A complete stationary-tower-free proof of the derived model theorem

Rui Zhou

1 Introduction

1.1 Background

The Derived Model Theorem is a fundamental theorem in the study of large cardinals and determinacy, originally due to Woodin in the 1980s. It establishes the consistency of AD^+, which is a strengthening of the Axiom of Determinacy, from the consistency of infinitely many Woodin cardinals via forcing. In the same result, Woodin also constructs an inner model of ZFC with infinitely many Woodin cardinals, under the assumption of $\mathrm{AD} + V = L(\mathbb{R})$. Taking these two together, we get the seminal result that AD is equiconsistent with infinitely many Woodin cardinals.

The proof that these notes use is outlined in Steel's stationary-tower-free proof of the derived model theorem (see [8]), with many of the details of the proof omitted there. These notes will fill in those details and give a clean, complete presentation of the proof. To make it as self-contained as possible, we will record the necessary definitions and preliminary results in Section 1.2. We will assume that readers are familiar with relevant set-theoretic notions such as forcing, direct limit, measurable cardinal, Woodin cardinal and so on. We use the following convention:

- OR denotes the class of ordinals, $\mathrm{ot}(X, R)$ denotes the order-type of a well-order $R \subseteq X^2$.
- For sets X, Y, ${}^X Y = Y^X$ denotes the collection of functions from X to Y, $\mathbb{R} = {}^\omega\omega$. For an ordinal α, $Y^{<\alpha} = \bigcup_{\beta<\alpha} Y^\beta$. For function f, $f \restriction A = f \cap (A \times \mathrm{ran}(f))$. For a family $\{Y_i : i \in I\}$ of non-empty sets,

$$\prod_{i\in I} Y_i = \{f \in (\bigcup_{i\in I} Y_i)^I : \forall i \in I, f(i) \in Y_i\}.$$

- We endow Y^ω with the product topology generated by the open basis of sets of the form $\mathcal{N}_s = \{f \in Y^\omega : s \subset f\}$ for $s \in Y^{<\omega}$.
- $T \subseteq X^{<\omega}$ is a tree on X if it is closed under taking initial segments. The branches through T are

$$[T] = \{x \in X^\omega : \forall n < \omega, x \restriction n \in T\}.$$

  For a tree T on $X \times Y$ and $s \in X^{<\omega}$, write $T_s = \{t : (s, t) \in T\}$, and we identify $[T]$ as a subset of $X^\omega \times Y^\omega$ in the natural way. Given $f \in \prod_{i \in I} X_i$ and $i \in I$, let $\mathrm{proj}_i(f) = f(i)$.
- If M is a class, μ is an M-ultrafilter on $Z \in M$, M is a model of enough ZFC to prove Łos' theorem for functions with domain Z, then $\mathrm{Ult}(M, \mu)$ is the ultrapower of M by μ. Given a direct system $\langle (M_i, \pi_{i,j}) : i \leqslant j \in I \rangle$, $\varinjlim \langle M_i : i \in I \rangle$ is the direct limit of the system.
- If $\mathrm{Ult}(M, \mu)$ is well-founded, then we identify it with the unique transitive class isomorphic to it and compose the ultrapower embedding with the isomorphism; similarly, if the direct limit of a directed system $\langle (M_i, \pi_{i,j}) : i \leqslant j \in I \rangle$ is well-founded, then we identify it with the corresponding isomorphic transitive class and compose the direct limit embedding with the isomorphism.

1.2 Preliminaries

1.2.1 Homogeneity Systems

The main tools that will be used to construct the derived model are homogeneity systems, which are sequences of indexed measures that interact with the indexing set in a coherent way, and the measures give rise to various notion of regularity properties.

Definition 1.1. For any set Z and $\kappa \geqslant \omega_1$, $\mathrm{meas}_\kappa(Z)$ denotes the set of κ-complete ultrafilters on Z. We abuse notation and write $\mathrm{meas}_\kappa(Z^{<\omega}) = \bigcup_{n<\omega} \mathrm{meas}_\kappa(Z^n)$. Also, $\mathrm{meas}_{\omega_1}(Z) = \mathrm{meas}(Z)$.

