} \text{ such that} \\
& \quad \mathcal{Q} \text{ is } (k-1, \Sigma)\text{-small above } \delta(\mathcal{T} \upharpoonright \gamma),\ \omega_1\text{-iterable above} \\
& \quad \delta(\mathcal{T} \upharpoonright \gamma), \text{ and a } \mathcal{Q}\text{-structure for } \mathcal{T} \upharpoonright \gamma, \text{ and} \\
& \exists \mathcal{P} \triangleright \mathcal{M}_\lambda^\mathcal{T} | i_{0\lambda}^\mathcal{T}(\delta) \text{ such that } \mathcal{P} \text{ is } (k-1, \Sigma)\text{-small above } i_{0\lambda}^\mathcal{T}(\delta), \\
& \quad \omega_1\text{-iterable above } i_{0\lambda}^\mathcal{T}(\delta), i_{0\lambda}^\mathcal{T}(\delta)\text{-sound}, \rho_\omega(\mathcal{P}) \leq i_{0\lambda}^\mathcal{T}(\delta), \text{ and} \\
& \quad i_{0\lambda}^\mathcal{T}(\delta) \text{ is not definably Woodin over } \mathcal{P}.
\end{aligned}$$

By Lemma 6.4.2 (i) the statement “P is ω_1-iterable” for some countable Σ-premouse P which is $(k-1, \Sigma)$-small, is Π_{2k-1}^Σ-definable uniformly in any real coding the premouse P relative to Σ. This yields that the statement $\phi_2(M)$ is Σ_{2k}^Σ-definable uniformly in any code for M (relative to Σ).
Now consider the two cases together again and let

$$\phi(M) = \phi_1(M) \vee \phi_2(M).$$

Then the iteration tree $\mathcal{T}$ witnesses that $\phi(M)$ holds in V (still assuming for simplicity that $\mathcal{T}$ has limit length if $\mathcal{T}$ is as in Case 1).
By our construction of the model $P^k(x; \Sigma)$, see Lemma 6.3.5 (ii), we have that $\phi(M)$ holds in the model $P^k(x; \Sigma)[G]$ for comeager many $G \in V$ which are $\mathrm{Col}(\omega, \delta)$-generic over $P^k(x; \Sigma)$ as $M \in P^k(x; \Sigma)$, M is countable in $P^k(x; \Sigma)[G]$ and $P^k(x; \Sigma)[G]$ is $\mathbf{\Sigma}_{2k+2}^\Sigma$-correct in V for comeager many such $G \in V$.
We want to reflect this counterexample to a countable hull. So we let $Hull_{m,\Sigma}^{P^k(x;\Sigma)}(\{\delta\})$ be the uncollapsed hull inside $P^k(x; \Sigma)$ for some large enough natural number m with the additional predicate Σ and we let H be its Mostowski collapse. Moreover let

$$\pi : H \rightarrow P^k(x; \Sigma)$$

denote the uncollapse map such that we have $\delta, M \in \mathrm{ran}(\pi)$ and let $\bar{\delta}, \bar{M} \in H$ be such that $\pi((\bar{\delta}, \bar{M})) = (\delta, M)$.
Let $g \in P^k(x;\Sigma)$ be $\mathrm{Col}(\omega, \bar{\delta})$-generic over H such that we have

$$H[g] \vDash \phi(\bar{M}).$$

Then we can pick $g \in P^k(x;\Sigma)$ because H is countable in $P^k(x;\Sigma)$.
Let $\bar{\mathcal{T}}$ be a tree on $\bar{M}$ witnessing that $\phi(\bar{M})$ holds in $H[g]$. Then we have that $\bar{\mathcal{T}} \in P^k(x;\Sigma)$. Moreover we have that $\bar{\mathcal{T}}$ also witnesses that $\phi(\bar{M})$ holds in V by absoluteness.
By Lemma 6.4.2 (ii) we have that the ω_1-iterable Σ-premouse $M_{k-1}^{\Sigma,\#}(\mathcal{N}, M|\delta)$ exists in $P^k(x;\Sigma)$. Moreover we have that $M_{k-1}^{\Sigma,\#}(\mathcal{N}, M|\delta)$ is not $(k-1,\Sigma)$-small above δ, because M is pre-(A,k)-suitable. We abbreviate

$$P = M_{k-1}^{\Sigma,\#}(\mathcal{N}, M|\delta).$$

Let $\bar{P}$ be the corresponding model to P inside H. So we have that $\bar{P}$ is not $(k-1,\Sigma)$-small above $\bar{\delta}$. Then we can lift the iteration tree $\bar{\mathcal{T}}$ on $\bar{M}$ to an iteration tree $\mathcal{T}^*$ on $\bar{P}$ inside $P^k(x;\Sigma)$ such that the branches choosen in $\mathcal{T}^*$ are the same as the ones choosen by the $\mathcal{Q}$-structure iteration strategy[1] in H while iterating $\bar{M}$, because the statement $\phi(\bar{M})$ guarantees that the $\mathcal{Q}$-structures for $\bar{\mathcal{T}}$ inside H are iterable enough to stay $\mathcal{Q}$-structures in the model $P^k(x;\Sigma)$.
Now we again distinguish two cases.

Case 1. $\bar{\mathcal{T}}$ witnesses that $\phi_1(\bar{M})$ holds in $H[g]$.

By the argument we gave above, we have that $\bar{\mathcal{T}}$ is in fact a short tree on $\bar{M}$ in $P^k(x;\Sigma)$.
Since M up to its largest cardinal was obtained from a $K^{c,\Sigma}$-construction inside the model $P^k(x;\Sigma)$, it is countably iterable by a generalization to this setting of the iterability proof in Chapter 9 in [St96] applied inside the model $P^k(x;\Sigma)$. Therefore there exists a cofinal well-founded branch $\bar{b}$ through $\bar{\mathcal{T}}$ in $P^k(x;\Sigma)$ such that $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$ is a Σ-premouse.
Assume first that there is a drop along the branch $\bar{b}$. Then it immediately follows that there exists a $\mathcal{Q}$-structure $\mathcal{Q}(\bar{\mathcal{T}}) \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$. Consider the statement

$$\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}})) \equiv \text{"there is a cofinal branch } b \text{ through } \bar{\mathcal{T}} \text{ such that } \mathcal{M}_b^{\bar{\mathcal{T}}} \text{ is a } \Sigma\text{-premouse and } \mathcal{Q}(\bar{\mathcal{T}}) \trianglelefteq \mathcal{M}_b^{\bar{\mathcal{T}}}\text{".}$$

[1]By $\mathcal{Q}$-structure iteration strategy we mean as before the possibly partial iteration strategy Λ such that for a tree $\mathcal{U}$ of limit length and a branch b through $\mathcal{U}$,

$$\Lambda(\mathcal{U}) = b \text{ iff } \mathcal{Q}(b,\mathcal{U}) = \mathcal{Q}(\mathcal{U}).$$

This statement $\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}))$ is Σ_1^Σ-definable uniformly in any code for the parameters $\bar{\mathcal{T}}$ and $\mathcal{Q}(\bar{\mathcal{T}})$ and holds in the model $P^k(x;\Sigma)$ as witnessed by the branch $\bar{b}$.

Let $\gamma < \omega_1^V$ be an ordinal such that $\bar{\mathcal{T}}$ and $\mathcal{Q}(\bar{\mathcal{T}})$ are countable inside the model $H[g]^{\mathrm{Col}(\omega,\gamma)}$. Recall that the model $P^k(x;\Sigma)$ was constructed such that it is $\mathbf{\Sigma}_{2k+2}^\Sigma$-absolute (in the sense of Lemma 6.3.5 (ii)) and recall that H is the Mostowski collapse of $Hull_{m,\Sigma}^{P^k(x;\Sigma)}(\{\delta\})$ for some large natural number m. So let h be $\mathrm{Col}(\omega,\gamma)$-generic over $H[g]$ such that $\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}))$ holds in $H[g][h]$.