Definition 1.2. A *homogeneity system* over Y with support Z is a sequene $\mu = \langle \mu_s : s \in Y^{<\omega} \rangle$ such that:

1. $\mu_s \in \mathrm{meas}(Z^{|s|})$, and

2. $\forall s \subseteq t$, μ_s is the projection of μ_t to $Z^{|s|}$, which means

$$A \in \mu_s \Leftrightarrow \{u \in Z^{|t|} : u \upharpoonright |s| \in A\} \in \mu_t.$$

μ is *κ-complete* if each μ_s is κ-complete.

Remark 1.3. Since each μ_s is countably complete, $\mathrm{Ult}(V, \mu_s) = M_s$ is well-founded with the associated ultrapower embedding $j_s : V \to M_s$. Also, for $s \subseteq t \in Y^{<\omega}, u \in Z^{|t|}, f : Z^{|t|} \to V$, put

$$u^{s,t}(i) = u(j) \Leftrightarrow s(i) = t(j),$$

$$f^{s,t}(u) = f(u^{s,t}).$$

So by clause 2 of the definition above, the map $j_{s,t} : M_s \to M_t$ via

$$j_{s,t}([f]_{\mu_s}) = [f^{s,t}]\mu_t$$

is well-defined and elementary. Moreover, the diagram below commutes

$$\begin{array}{ccc} V & \xrightarrow{j_t} & M_t \\ & \searrow^{j_s} & \uparrow j_{s,t} \\ & & M_s \end{array}$$

so that each $x \in Y^\omega$ corresponds canonically to a direct limit given by

$$M_x = \varinjlim \langle (M_{x \upharpoonright m}, j_{x \upharpoonright m, x \upharpoonright n}) : m \leqslant n < \omega \rangle.$$

Definition 1.4. A *tower* is a sequence $\langle \mu_n : n < \alpha \rangle$ for some $\alpha \leqslant \omega$ so that each μ_n is the projection of μ_{n+1}. If $\alpha < \omega$ then it is a finite tower, otherwise we call it an infinite tower (or simply a tower). If $Y \subseteq \mathrm{meas}_\kappa(Z^{<\omega})$, then TW_Y is the collection of towers whose range is contained in Y. $\mathrm{TW}_Y^{<\omega}$ is the collection of finite towers whose range is contained in Y.

Definition 1.5. A *weak homogeneity system* over Y with support Z is a sequence $\mu = \langle \mu_s : s \in Y^{<\omega} \rangle$ such that for every $s \in Y^{<\omega}$

1. μ_s is a countably complete ultrafilter on Z^n for some $n \leqslant |s|$, and
2. if ν is the projection of μ_s to Z^m with $m < n$, then there is $i < |s|$ such that $\mu_{s \upharpoonright i} = \nu$.

μ is *κ-complete* if each μ_s is κ-complete.

Remark 1.6. As in the previous remark, if $x, y \in \omega^\omega$ are such that each $\mu_{x\upharpoonright y(i+1)}$ is a projection of $\mu_{x\upharpoonright y(i)}$, then we can form the tower

$$M_{x,y} = \varinjlim \langle (M_{x\upharpoonright y(m)}, j_{x\upharpoonright y(m), x\upharpoonright y(n)}) : m \leqslant n < \omega \rangle.$$

Definition 1.7. A tower $\langle \mu_n : n < \omega \rangle$ is *countably complete* if whenever $\{A_n \subseteq Z^n : n < \omega\}$ is such that $A_n \in \mu_n$ for each $n < \omega$, there is $f : \omega \to Z$ with $f \upharpoonright n \in A_n$.

The following is due to Martin and Steel [4].