Since the branch b witnessing that $\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}))$ holds in $H[g][h]$ is uniquely definable from $\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}) \in H[g]$, it follows by homogeneity of the forcing $\mathrm{Col}(\omega,\gamma)$ that the branch b witnessing $\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}))$ is in fact already an element of $H[g]$. This contradicts the fact that $\bar{\mathcal{T}}$ witnesses in $H[g]$ that $\bar{M}$ is not short tree iterable.

Therefore we can assume that $\bar{b}$ does not drop.

Since $\bar{\mathcal{T}}$ witnesses that $\phi_1(\bar{M})$ holds in $H[g]$, we have that there exists a $\mathcal{Q}$-structure $\mathcal{Q}_\lambda$ for $\bar{\mathcal{T}}$ with $\mathrm{lh}(\bar{\mathcal{T}}) = \lambda$ as in $\phi_1(\bar{M})$. In particular $\mathcal{Q}_\lambda$ is $(k-1,\Sigma)$-small above $\delta(\bar{\mathcal{T}})$ and ω_1-iterable above $\delta(\bar{\mathcal{T}})$ in $H[g]$.

Case 1.1. $\delta(\bar{\mathcal{T}}) = i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$.

Consider the comparison of $\mathcal{Q}_\lambda$ with $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ inside $P^k(x;\Sigma)$.

The comparison takes place above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta}) = \delta(\bar{\mathcal{T}})$. Moreover we have that $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ is (k,Σ)-small above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$ because $P = M_{k-1}^{\Sigma,\#}(\mathcal{N}, M|\delta)$ is (k,Σ)-small above δ.

So the Σ-premouse $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ is iterable above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$ in $P^k(x;\Sigma)$ via the realization strategy. Moreover we have that $\mathcal{Q}_\lambda$ is ω_1-iterable above $\delta(\bar{\mathcal{T}})$ in $H[g]$ thus by $\mathbf{\Sigma}_{2k}^\Sigma$-correctness also inside $P^k(x;\Sigma)$, as the premouse $\mathcal{Q}_\lambda$ is $(k-1,\Sigma)$-small above $\delta(\bar{\mathcal{T}})$.

We have that $\bar{P}$ is sound by construction and thus the non-dropping iterate $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ is sound above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$. Moreover we have that $\rho_\omega(\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}) \leq i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$. In addition $\mathcal{Q}_\lambda$ is also sound above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$ and we have that $\rho_\omega(\mathcal{Q}_\lambda) \leq \delta(\bar{\mathcal{T}}) = i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$. Hence Lemma 6.4.2 (iii) implies that we have

$$\mathcal{Q}_\lambda \lhd \mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \text{ or } \mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \unlhd \mathcal{Q}_\lambda.$$

So we again distinguish two different cases.

Case 1.1.1. $\mathcal{Q}_\lambda \lhd \mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$.

By assumption $\bar{\delta}$ is a Woodin cardinal in $\bar{P}$, because M is pre-(A,k)-suitable and thus δ is a Woodin cardinal in $P = M_{k-1}^{\Sigma,\#}(\mathcal{N}, M|\delta)$. Therefore we have by elementarity that

$$\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \models \text{“}i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta}) \text{ is Woodin”}.$$

But since $\mathcal{Q}_\lambda$ is a $\mathcal{Q}$-structure for $\bar{\mathcal{T}}$, we have that $\delta(\bar{\mathcal{T}}) = i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$ is not definably Woodin over $\mathcal{Q}_\lambda$. This contradicts $\mathcal{Q}_\lambda \lhd \mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$.

Case 1.1.2. $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \trianglelefteq \mathcal{Q}_\lambda$.

The Σ-premouse $\bar{P}$ is not $(k-1, \Sigma)$-small above $\bar{\delta}$ and therefore it follows that $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ is not $(k-1, \Sigma)$-small above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$. But then $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \trianglelefteq \mathcal{Q}_\lambda$ contradicts the fact that $\mathcal{Q}_\lambda$ is $(k-1, \Sigma)$-small above $\delta(\bar{\mathcal{T}}) = i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$. This contradiction finishes Case 1.1.

Case 1.2. $\delta(\bar{\mathcal{T}}) < i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$.

We have that $P = M_{k-1}^{\Sigma,\#}(\mathcal{N}, M|\delta)$ is $(k+1, \Sigma)$-small and (k, Σ)-small above δ because M is pre-(A, k)-suitable. Therefore $\mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ is $(k+1, \Sigma)$-small and (k, Σ)-small above $i_{\bar{b}}^{\mathcal{T}^*}(\bar{\delta})$. Hence it follows that

$$\mathcal{M}_{\bar{b}}^{\mathcal{T}^*} \models \text{“}\delta(\bar{\mathcal{T}}) \text{ is not Woodin”}.$$

This implies that $\mathcal{Q}(\bar{\mathcal{T}}) \lhd \mathcal{M}_{\bar{b}}^{\mathcal{T}^*}$ and therefore we have that

$$\mathcal{Q}(\bar{\mathcal{T}}) \trianglelefteq \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}.$$

Now we can again consider the statement

$$\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}})) \equiv \text{ “there is a cofinal branch } b \text{ through } \bar{\mathcal{T}} \text{ such that } \mathcal{M}_b^{\bar{\mathcal{T}}} \text{ is a } \Sigma\text{-premouse and } \mathcal{Q}(\bar{\mathcal{T}}) \trianglelefteq \mathcal{M}_b^{\bar{\mathcal{T}}}\text{”}.$$

Again we have that the statement $\psi(\bar{\mathcal{T}}, \mathcal{Q}(\bar{\mathcal{T}}))$ holds in the model $P^k(x; \Sigma)$ as witnessed by the branch $\bar{b}$. By an absoluteness argument as above it also holds in $H[g]$, which contradicts the fact that $\bar{\mathcal{T}}$ witnesses in $H[g]$ that $\bar{M}$ is not short tree iterable.

Case 2. $\bar{\mathcal{T}}$ witnesses that $\phi_2(\bar{M})$ holds in $H[g]$.

Recall that in this case $\bar{\mathcal{T}}$ is a tree of successor length and let $\bar{\lambda} < \omega_1^V$ be such that $\text{lh}(\mathcal{T}) = \bar{\lambda} + 1$.
Since $\phi_2(\bar{M})$ holds in $H[g]$, there exists a model

$$\mathcal{R} \trianglerighteq \mathcal{M}_{\bar{\lambda}}^{\bar{\mathcal{T}}} | i_{0\bar{\lambda}}^{\bar{\mathcal{T}}}(\bar{\delta}) = \mathcal{M}_{\bar{\lambda}}^{\mathcal{T}^*} | i_{0\bar{\lambda}}^{\mathcal{T}^*}(\bar{\delta})$$

such that $\mathcal{R}$ is $(k-1, \Sigma)$-small above $i_{0\bar{\lambda}}^{\mathcal{T}^*}(\bar{\delta})$ and ω_1-iterable above $i_{0\bar{\lambda}}^{\mathcal{T}^*}(\bar{\delta})$, where $i_{0\bar{\lambda}}^{\mathcal{T}^*} : \bar{P} \to \mathcal{M}_{\bar{\lambda}}^{\mathcal{T}^*}$ denotes the usual iteration embedding.
Consider the comparison of $\mathcal{R}$ with $\mathcal{M}_{\bar{\lambda}}^{\mathcal{T}^*}$ inside $H[g]$. We have that $\mathcal{M}_{\bar{\lambda}}^{\mathcal{T}^*}$ is ω_1-iterable above $i_{0\bar{\lambda}}^{\mathcal{T}^*}(\bar{\delta})$ inside $H[g]$, as $\mathcal{M}_{\bar{\lambda}}^{\mathcal{T}^*}$ is an iterate of $\bar{P}$.