Proposition 1.8. *Let $\langle \mu_n : n < \omega \rangle$ be a tower. Then $\varinjlim \langle \mu_n : n < \omega \rangle$ is well-founded if and only if $\langle \mu_n : n < \omega \rangle$ is countably complete.*

Definition 1.9. For a homogeneity system μ with $Y = \operatorname{ran}(\mu)$ we write $W_\mu = \{\nu \in \mathrm{TW}_Y : \nu \text{ is countably complete}\}$. For a weak homogeneity system we write S_μ instead of W_μ. $A \subseteq Y^\omega$ is *κ-homogeneous* iff there is a κ-complete homogeneity system μ such that $A = W_\mu$. A is *κ-weakly homogeneous* iff there is a κ-weak homogeneity system μ such that $A = S_\mu$. A tree T on $Y \times Z$ is *κ-homogeneous* iff there is a κ-homogeneity system μ satisfying

1. $x \in \mathrm{proj}_0[T] \Leftrightarrow \varinjlim \langle \mathrm{Ult}(V, \mu_{x\upharpoonright n}) : n < \omega \rangle$ is well-founded, and
2. $\forall s \in Y^{<\omega}\ T_s \in \mu_s$.

T is *κ-weakly homogeneous* iff there is a κ-weak homogeneity system μ satisfying

1. $x \in \mathrm{proj}_0[T] \Leftrightarrow \exists y \in {}^\omega\omega\ \ \varinjlim \langle \mathrm{Ult}(V, \mu_{x\upharpoonright y(n)}) : n < \omega \rangle$ is well-founded, and
2. $\forall s \in Y^{<\omega} \exists n \leqslant |s|\ T_s \in \mu_{s\upharpoonright n}$.

A is κ-(weakly)-homogeneously Suslin iff there is a κ-(weakly)-homogeneous tree T with $A = \mathrm{proj}_0[T]$. Let

$$\mathrm{Hom}^Y_\kappa = \{A \subseteq Y^\omega : A \text{ is } \kappa\text{-homogeneous}\},$$

$$\mathrm{Hom}^Y_{<\kappa} = \bigcap_{\alpha<\kappa} \mathrm{Hom}^Y_\alpha,$$

$$\mathrm{Hom}^Y_\infty = \bigcap_{\alpha\in \mathrm{OR}} \mathrm{Hom}^Y_\alpha.$$

When $Y = \omega$ we shall suppress the superscript Y.

So for weakly homogeneous A witnessed by μ, $x \in A$ if and only if there is an increasing sequence $\langle i_n : n < \omega \rangle$ such that $\langle \mu_{x\upharpoonright i_n} : n < \omega \rangle$ is countably complete. The proof of the equivalence below is left as an exercise in [9], which we fill in the detail.

Proposition 1.10. *Suppose $\kappa \geqslant \omega_1$ and there is a measurable cardinal at least κ. Then, for $A \subseteq Y^\omega$, the following are equivalent:*

1. *A is κ-weakly homogeneous.*

2. *$A = proj_0[B]$ for some homogeneous $B \subseteq Y^\omega \times {}^\omega\omega$.*

Proof. ($\Leftarrow$) : Fix an effective enumeration $\{r_n : n < \omega\} = \omega^{<\omega}$ with prefixes listed first, and let $\nu_s = \mu_{(s\restriction|r_n|, r_n)}$ where $n = |s|$. It's not hard to see that ν is a weak homogeneity system. If $x \in A = \mathrm{proj}_0[B]$ so that $(x, y) \in B$ for some $y \in {}^\omega\omega$ then, putting $i_n = k \Leftrightarrow y\restriction n = r_k$, we see that $\langle \mu_{x\restriction i_n} : n < \omega\rangle$ forms a countably complete tower; if $\langle \mu_{x\restriction i_n} : n < \omega\rangle$ is countably complete then, putting $y = \bigcup_{n<\omega} r_{i_n}$, we see that $\langle \mu_{(x\restriction n, y\restriction n)} : n < \omega\rangle$ is countably complete, so $(x, y) \in B$, and $x \in A$.