Moreover we have that $\mathcal{M}_{\bar\lambda}^{\mathcal{T}^*}$ is sound above $i_{0\bar\lambda}^{\mathcal{T}^*}(\bar\delta)$ and $\rho_\omega(\mathcal{M}_{\bar\lambda}^{\mathcal{T}^*}) \leq i_{0\bar\lambda}^{\mathcal{T}^*}(\bar\delta)$. We have that $\phi_2(\bar M)$ additionally gives that $\mathcal{R}$ is sound above $i_{0\bar\lambda}^{\mathcal{T}^*}(\bar\delta)$ and that we have $\rho_\omega(\mathcal{R}) \leq i_{0\bar\lambda}^{\mathcal{T}^*}(\bar\delta)$. Therefore the proof of Lemma 6.4.2 yields that we in fact have

$$\mathcal{R} \lhd \mathcal{M}_{\bar\lambda}^{\mathcal{T}^*} \text{ or } \mathcal{M}_{\bar\lambda}^{\mathcal{T}^*} \unlhd \mathcal{R}.$$

So we consider two different cases.

Case 2.1. $\mathcal{R} \lhd \mathcal{M}_{\bar\lambda}^{\mathcal{T}^*}$.

By our assumptions $\bar\delta$ is a Woodin cardinal in $\bar P$ and therefore $i_{0\bar\lambda}^{\mathcal{T}^*}(\bar\delta)$ is a Woodin cardinal in $\mathcal{M}_{\bar\lambda}^{\mathcal{T}^*}$ by elementarity. On the other hand $\phi_2(\bar M)$ implies that $i_{0\bar\lambda}^{\mathcal{T}^*}(\bar\delta)$ is not definably Woodin over $\mathcal{R}$. This contradicts our case assumption that $\mathcal{R} \lhd \mathcal{M}_{\bar\lambda}^{\mathcal{T}^*}$.

Case 2.2. $\mathcal{M}_{\bar\lambda}^{\mathcal{T}^*} \unlhd \mathcal{R}$.

We have that $\bar P$ is not $(k-1,\Sigma)$-small above $\bar\delta$ and therefore by elementarity $\mathcal{M}_{\bar\lambda}^{\mathcal{T}^*}$ is not $(k-1,\Sigma)$-small above $i_{0\bar\lambda}^{\mathcal{T}^*}(\bar\delta)$. But this contradicts the fact that $\mathcal{R}$ is $(k-1,\Sigma)$-small above $i_{0\bar\lambda}^{\mathcal{T}^*}(\bar\delta)$. □

So all in all we proved that there exists a Σ-premouse M which is (A,k)-suitable. □

7.3. $M_k^{\Sigma,\#}$ from a Level of Determinacy

In this section we will finally construct the ω_1-iterable Σ-premouse $M_k^{\Sigma,\#}$ from a level of determinacy. In the previous sections we constructed a Σ-premouse M which is (A,k)-suitable. We were able to prove short tree iterability for this premouse M, but we do not know how to prove ω_1-iterability for such a model outright.

For the construction of $M_k^{\Sigma,\#}$ we will prove iterability for a different premouse, but we will use what we have done so far for this proof. For our argument, which is along the lines of the proof of Theorem 3.4.1, we need to assume one projective level of determinacy above Γ more than we did in the previous sections.

Some parts of the argument will be similar to the proof of Theorem 3.4.1 in the projective hierarchy we gave earlier in Section 3.4, but we will give most of the details again to make sure that the argument works in this context with the adaptations we need to make.

Theorem 7.3.1. *Let $k < \omega$ and let Γ, Σ be as above. Assume that every $\boldsymbol{\Sigma}^1_{2k+2}\Gamma$-definable set of reals is determined. Moreover assume that there is no $\boldsymbol{\Sigma}^1_{2k+4}\Gamma$-definable ω_1-sequence of pairwise distinct reals. Then the Σ-premouse $M_k^{\Sigma,\#}$ exists and is ω_1-iterable.*

Using Theorem 6.2.1 we can immediately obtain the following corollary.

COROLLARY 7.3.2. *Let $k < \omega$ and let Γ, Σ be as above. Assume that every $\mathbf{\Pi}^1_{2k+3}\Gamma$-definable set of reals is determined. Then the Σ-premouse $M_k^{\Sigma,\#}(x)$ exists and is ω_1-iterable for all reals x.*

The rest of this section is devoted to the proof of Theorem 7.3.1.

PROOF OF THEOREM 7.3.1. Fix an (A,k)-suitable Σ-premouse M as constructed in Theorem 7.2.1.

The following proof is inductive and devides into three parts. So assume as before that we already proved Theorem 7.3.1, or in fact Corollary 7.3.2, for $k-1$. Then we first construct a powerful model W_x which contains a real x and the countable Σ-premouse M we fixed above similar to the construction of $P^k(x;\Sigma)$. Afterwards we show that the $K^{c,\Sigma}$-construction in that model reaches a level which is not (k,Σ)-small. The last step will be to show that this is enough to obtain an ω_1-iterable Σ-premouse $M_k^{\Sigma,\#}$ in V.

Step 1: We fix an arbitrary real x and start with the construction of the model W_x. We aim to construct the model W_x such that it satisfies the following properties.

(1) $x, M \in W_x$,
(2) $W_x \vDash \mathrm{ZFC}$, $W_x \cap \mathrm{Ord} = \omega_1^V$,
(3) $W_x \prec_{\mathbf{\Sigma}^\Sigma_{2k+3}} V$, and
(4) W_x is closed under the operation $a \mapsto M_{k-1}^{\Sigma,\#}(a)$.

We will construct W_x level-by-level by constructing a sequence of models $(W_\alpha \mid \alpha < \omega_1^V)$ such that we add Σ-premice $M_{k-1}^{\Sigma,\#}(a)$ and witnesses for the statement

$$W_x \prec_{\mathbf{\Sigma}^\Sigma_{2k+3}} V$$

along the way. During the construction we will ensure that the resulting model will be nicely definable, namely $\Sigma^1_{2k+4}\Gamma$-definable from the real x and a code for M (relative to Σ). Moreover we will define the order of construction for elements of the model W_x along the way.

Start from $W_0 = \{x, M\}$. We will use the odd successor steps of our construction to ensure that property (3) holds for W_x. Then we will use the even successor steps to add witnesses for property (4).

Before we are going to describe the construction at the successor levels in more detail we fix a $\Pi^1_{2k+3}\Gamma$-definable set U which is universal for $\mathbf{\Pi}^1_{2k+3}\Gamma$-definable sets in V. Moreover let $U^\Gamma \in \Gamma$ be a universal set for the pointclass Γ, which exists as Γ is $\mathbb{R}$-parametrized. We can pick U such that we have $U_{\ulcorner\varphi\urcorner\frown(a\oplus b)} = A^\Gamma_{\varphi,a,b}$ for every Π^1_{2k+3}-formula φ and every $a, b \in {}^\omega\omega$, where $\ulcorner\varphi\urcorner$ denotes the Gödel number of the formula φ and

$$A^\Gamma_{\varphi,a,b} = \{x \mid \varphi(x, a, U^\Gamma_b, {}^\omega\omega \setminus U^\Gamma_b)\},$$

where as usual this means that U_b^Γ is allowed to occur positively and negatively in φ.

Moreover we fix a $\Pi^1_{2k+3}\Gamma$-definable uniformizing function F for U. That means for all $z \in \text{dom}(F)$ we have that

$$(z, F(z)) \in U,$$

where $\text{dom}(F) = \{z \mid \exists y\, (z,y) \in U\}$. We have that U and F as above exist by our assumptions on Γ (see second remark after Definition 6.1.7), because we additionally assume that every set of reals in $\boldsymbol{\Sigma}^1_{2k+2}\Gamma$ is determined.

Odd successor steps: Assume that we already constructed the model W_α for some ordinal $\alpha < \omega_1^V$ such that $\alpha = 0$, α is a limit ordinal or α is an even successor ordinal. Then we close W_α under the function F we fixed above. That means we let

$$W_{\alpha+1} = rud(W_\alpha \cup \{y \in {}^\omega\omega \mid \exists z \in W_\alpha \cap {}^\omega\omega\ \varphi_F(z, y, U_b^\Gamma, {}^\omega\omega \setminus U_b^\Gamma)\}),$$

where φ_F is a Π^1_{2k+3}-formula and b is a fixed real such that for all $z, y \in {}^\omega\omega$

$$F(z) = y \text{ iff } \varphi_F(z, y, U_b^\Gamma, {}^\omega\omega \setminus U_b^\Gamma).$$

Order of construction: We say that $F(z)$ is constructed before $F(z')$ for $F(z) \neq F(z')$ with $z, z' \in \text{dom}(F) \cap W_\alpha$ if z is constructed before z' in the order of construction for elements of W_α and z, z' are the minimal $y, y' \in \text{dom}(F) \cap W_\alpha$ such that $F(z) = F(y)$ and $F(z') = F(y')$. Moreover we define the order of construction for elements added by the closure under rudimentary functions analogous to the order of construction for L.

Even successor steps: Assume that we already constructed the model W_α for some odd successor ordinal $\alpha < \omega_1^V$. Let $a \in W_\alpha$ be such that $M_{k-1}^{\Sigma,\#}(a)$ does not exist in W_α. The (Σ, a)-premouse $M_{k-1}^{\Sigma,\#}(a)$ exists in V and is (k, Σ)-small and ω_1-iterable there, because we inductively assume that this follows from our hypothesis that $\boldsymbol{\Pi}^1_{2k+1}\Gamma$ determinacy holds, i.e. we inductively assume that Corollary 7.3.2 holds. Let $\mathcal{M}$ be a countable (Σ, a)-premouse in V with the following properties.

- (i) $\mathcal{M}$ is (k, Σ)-small, but not $(k-1, \Sigma)$-small,
- (ii) all proper initial segments of $\mathcal{M}$ are $(k-1, \Sigma)$-small,
- (iii) $\mathcal{M}$ is a-sound, and $\rho_\omega(\mathcal{M}) = a$, and
- (iv) $\mathcal{M}$ is ω_1-iterable.

These properties uniquely determine the premouse $M_{k-1}^{\Sigma,\#}(a)$ in V using Lemma 6.4.2. We let $W_{\alpha+1}$ be the model obtained by taking the closure under rudimentary functions of W_α together with all such (Σ, a)-premice $\mathcal{M}$ as above for all $a \in W_\alpha$.

Order of construction: For a (Σ, a)-premouse $\mathcal{M}_a$ and a (Σ, b)-premouse $\mathcal{M}_b$ satifying properties $(i) - (iv)$ with $\mathcal{M}_a \neq \mathcal{M}_b$ for $a, b \in W_\alpha$, we say that $\mathcal{M}_a$ is defined before $\mathcal{M}_b$ if a is defined before b in the order of construction for elements of W_α, which exists inductively, and a, b are the minimal $a', b' \in W_\alpha$ such that $\mathcal{M}_a = \mathcal{M}_{a'}$ and $\mathcal{M}_b = \mathcal{M}_{b'}$ where $\mathcal{M}_{a'}$ and $\mathcal{M}_{b'}$ also satisfy

$(i) - (iv)$. For elements added by the closure under rudimentary functions we define the order of construction analogous to the order of construction for L.

Limit steps: At limit steps of the construction we take unions. That means if $\lambda \leq \omega_1^V$ is a limit ordinal and we already constructed W_α for all $\alpha < \lambda$, then we let

$$W_\lambda = \bigcup_{\alpha<\lambda} W_\alpha.$$

Finally we let

$$W_x = W_{\omega_1^V} = \bigcup_{\alpha<\omega_1^V} W_\alpha.$$

Order of construction: At limit steps we define the order of construction analogous to the order of construction for L.

For our proof of the following claim we now need to use the additional hypothesis that there exists no $\mathbf{\Sigma}^1_{2k+4}\Gamma$-definable ω_1-sequence of pairwise distinct reals. This is done in the same way as before, for example for the model $P^k(x;\Sigma)$.

CLAIM 1. $W_x \vDash$ ZFC.

PROOF. Assume this is not the case. Then the power set axiom has to fail. So let γ be a countable ordinal such that

$$\mathcal{P}(\gamma) \cap W_x \notin W_x.$$

This implies that the set $\mathcal{P}(\gamma) \cap W_x$ has size $\aleph_1$.

Let $W_\gamma = W_x|\gamma$ be the γ-th level in the construction of W_x. Then we can fix a real a in V which codes the countable set W_γ.

If it exists, we let A_ξ for $\gamma < \xi < \omega_1^V$ be the smallest subset of γ in

$$W_x \,|\, (\xi + 1) \ \setminus \ W_x \,|\, \xi$$

according to the order of construction. Moreover we let X be the set of all ordinals ξ with $\gamma < \xi < \omega_1^V$ such that A_ξ exists. Then X is cofinal in ω_1^V.

Finally we let a_ξ again be a real coding A_ξ relative to the code a we fixed for W_γ. For $\xi \in X$ we have that $A_\xi \in \mathcal{P}(\gamma) \cap W_x$ and thus $A_\xi \subseteq W_\gamma$, so the canonical code a_ξ for A_ξ relative to a exists.

Now we consider the following ω_1^V-sequence of reals

$$A = (a_\xi \in {}^\omega\omega \mid \xi \in X).$$

Analogous to the proof of the levels $\alpha + 2$ in the proof of Lemma 6.3.4 in Section 6.3 and to the proof of Claim 3 in the proof of Theorem 3.4.1 (using Lemma 6.4.2 (i) in this setting) it follows that the sequence A as defined above is $\mathbf{\Sigma}^1_{2k+4}\Gamma$-definable in the parameters a, x and a code for M. Hence A contradicts the additional hypothesis that there exists no $\mathbf{\Sigma}^1_{2k+4}\Gamma$-definable ω_1-sequence of pairwise distinct reals. $\square$

Moreover the construction gives the following claim.

Claim 2. *We have that*

(*i*) $x, M \in W_x$,
(*ii*) $W_x \cap \mathrm{Ord} = \omega_1^V$,
(*iii*) $W_x \prec_{\mathbf{\Sigma}^{\Sigma}_{2k+3}} V$, *and*
(*iv*) *the model* W_x *is closed under the operation*

$$a \mapsto M_{k-1}^{\Sigma,\#}(a).$$

Proof. Properties (*i*) and (*ii*) immediately follow from the construction. Moreover we have that property (*iii*) follows from the construction at the odd successor levels exactly as in the proof of property (*i*) in Lemma 6.3.5 and property (*iv*) follows from the construction at the even successor levels analogous to the proof of Claim 4 (3) in the proof of Theorem 3.4.1 using Lemma 6.4.2. □

So we constructed the model W_x as desired.
Step 2: In the next step we want to prove that a $K^{c,\Sigma}$-construction inside the model W_x reaches the Σ-premouse $M_k^{\Sigma,\#}$. For this reason we assume toward a contradiction that

$$W_x \vDash \text{“}M_k^{\Sigma,\#} \text{ does not exist”}.$$

As usual by “$M_k^{\Sigma,\#}$ exists” we mean that “$M_k^{\Sigma,\#}$ exists and is ω_1-iterable”. With $(K^{c,\Sigma})^{W_x}$ we denote the result of a K^c-construction from [MSch04] generalized to the hybrid context as in Section 5.2 and furthermore generalized to (k,Σ)-small Σ-premice. Moreover the $K^{c,\Sigma}$-construction is performed inside the model W_x. The following claim is now crucial to prove that $M_k^{\Sigma,\#}$ exists in V.

Claim 3. $(K^{c,\Sigma})^{W_x}$ *is not* (k,Σ)*-small.*

Proof. Work inside the model W_x for the whole proof of this claim and distinguish the following cases.