($\Rightarrow$) : Intuitively, the second coordinate of B should encode the increasing enumeration of the lengths of the initial segments of x we attempt to take, so a member $(x, f) \in B$ should be so that f is increasing with $\langle \mu_{x\restriction f(n)} : n < \omega\rangle$ countably complete (where the μ's come from the weak homogeneity system).

For two measures M_0, M_1 on Z^m, Z^n, let $M_0 \frown M_1$ be the measure on Z^{m+n} gluing together M_0, M_1 via

$$X \in M_0 \frown M_1 \Leftrightarrow \{(x_0, ..., x_m) : x \in X\} \in M_0 \wedge \{(x_{m+1}, ..., x_{m+n}) : x \in X\} \in M_1.$$

Fix $z \in Z$ and for each $n < \omega$, let P^n be the principal measure generated by $\{z\}^n$, Q^n generated by $\{0\}^n$, and U^n the non-principal ultrafilter on decreasing n-tuples of κ, derived from the n-fold product measure on κ^n given by the non-principal ultrafilter U on κ as a $\{0, 1\}$-valued measure. Given finite sequence (s, t), if $i < |t| = n$ is the first place with $t(i+1) \leqslant t(i)$ then let $\nu_{(s,t)}$ be measure on $(Z \times \kappa)^n$ via

$$\nu_{(s,t)} = \nu_{(s,t)\restriction(i+1)} \frown (P^{n-i} \times U^{n-i}).$$

If i is the first place with $t(i+1) \leqslant n$ and $\mu_{s\restriction t(i+1)}$ does not project to $\mu_{s\restriction t(i)}$ then change the $i+1$ above to $t(i+1)$. Otherwise, we recursively keep an index set $I_{(s,t)}$ where the actual measures are "activated": let i be the largest such that

$$t(i) \leqslant n, I_{(s,t)} = I_{(s,t)\restriction n} \cup \{i\} = \{i_0, ..., i_m\}, (n+1)\backslash I_{(s,t)} = \{j_0, ..., j_{n-m-1}\}$$

and set

$$X \in \nu_{(s,t)} \Leftrightarrow \{(u_{i_0}, ..., u_{i_m}) : (u, r) \in X\} \in \mu_{s \upharpoonright (t(i)+1)}$$
$$\wedge \{((u_{j_0}, ..., u_{j_{n-m-1}}), (r_{j_0}, ..., r_{j_{n-m-1}})) : (u, r) \in X\} \in P^{n-m} \times Q^{n-m}$$

So we measure the indices on which the actual measures are activated and fill in the rest with dummy variables.

It's not hard to see that if $(s_0, t_0) \subseteq (s_1, t_1)$ then $\nu_{(s_1,t_1)}$ projects to $\nu_{(s_0,t_0)}$. For the countable completeness, if $(x, f) \in B$ then we can easily translate the witnesses; if $(x, f) \notin B$ then there is some i after which the second coordinate cannot encode a valid enumeration of the lengths of initial segments of x, so we are tagging along $P^{n-i} \times U^{n-i}$. Let $A_j \in \nu_{(x,f)\upharpoonright j}$ be arbitrary, let $X^{n-i} \in U^{n-i}$ be the set of decreasing $(n-i)$-tuples of κ for $n > i$, and put $A_n = A_i \frown (\{z\}^{n-i} \times X^{n-i})$. Then each $A_n \in \nu_{(x,f)\upharpoonright n}$, but if there is (g_0, g_1) so that each $(g_0, g_1)\upharpoonright n \in A_n$ then $g_1(n+1) > g_1(n)$ for $n > i$ and we get a decreasing sequence of ordinals, which is impossible. Thus the tower is not countably complete. □

Remark 1.11. The only reason that we need the measurable cardinal here is to rule out the indexing sequences that are not valid encodings. More generally, using the same trick, if $A \subseteq Y^\omega$, $S \subseteq Y^{<\omega}$ satisfies $\bigcup_{s\in S} \mathcal{N}_s \cap A = \varnothing$, then defining only μ_t for $t \notin S$ suffices to produce a homogeneity system for A.