Case 1. Assume that

$$K^{c,\Sigma} \vDash \text{“there is a Woodin cardinal”}.$$

This means that there exists a largest Woodin cardinal, say δ, in $K^{c,\Sigma}$, because otherwise $K^{c,\Sigma}$ would certainly not be (k,Σ)-small.
Now let

$$\mathcal{M} = M_{k-1}^{\Sigma,\#}(\mathcal{N}, (K^{c,\Sigma})|\delta)$$

be the model as constructed in the sense of the remark after Definition 7.1.1. Then $\mathcal{M}$ exists in W_x by Claim 2, so we can consider the coiteration of $K^{c,\Sigma}$ with $\mathcal{M}$ inside W_x.

This coiteration is successful, because it takes place above δ and furthermore $\mathcal{M}$ is iterable enough for a successful comparison above δ by construction (see also the proof of Lemma 6.4.2) and $K^{c,\Sigma}$ is iterable enough for a successful comparison above δ by the straightforward generalization to the hybrid context of Subclaim 1 in the proof of Claim 5 in the proof of Theorem 3.4.1. Therefore there exists an iterate $\mathcal{R}$ of $K^{c,\Sigma}$ and an iterate $\mathcal{M}^*$ of $\mathcal{M}$ such that we have

$$\mathcal{R} \trianglelefteq \mathcal{M}^* \text{ or } \mathcal{M}^* \trianglelefteq \mathcal{R}.$$

By universality above δ inside the model W_x (which follows from a generalization of Section 3 in [MSch04]) the $K^{c,\Sigma}$-side has to win the comparison, that means there is no drop on the $\mathcal{M}$-side of the coiteration and we have that

$$\mathcal{M}^* \trianglelefteq \mathcal{R}.$$

This implies in particular that the construction of the Σ-premouse $\mathcal{M} = M_{k-1}^{\Sigma,\#}(\mathcal{N}, (K^{c,\Sigma})|\delta)$ does not stop, that means we have that $\mathcal{M}$ is not $(k-1, \Sigma)$-small above δ because otherwise $\mathcal{M}$ is not fully sound and since $\mathcal{M}^* \trianglelefteq \mathcal{R}$ this yields a contradiction because of soundness.

Therefore it follows that the Σ-premouse $\mathcal{M}$ is not (k, Σ)-small and thus $\mathcal{M}^*$ is also not (k, Σ)-small. Hence we have that $\mathcal{R}$ and finally $K^{c,\Sigma}$ is not (k, Σ)-small as claimed.

Case 2. Assume that

$$K^{c,\Sigma} \models \text{“there is no Woodin cardinal”}.$$

Recall that we fixed an (A, k)-suitable Σ-premouse M at the beginning of this proof and consider the coiteration of the Σ-premice $K^{c,\Sigma}$ and M inside the model W_x. Let $\mathcal{T}$ and $\mathcal{U}$ be the iteration trees on $K^{c,\Sigma}$ and M respectively resulting from the coiteration. We stop the coiteration if it terminates successfully or if it reaches a maximal tree $\mathcal{U}$ on M. As in Case 1 we have that the coiteration cannot fail on the $K^{c,\Sigma}$-side by the straightforward generalization to the hybrid context of Subclaim 1 in the proof of Claim 5 in the proof of Theorem 3.4.1.

Then we distinguish the following two subcases.

Case 2.1. Assume that $\mathcal{U}$ is a short tree on M.

Then the coiteration of $K^{c,\Sigma}$ and M is successful because M is short tree iterable by definition. In particular $\mathcal{T}$ and $\mathcal{U}$ are iteration trees of length $\lambda + 1$ for some ordinal λ and if we let $\mathcal{R} = \mathcal{M}_\lambda^\mathcal{T}$ and $M^* = \mathcal{M}_\lambda^\mathcal{U}$, then we have that

$$\mathcal{R} \trianglelefteq M^* \text{ or } M^* \trianglelefteq \mathcal{R}.$$

By universality of $K^{c,\Sigma}$ inside W_x (which again follows from a generalization of Section 3 in [MSch04]) we have that the $K^{c,\Sigma}$-side wins the comparison.

That means we have that

$$M^* \trianglelefteq \mathcal{R}$$

and there is no drop on the M-side of the coiteration.

Since M is (A,k)-suitable, the fact that $\mathcal{U}$ is a short tree implies that M^* is also pre-(A,k)-suitable by fullness preservation for non-dropping short trees. Let δ^* be the largest cardinal in M^*. Then we have in particular that $M_{k-1}^{\Sigma,\#}(\mathcal{N}, M^*|\delta^*)$ is a Σ-premouse with k Woodin cardinals which is ω_1-iterable above δ^*. Therefore we can consider the coiteration of the Σ-premice

$$\mathcal{R} \text{ and } M_{k-1}^{\Sigma,\#}(\mathcal{N}, M^*|\delta^*).$$

We have that $M^* \trianglelefteq \mathcal{R}$ and that both $M_{k-1}^{\Sigma,\#}(\mathcal{N}, M^*|\delta^*)$ and $\mathcal{R}$ are iterable above δ^*, so the coiteration is successful. If there is no drop on the main branch through $\mathcal{T}$, then $\mathcal{R} = \mathcal{M}_\lambda^\mathcal{T}$ has to win this comparison by universality of $K^{c,\Sigma}$ again. If there is a drop on the main branch through $\mathcal{T}$, then we have

$$\rho_\omega(\mathcal{R}) < \delta^* \text{ and } \rho_\omega(M_{k-1}^{\Sigma,\#}(\mathcal{N}, M^*|\delta^*)) = \delta^*,$$

so $\mathcal{R}$ again wins the comparison with $M_{k-1}^{\Sigma,\#}(\mathcal{N}, M^*|\delta^*)$. Therefore there exists an iterate $\mathcal{R}^*$ of $\mathcal{R}$ and a non-dropping iterate M^{**} of $M_{k-1}^{\Sigma,\#}(\mathcal{N}, M^*|\delta^*)$ such that we have

$$M^{**} \trianglelefteq \mathcal{R}^*.$$

As argued above we have that $M_{k-1}^{\Sigma,\#}(\mathcal{N}, M^*|\delta^*)$ is not (k,Σ)-small. Therefore it follows by elementarity that M^{**} cannot be (k,Σ)-small and thus we have that $\mathcal{R}^*$ is not (k,Σ)-small. After all $\mathcal{R}$ and thus $K^{c,\Sigma}$ is not (k,Σ)-small, because $\mathcal{R}$ is an iterate of $K^{c,\Sigma}$. This finishes the case that $\mathcal{U}$ is a short tree on M.

Case 2.2. Assume that $\mathcal{U}$ is a maximal tree on M.

Then we have that $\mathcal{U}$ has limit length and

$$M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{U})) \vDash \text{“}\delta(\mathcal{U}) \text{ is Woodin”}.$$

The Σ-premouse $M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{U}))$ is not $(k-1,\Sigma)$-small above $\delta(\mathcal{U})$ by maximality of $\mathcal{U}$, because otherwise $M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{U}))$ would provide a $\mathcal{Q}$-structure $\mathcal{Q}$ for $\mathcal{U}$ which is $(k-1,\Sigma)$-small above $\delta(\mathcal{U})$, witnessing that $\mathcal{U}$ is short.

As above we can write $\mathcal{R} = \mathcal{M}_\lambda^\mathcal{T}$ as $K^{c,\Sigma}$ is iterable enough for the comparison and therefore we can extend the iteration tree on $K^{c,\Sigma}$ one more step by a cofinal well-founded branch to obtain the limit model $\mathcal{M}_\lambda^\mathcal{T}$. Now consider the coiteration of the Σ-premice

$$M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{U})) \text{ and } \mathcal{R} = \mathcal{M}_\lambda^\mathcal{T}.$$

This coiteration takes place above $\delta(\mathcal{U}) = \delta(\mathcal{T} \upharpoonright \lambda)$ and since both Σ-premice $M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{U}))$ and $\mathcal{R}$ are iterable enough above $\delta(\mathcal{U})$, the coiteration is successful.

As before, if there is no drop on the main branch through $\mathcal{T}$, then $\mathcal{R} = \mathcal{M}_\lambda^\mathcal{T}$ wins the comparison by universality of $K^{c,\Sigma}$. If there is a drop on the main branch through $\mathcal{T}$, then we have

$$\rho_\omega(\mathcal{R}) < \delta(\mathcal{U}) = \mathcal{M}(\mathcal{U}) \cap \mathrm{Ord},$$

so $\mathcal{R}$ again wins the comparison. That means there is an iterate $\mathcal{R}^*$ of $\mathcal{R}$ and a non-dropping iterate $\mathcal{M}^*$ of $M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{U}))$ such that we have

$$\mathcal{M}^* \trianglelefteq \mathcal{R}^*.$$

As argued above the Σ-premouse $M_{k-1}^{\Sigma,\#}(\mathcal{N}, \mathcal{M}(\mathcal{U}))$ is not (k, Σ)-small. So we can argue as in the previous case to conclude that $\mathcal{R}$ is not (k, Σ)-small and therefore $K^{c,\Sigma}$ is not (k, Σ)-small.

This finishes the case that $\mathcal{U}$ is a maximal tree on M and thus finishes the proof of Claim 3. □

Step 3: In the last step of the proof of Theorem 7.3.1 we now want to show that $M_k^{\Sigma,\#}$ exists in V and is ω_1-iterable. For this purpose we now work in V again.

Let x be an arbitrary real. Then Claim 3 implies that

$$W_x \vDash \text{“}M_k^{\Sigma,\#} \text{ exists and is } \omega_1\text{-iterable”}.$$

Since $M_k^{\Sigma,\#}$ is $(k+1, \Sigma)$-small, the statement “$M_k^{\Sigma,\#}$ is ω_1-iterable” is Π^Σ_{2k+3}-definable uniformly in any code for $M_k^{\Sigma,\#}$ (relative to Σ) by Lemma 6.4.2 (i). Here $\mathbf{\Pi}^1_{2k+1}\Gamma$ determinacy suffices to prove this as remarked in the proof of Lemma 6.4.2. Since the model W_x is $\mathbf{\Sigma}^\Sigma_{2k+3}$-correct in V this implies that

$$V \vDash \text{“}(M_k^{\Sigma,\#})^{W_x} \text{ is } \omega_1^{W_x}\text{-iterable”}.$$

Let y be a real such that $(M_k^{\Sigma,\#})^{W_x} \in W_y$ and $\omega_1^{W_x} \leq \omega_1^{W_y}$. Then it follows by correctness again that

$$W_y \vDash \text{“}(M_k^{\Sigma,\#})^{W_x} \text{ is } \omega_1^{W_x}\text{-iterable”}.$$

This suffices to successfully compare the Σ-premice $(M_k^{\Sigma,\#})^{W_x}$ and $(M_k^{\Sigma,\#})^{W_y}$ and therefore we have as argued several times before that $(M_k^{\Sigma,\#})^{W_x} = (M_k^{\Sigma,\#})^{W_y}$. So we have that $(M_k^{\Sigma,\#})^{W_x}$ is the same for a cone of reals x and we call this Σ-premouse $M_k^{\Sigma,\#}$.

Now an easy argument shows that $M_k^{\Sigma,\#}$ is ω_1^V-iterable in V as

$$V \vDash \text{“}M_k^{\Sigma,\#} \text{ is } \omega_1^{W_x}\text{-iterable”}$$

for a cone of reals x. Therefore it follows that we have

$$V \vDash \text{“}M_k^{\Sigma,\#} \text{ exists and is } \omega_1\text{-iterable”}$$

and we finished the proof of Theorem 7.3.1. □

CHAPTER 8

Applications to the Core Model Induction

In this chapter we will finally outline some applications of the results in Part 2 of this thesis and mention related open problems. In particular we will sketch how to obtain premice $\mathcal{N}$ and pointclasses Γ as used in Chapters 6 and 7.

8.1. The $L(\mathbb{R})$-hierarchy

The application we are interested in is related to determinacy at the individual levels of the $L(\mathbb{R})$-hierarchy. Thus we use this section to briefly remind the reader of the definitions and the relevant properties of the $L(\mathbb{R})$-hierarchy as developed in [St08]. Therefore if not specified otherwise all definitions and results in this section are due to John R. Steel and can be found in [St08].

Definition 8.1.1. *We define the following hierarchy in $L(\mathbb{R})$. Let*

$$J_1(\mathbb{R}) = V_{\omega+1},$$

$$J_{\alpha+1}(\mathbb{R}) = rud(J_\alpha(\mathbb{R})), \text{ for } \alpha > 0,$$

and

$$J_\lambda(\mathbb{R}) = \bigcup_{\alpha<\lambda} J_\alpha(\mathbb{R}), \text{ for } \lambda \text{ limit.}$$

Then we let $L(\mathbb{R}) = \bigcup_{\alpha\in\mathrm{Ord}} J_\alpha(\mathbb{R})$.

Section 1 of [St08] describes the fine structure of this hierarchy. We will not go into any details here anyway, so we omit the presentation of the fine structure of $L(\mathbb{R})$.

We focus on the scale property in the $L(\mathbb{R})$-hierarchy. To outline the situation there, it is important to notice that this hierarchy is divided into gaps which are defined as follows.

Definition 8.1.2. *For ordinals $\alpha \leq \beta$ we say that the interval $[\alpha, \beta]$ is a* Σ_1-gap *iff*

(i) $J_\alpha(\mathbb{R}) \prec_1^{\mathbb{R}} J_\beta(\mathbb{R})$,
(ii) $\forall \alpha' < \alpha \, (J_{\alpha'}(\mathbb{R}) \not\prec_1^{\mathbb{R}} J_\alpha(\mathbb{R}))$, *and*
(iii) $\forall \beta' > \beta \, (J_\beta(\mathbb{R}) \not\prec_1^{\mathbb{R}} J_{\beta'}(\mathbb{R}))$.

REMARK. As in [St08] we write $M \prec_1^{\mathbb{R}} N$ for models M and N iff for all parameters $a \in \mathbb{R}$ and all Σ_1-formulae φ,

$$M \vDash \varphi(a) \text{ iff } N \vDash \varphi(a).$$

Let us fix the following notation.

DEFINITION 8.1.3. *Let $n \geq 1$ and let α be an ordinal. Then we let $\boldsymbol{\Sigma}_n(J_\alpha(\mathbb{R}))$ denote the collection of all sets which are definable over the model $(J_\alpha(\mathbb{R}), \in)$ by a Σ_n-formula with parameters from $J_\alpha(\mathbb{R})$.*
Moreover the lightface version $\Sigma_n(J_\alpha(\mathbb{R}))$ denotes the collection of all sets which are definable over the model $(J_\alpha(\mathbb{R}), \in)$ by a Σ_n-formula with parameters from $V_{\omega+1}$.

REMARK. Whenever we say for example "$\boldsymbol{\Sigma}_n(J_\alpha(\mathbb{R}))$ has the scale property" we in fact mean that "$\boldsymbol{\Sigma}_n(J_\alpha(\mathbb{R})) \cap \mathcal{P}(\mathbb{R})$ has the scale property".

At the beginning of a gap the picture is as follows.

LEMMA 8.1.4. *If an ordinal $\alpha > 1$ begins a Σ_1-gap and every set in $J_\alpha(\mathbb{R})$ is determined, then $\boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R}))$ has the scale property.*

The picture looks different depending on the admissibility of $J_\alpha(\mathbb{R})$. Option (2) in the following lemma is due to Martin in [Ma08].

LEMMA 8.1.5. *Let $\alpha > 1$ be an ordinal such that α begins a Σ_1-gap. Suppose that every set in $J_{\alpha+1}(\mathbb{R})$ is determined.*

(1) *If the set $J_\alpha(\mathbb{R})$ is not admissible, then for all $n < \omega$, the pointclasses $\boldsymbol{\Sigma}_{2n+1}(J_\alpha(\mathbb{R}))$ and $\boldsymbol{\Pi}_{2n+2}(J_\alpha(\mathbb{R}))$ have the scale property.*
(2) *If $J_\alpha(\mathbb{R})$ is admissible, then none of the pointclasses $\boldsymbol{\Sigma}_n(J_\alpha(\mathbb{R}))$ and $\boldsymbol{\Pi}_n(J_\alpha(\mathbb{R}))$ for $n > 1$ has the scale property.*

The following lemma states that scales do not appear inside a gap.