A detailed proof of the following fact can be found in [9].

Proposition 1.12. *A is κ-homogeneous iff A is κ-homogeneously Suslin, and A is κ-weakly homogeneous iff A is κ-weakly homogeneously Suslin.*

1.2.2 Universally Baire

The derived model is a forcing construction, and in order to have canonical names for the homogeneous sets, we need to ensure a certain degree of forcing absoluteness; in particular, we will use universally Baire sets, which we now define.

Definition 1.13. For cardinal κ, we say G is $< \kappa$*-generic* if G is a V-generic filter for some poset $\mathbb{P}$ with $|\mathbb{P}|^V < \kappa$.

Definition 1.14. A tree T on $X \times Y$ is κ*-absolutely complemented* if there is a tree U on $X \times Z$ such that for every poset $\mathbb{P}$ with $|\mathbb{P}| < \kappa$,

$$\Vdash_V^{\mathbb{P}} \text{proj}_0[T] = Y^\omega - \text{proj}_0[U].$$

$A \subseteq X^\omega$ is *κ-universally Baire* if there is a κ-absolutely complemented tree T so that $A = \text{proj}_0[T]$. We write

$$\text{UB}_\kappa^X = \{A \subseteq X^\omega : A \text{ is } \kappa\text{-universally Baire}\}, \text{UB}_\kappa = \text{UB}_\kappa^\omega.$$

The above definition is the most sensible one for our purposes; in [1] Feng, Magidor, and Woodin give an equivalent characterization as stated below, which we will also use for a proposition later.

Theorem 1.15. *Let $\kappa \geqslant \omega$ be a cardinal and $A \subseteq \mathbb{R}$; the following are equivalent:*

1. *A is κ-universally Baire.*
2. *For every topological space X with regular open basis of size $\leqslant \kappa$ and every continuous $f : X \to \mathbb{R}$, $f^{-1}[A]$ has the Baire property.*
3. *For every continuous $f : \kappa^\omega \to \mathbb{R}$, $f^{-1}[A]$ has the Baire property.*

In particular, every κ-universally Baire A has the Baire property. Martin and Solovay have shown that κ-weakly homogeneous implies κ-universally Baire via the following construction. We start by a simpler construction; the essential idea is that the tree should be searching for an infinite decreasing chain in the direct limit of the ultrapowers indexed by initial segments of $x \notin A$.

Definition 1.16. For a homogeneity system μ over Y with associated embeddings $j_{s,t}$ for $s \subseteq t \in Y^{<\omega}$ and ordinal θ, write $\theta' = \sup\{\bigcup j_{s,t}[\theta] : s \subseteq t\}$; its Martin-Solovay tree $ms(\mu, \theta)$ on $Y \times \theta'$ is given by

$$(s,t) \in ms(\mu,\theta) \Leftrightarrow t(0) < \theta \wedge \forall n < |s| - 1[j_{s\restriction n, s\restriction(n+1)}(t(n)) > t(n+1)].$$

Remark 1.17. It is not hard to see that

1. If (x, f) is a branch through $ms(\mu, \theta)$ with direct limit embedding $j_{x\restriction i,x}$, then $\langle j_{x\restriction n,x}(f(n)) : n < \omega\rangle$ is an infinite decreasing sequence of ordinals in M_x.
2. If the direct limit is ill-founded, then the witness to ill-foundedness can be found below $j[|Z|^+]$ where j is the embedding from V to the direct limit, so when $\theta > |Z|^+$ we have $\text{proj}_0[ms(\mu, \theta)] = Y^\omega - S_\mu$.

We may extend the construction to weak homogeneity systems.

Definition 1.18. For a weak homogeneity system μ over Y with associated ultrapower embeddings $j_{s,t}$ for $s, t \in Y^{<\omega}$ such that μ_t projects to μ_s, and ordinal θ, write $\theta' = \sup\{\bigcup j_{s,t}[\theta] : \mu_t \text{ projects to } \mu_s\}$; its Martin-Solovay tree $ms(\mu, \theta)$ on $Y \times \theta'$ is given by

$$(s, t) \in ms(\mu, \theta) \Leftrightarrow t(0) < \theta \wedge \forall m < n \leqslant |s|$$
$$(\mu_{s \upharpoonright n} \text{ projects to } \mu_{s \upharpoonright m} \Rightarrow j_{s \upharpoonright m, s \upharpoonright n}(t(n-1)) > t(m)).$$

As the name suggests, we have the following result due to Martin and Solovay.

Theorem 1.19. *Let $T \subseteq (X \times Z)^{<\omega}$ be a κ-weakly homogeneous tree as witnessed by μ, and $\theta > |T|^+$; then T and $ms(\mu, \theta)$ are κ-absolute complements. In particular, $proj_0[T]$ is κ-universally Baire.*

To prove the theorem above, we need some elementary results about measures with respect to small forcing. Let μ be a κ-complete measure on I in V and G be $< \kappa$-generic over V; in $V[G]$ we define $\mu^* = \{B \subseteq I : \exists A \in \mu, A \subseteq B\}$.

Proposition 1.20. *Let $I \in V$ and let $\nu \in \mathcal{P}(\mathcal{P}(I))^{V[G]}$. The following are equivalent:*

1. *$\nu = \mu^*$ for some $\mu \in V$.*
2. *In $V[G]$, ν is a κ-complete measure on I.*

Furthermore, if (1) or (2) holds then for any $f : I \to V$ with $f \in V[G]$, there is $g \in V$ and $A \in \nu \cap V$ such that $f(i) = g(i)$ for $i \in I$.

Proof. ($\Rightarrow$): Let $B \subseteq I, B \in V[G]$. Let $p \in G, \dot{B} \in V$ with $p \Vdash \dot{B} \subseteq \check{I}$; work in V, we say r decides $\varphi(\tau)$ if either $r \Vdash \varphi(\tau)$ or $r \Vdash \neg\varphi(\tau)$. For each $q \leqslant p$ and $r \leqslant q$, let

$$B_r = \{i \in I : r \text{ decides } \check{i} \in \dot{B}\}.$$

Note that for each q, $\bigcup_{r \leqslant q} B_r = I \in \mu$, so some $r \leqslant q$ has $B_r \in \mu$. Then $\{r \leqslant p : B_r \in \mu\}$ is dense below p, so there is $q \leqslant p$ with $q \in G$ and $B_q \in \mu$. Let

$$B_0 = \{i \in B_q : q \Vdash \check{i} \notin \dot{B}\}, B_1 = \{x \in B_q : q \Vdash \check{i} \in \dot{B}\}$$

and note that either $B_0 \in \mu$ or $B_1 \in \mu$, with $B_0 \subseteq I - B, B_1 \subseteq B$. Similarly, let $\tilde{\mu} \in V$ be the canonical name for ν^*, if $p \Vdash \dot{B} = \langle \dot{B}_\alpha : \alpha < \check{\gamma} \rangle, \dot{B}[\check{\gamma}] \subseteq \tilde{\mu}$ for some $p \in G, \gamma < \kappa$, then working in V, we fix $A_{q,\alpha} \in \mu$ for each $q \leqslant p, \alpha < \gamma$ such that $q \Vdash \check{A}_{q,\alpha} \subseteq \dot{B}(\check{\alpha})$ whenever it exists (else let $A_{q,\alpha} = X$), and note that

$A = \bigcap_{q \leqslant p, \alpha < \gamma} A_{q,\alpha} \in \nu$ by the completeness of μ, and $p \Vdash \check{A} \subseteq \bigcap_{\alpha < \check{\gamma}} \dot{B}(\alpha)$, so $\bigcap_{\alpha<\gamma} B_\alpha \in \mu^*$.
$(\Leftarrow)$: First fix a name $\dot{\nu} \in V$ for ν.