LEMMA 8.1.6. *If $\alpha < \gamma < \beta$ are ordinals such that $[\alpha, \beta]$ is a Σ_1-gap, then none of the pointclasses $\boldsymbol{\Sigma}_n(J_\gamma(\mathbb{R}))$ or $\boldsymbol{\Pi}_n(J_\gamma(\mathbb{R}))$ for $n < \omega$ has the scale property.*

The most interesting behavior happens at the end of a gap. There we again have to distinguish two different cases, which are given by the following definitions.

DEFINITION 8.1.7. *We say that an ordinal β is* strongly Π_n-reflecting *iff every Σ_n-type realized in $J_\beta(\mathbb{R})$ is realized in $J_\alpha(\mathbb{R})$ for some $\alpha < \beta$, where the Σ_n-type realized by some a in $J_\beta(\mathbb{R})$ is defined as*

$$\{\varphi \mid \varphi \textit{ is a } \Sigma_n\textit{- or a } \Pi_n\textit{-formula and } J_\beta(\mathbb{R}) \vDash \varphi(a)\}.$$

DEFINITION 8.1.8. *Let $\alpha < \beta$ be ordinals such that $[\alpha, \beta]$ is a Σ_1-gap. We say that $[\alpha, \beta]$ is a* strong Σ_1-gap *iff β is strongly Π_n-reflecting, where $n < \omega$ is the least natural number such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$. Otherwise we call $[\alpha, \beta]$ a* weak Σ_1-gap.

Then we get the following picture at the end of a gap. Here option (2) is again a corollary of Martin's work in [Ma08].

LEMMA 8.1.9. *Let $\alpha < \beta$ be ordinals such that $[\alpha, \beta]$ is a Σ_1-gap and suppose that every set in $J_\alpha(\mathbb{R})$ is determined.*

(1) *If $[\alpha, \beta]$ is a weak Σ_1-gap and $n < \omega$ is least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$, then we have that for $k < \omega$ each of the pointclasses $\boldsymbol{\Sigma}_{n+2k}(J_\beta(\mathbb{R}))$ and $\boldsymbol{\Pi}_{n+2k+1}(J_\beta(\mathbb{R}))$ has the scale property.*
(2) *If $[\alpha, \beta]$ is a strong Σ_1-gap and every set in $J_{\alpha+1}(\mathbb{R})$ is determined, then we have that none of the pointclasses $\boldsymbol{\Sigma}_n(J_\beta(\mathbb{R}))$ or $\boldsymbol{\Pi}_n(J_\beta(\mathbb{R}))$ for $n < \omega$ has the scale property. That means at strong gaps $[\alpha, \beta]$ the scale property first appears again at the pointclass $\boldsymbol{\Sigma}_1(J_{\beta+1}(\mathbb{R}))$.*

The proof of this lemma in [St08] yields the following corollary which will be useful to obtain hybrid mice capturing sets of reals at the end of a weak Σ_1-gap.

COROLLARY 8.1.10. *Let $\alpha < \beta$ be ordinals such that $[\alpha, \beta]$ is a weak Σ_1-gap and suppose that every set in $J_\alpha(\mathbb{R})$ is determined. If $n < \omega$ is least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$, then*

$$A \in \boldsymbol{\Sigma}_n(J_\beta(\mathbb{R})) \cap \mathcal{P}(\mathbb{R}) \text{ iff } A = \bigcup_{i<\omega} A_i,$$

where $A_i \in J_\beta(\mathbb{R})$ for all $i < \omega$.

Concerning the connection of the scale property and the uniformization property we have the following general result which is Theorem 3.1 in [KM08]. For the notion of an adequate pointclass see Section 1 in [KM08]. In what follows all pointclasses we consider are adequate.

THEOREM 8.1.11. *Assume that Γ is a pointclass which is adequate and closed under $\forall^{\mathbb{R}}$. Then the scale property for Γ implies the uniformization property for Γ.*

8.2. Capturing Sets of Reals with Hybrid Mice

In this section we will describe how to obtain a possibly hybrid mouse $\mathcal{N}$ satisfying the assumption of Lemma 5.3.4 starting from a set A. Most of the details can be found in [SchSt]. Once we have that, we can use Chapters 6 and 7 to construct a hybrid mouse which captures sets of reals in the pointclasses $\Sigma^1_n(A)$ and $\Pi^1_n(A)$ for some $n < \omega$ as in Corollary 5.3.5.

Chapter 5 in [SchSt] proves the following theorem which is due to W. Hugh Woodin. We phrase everything here for the end of a weak gap, but it is possible to get a similar picture at the end of a strong gap.

THEOREM 8.2.1 (Woodin). *Let $\alpha < \beta$ be ordinals such that $[\alpha, \beta]$ is a weak Σ_1-gap and suppose that every set of reals in $J_\alpha(\mathbb{R})$ is determined. Moreover* *let $n < \omega$ be least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$ and let A be a set of reals in $\boldsymbol{\Sigma}_n(J_\beta(\mathbb{R}))$. Let $A_i \in J_\beta(\mathbb{R})$ for $i < \omega$ be such that*

$$A = \bigcup_{i<\omega} A_i.$$

Then there exists a premouse $\mathcal{N}$ with a Woodin cardinal δ such that there is an iteration strategy Σ for $\mathcal{N}$ which condenses well and witnesses that $\mathcal{N}$ captures the set A_i at δ for every $i < \omega$.

This result yields the following theorem.

THEOREM 8.2.2. *Let $\alpha < \beta$ be ordinals such that $[\alpha, \beta]$ is a weak Σ_1-gap and suppose that every set in $J_\alpha(\mathbb{R})$ is determined. Moreover let $n < \omega$ be least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$ and let A be a set of reals in the pointclass*

$$\Gamma = \boldsymbol{\Sigma}_n(J_\beta(\mathbb{R})).$$

Then if the premouse $\mathcal{N}$ with iteration strategy Σ is as in Theorem 8.2.1, we have that $\Sigma \in \forall^{\mathbb{N}}\Gamma$ and the pointclass Γ has the scale property, is $\mathbb{R}$-parametrized and adequate.

PROOF. It is left to prove that $\Sigma \in \forall^{\mathbb{N}}\Gamma$, because for $\Gamma = \boldsymbol{\Sigma}_n(J_\beta(\mathbb{R}))$ the rest is clear by the results in [St08] quoted in the previous section.
We can write

$$A = \bigcup_{i<\omega} A_i$$

for $A_i \in J_\beta(\mathbb{R})$ for $i < \omega$. Let $\mathcal{N}$ and Σ be as in Theorem 8.2.1 and let $\mathcal{T}$ be an arbitrary iteration tree on $\mathcal{N}$ which is based on $\mathcal{N}|\delta$, where δ is the Woodin cardinal in $\mathcal{N}$ from Theorem 8.2.1. For all $i < \omega$, let $\tau_i^{\mathcal{N}}$ denote the $(\mathcal{N}, \Sigma)$-term for A_i at δ. Moreover assume that $\mathcal{T}$ has limit length $\lambda < \omega_1^V$. Then we have that

$$\begin{aligned} \Sigma(\mathcal{T}) = b \quad \text{iff} \quad & \forall \gamma < \lambda \text{ limit } \mathcal{T} \upharpoonright \gamma \text{ is according to } \Sigma, \text{ and} \\ & \text{let } M = \mathcal{M}_b^{\mathcal{T}} \trianglerighteq \mathcal{M}(\mathcal{T}),\ i_b^{\mathcal{T}} : \mathcal{N} \to M, \\ & \text{and let } i_b^{\mathcal{T}}(\delta) = \delta^*, \text{ then} \\ & \forall k < \omega,\ i_b^{\mathcal{T}}(\tau_k^{\mathcal{N}}) = \tau_k^*, \end{aligned}$$

where for $k < \omega$, τ_k^* is a canonical term for the set A_k in M at δ^*. That means we have

$$\begin{aligned} (\sigma, p) \in \tau_k^* \quad \text{iff} \quad & p \in \mathrm{Col}(\omega, \delta^*), \sigma \in (H_{(\delta^*)^+})^M \text{ is a } \mathrm{Col}(\omega, \delta^*) - \\ & \text{standard term for a real, and for comeager many } g \\ & \text{being } \mathrm{Col}(\omega, \delta^*)\text{-generic over } M, \text{ if } p \in g, \\ & \text{then } \sigma^g \in A_k. \end{aligned}$$

Therefore the statement "$\Sigma(\mathcal{T}) = b$" is $\forall^{\mathbb{N}}\boldsymbol{\Sigma}_n(J_\beta(\mathbb{R}))$-definable because the sequence $(A_k \mid k < \omega)$ is $\boldsymbol{\Sigma}_n(J_\beta(\mathbb{R}))$-definable. □

8.3. Conclusion

For sets of reals in $L(\mathbb{R})$ we finally proved the following theorem in this part of the thesis.

THEOREM 8.3.1. *Let $\alpha < \beta$ be ordinals such that $[\alpha, \beta]$ is a weak Σ_1-gap and let*

$$A \in \Gamma = \mathbf{\Sigma}_n(J_\beta(\mathbb{R})) \cap \mathcal{P}(\mathbb{R}),$$

where $n < \omega$ is the least natural number such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$. Moreover assume that every $\mathbf{\Pi}^1_5\Gamma$-definable set of reals is determined. Then there exists an ω_1-iterable hybrid Σ-premouse $\mathcal{N}$ which captures A.

PROOF. If we let $\mathcal{N} = M_1^{\Sigma,\#}$ this follows from Corollary 7.3.2 together with Corollary 5.3.5 and Theorem 8.2.2. □

In fact we showed the following more general theorem, which also follows from Corollary 7.3.2 together with Corollary 5.3.5 and Theorem 8.2.2.

THEOREM 8.3.2. *Let $\alpha < \beta$ be ordinals such that $[\alpha, \beta]$ is a weak Σ_1-gap, let $k \geq 0$, and let*

$$A \in \Gamma = \mathbf{\Sigma}_n(J_\beta(\mathbb{R})) \cap \mathcal{P}(\mathbb{R}),$$

where $n < \omega$ is the least natural number such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$. Moreover assume that every $\mathbf{\Pi}^1_{2k+5}\Gamma$-definable set of reals is determined. Then there exists an ω_1-iterable hybrid Σ-premouse $\mathcal{N}$ which captures every set of reals in the pointclass $\Sigma^1_k(A)$ or $\Pi^1_k(A)$.

Here we again denote by $\Sigma^1_n(A)$ and $\Pi^1_n(A)$ for $n < \omega$ and a set of reals A the set of all reals which are definable over the model $(V_{\omega+1}, \in, A)$ by a Σ_n- or a Π_n-formula respectively from the parameter $V_{\omega+1}$.

Converse directions of the results presented in this part of the thesis, for example obtaining determinacy for sets of reals in levels of the $L(\mathbb{R})$-hierarchy from the existence and ω_1-iterability of the Σ-premouse $M_k^{\Sigma,\#}$ for $k < \omega$ and an iteration strategy Σ which condenses well for some appropriate countable premouse $\mathcal{N}$, are due to Itay Neeman and follow from [Ne02] (see Definition 2.10 and Theorem 2.14 in [Ne02]).

8.4. Open Problems

We close with mentioning some related open problems. The result we proved in this part of the thesis concerning sets in the $L(\mathbb{R})$-hierarchy is most likely not optimal, but at some levels in the $L(\mathbb{R})$-hierarchy it it is not even clear which mice to consider when aiming for an optimal result.

A promising candidate would be a mouse whose *mouse set* is definable in a countable ordinal over a level of the $L(\mathbb{R})$-hierarchy, where a mouse set as defined by Mitchell Rudominer is the set of reals of a mouse. This would mimic the situation in the projective hierarchy since there we have the following theorem.

THEOREM 8.4.1 (Martin, Steel, Woodin in [St95]). *Let $n \geq 1$ and assume that $M_n^\#$ exists and is ω_1-iterable. Then $\mathbb{R} \cap M_n^\#$ is exactly the set of reals which are Δ^1_{n+2}-definable in a countable ordinal.*

Mitch Rudominer investigated mouse sets in the $L(\mathbb{R})$-hierarchy in [Ru97]. But for some levels of the $L(\mathbb{R})$-hierarchy, for example for the set of reals which are Δ_2-definable in a countable ordinal over the first level after the projective sets $J_2(\mathbb{R})$, the question what the right mouse to consider is remains open.
All definitions and results which we describe concerning this question are due to Rudominer and can be found in [Ru97]. It is conjectured that the right candidate at this level is a so called ladder mouse, which is defined as follows.

DEFINITION 8.4.2. *A mouse $\mathcal{P}$ is called a* ladder mouse *iff there exists a sequence $(\delta_n \mid n < \omega)$ of cardinals in $\mathcal{P}$ such that $\mathcal{P} \cap \mathrm{Ord} = \sup_{n<\omega} \delta_n$ and we have for all $n < \omega$ that*

(1) $M_n^\#(\mathcal{P}|\delta_n) \vDash$ *"δ_n is a Woodin cardinal", and*
(2) $M_n^\#(\mathcal{P}|\delta_n) \trianglelefteq \mathcal{P}$.

So the following is conjectured.

CONJECTURE (Steel, Woodin, 2015). *Let $\mathcal{P}$ be the minimal ladder mouse. Then for all reals x, we have that*

$$x \in \mathcal{P} \cap \mathbb{R} \text{ iff } x \text{ is } \Delta_2(J_2(\mathbb{R}))\text{-definable in a countable ordinal.}$$

What is known so far is, that the minimal ladder mouse as defined above is a lower bound for the set of reals which are $\Delta_2(J_2(\mathbb{R}))$-definable in a countable ordinal.

THEOREM 8.4.3 (Rudominer in [Ru97]). *Let $\mathcal{P}$ be the minimal ladder mouse and let $x \in \mathcal{P} \cap \mathbb{R}$. Then x is $\Delta_2(J_2(\mathbb{R}))$-definable in a countable ordinal.*

The known upper bound for the set of reals which are $\Delta_2(J_2(\mathbb{R}))$-definable in a countable ordinal is an admissible ladder mouse, which is defined as follows.

DEFINITION 8.4.4. *A mouse $\mathcal{P}_a$ is called an* admissible ladder mouse *iff there exists a sequence $(\delta_n \mid n < \omega)$ of cardinals in $\mathcal{P}_a$ such that $\mathcal{P}_a \cap \mathrm{Ord}$ is the least admissible ordinal above $\sup_{n<\omega} \delta_n$ and we have for all $n < \omega$ that*

(1) $M_n^\#(\mathcal{P}_a|\delta_n) \vDash$ *"δ_n is a Woodin cardinal", and*
(2) $M_n^\#(\mathcal{P}_a|\delta_n) \trianglelefteq \mathcal{P}_a$.

THEOREM 8.4.5 (Rudominer in [Ru97]). *Let $\mathcal{P}_a$ be the minimal admissible ladder mouse and let x be a real which is $\Delta_2(J_2(\mathbb{R}))$-definable in a countable ordinal. Then $x \in \mathcal{P}_a$.*

So there are still lots of interesting questions concerning generalizations of results in the projective hierarchy to levels of the $L(\mathbb{R})$-hierarchy which remain open and further research has to be done to analyse the full picture of the sets of reals in the $L(\mathbb{R})$-hierarchy and their connection to mice.

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