Claim 1. *For all $A \in \nu$, there is $B \subseteq A$ with $B \in \nu \cap V$.*

Proof of claim: For each A in the claim, fix a name $\dot{A} \in V$ and $q \in G$ forcing $\dot{A} \in \dot{\nu}$. Let A_p be defined as before for $p \in \mathbb{P}$, so each $A_p \in V$. For each $q \leqslant p$, we have $\bigcup_{r \leqslant q} A_r = I \in \nu$, so some $A_r \in \nu$. Then $\{r : \exists B, r \Vdash \check{B} \subseteq \dot{A}, \check{B} \in \dot{\nu}\}$ is dense below p, thus is forced by some condition in G. △

Let $\mathbb{Q}$ be the Boolean completion of the forcing poset $\mathbb{P}$ computed in V. Then $\mathbb{Q}$ has no κ-length decreasing chain in $V[G]$, since such a chain corresponds to a collection of incompatible conditions of $\mathbb{P}$ of size κ, which is impossible since κ is still a cardinal in $V[G]$.

Claim 2. *There is $B \in \nu \cap V$ such that for every $C \subseteq B$ with $C \in V$, either $\|\check{C} \in \dot{\nu}\| = \|\check{B} \in \dot{\nu}\|$ or $\|\check{C} \in \dot{\nu}\| = \mathbb{0}$*

Proof of claim: Suppose this is not true. Work in $V[G]$, put $B_0 = I$. For each $\alpha < \kappa$, we can inductively find some $B_{\alpha+1} \subseteq B_\alpha$ such that $B_{\alpha+1} \in V$, $\|\check{B}_{\alpha+1} \in \dot{\nu}\| < \|\check{B}_\alpha \in \dot{\nu}\|$, and $\|\check{B}_{\alpha+1} \in \dot{\nu}\| \in G$. For limit $\beta < \kappa$, let $A_\beta = \bigcap_{\alpha<\beta} B_\alpha$ and let $B_\beta \subseteq A_\beta$ with $B_\beta \in \nu \cap V$ be given by Claim 1. But note now that $\langle \|\check{B}_\alpha \in \dot{\nu}\| : \alpha < \kappa \rangle$ is a decreasing κ-sequence of Boolean values, which is impossible. △

Finally, work in V, let

$$\mu = \{C \subseteq I : \|\check{B} \in \dot{\nu}\| \leqslant \|\check{C} \in \dot{\nu}\|\}.$$

It's not hard to see that $\mu^* = \nu$ in $V[G]$. μ is a κ-complete measure in V, since for every $C \subseteq I$ with $C \in V$ if $C \in \nu$ then $\|\check{B} \in \dot{\nu}\| = \|\check{C} \cap \check{B} \in \dot{\nu}\| \leqslant \|\check{C} \in \dot{\nu}\|$, and otherwise the same applies to $I - C$; upward closure is also trivial, and the κ-completeness follows from the same calculation as above by replacing C with $\bigcap_{\alpha<\gamma} C_\alpha$ and noting that each $C_\alpha \in \nu$.
The moreover part follows from essentially the same calculation: let B_r be the collection of $i \in I$ for which $f(i)$ is decided by r, and find $r \in \mathbb{P}, A \in \mu$ so that $f(i)$ is decided by r for every $i \in A$. This proves the proposition. □

Before working in a generic extension, let us first see that $ms(\mu, \theta)$ and T are complementing in V:

Lemma 1.21. *Let $T \subseteq (Y \times Z)^{<\omega}$ be a κ-weakly homogeneous tree witnessed by μ and $\theta \geqslant |T|^+$. Then $proj_0[T] = {}^\omega Y - proj_0[ms(\mu, \theta)]$*

Proof. If $(x, f) \in [T]$ then f witnesses that $\varinjlim\langle \mathrm{Ult}(V, \mu_{x\restriction y(n)}) : n < \omega\rangle$ is ill-founded for all towers $\langle \mu_{x\restriction y(n)} : n < \omega\rangle$. Conversely, suppose $x \notin \mathrm{proj}_0[T]$. For each $y \in \mathbb{R}$ such that $\langle \mu_{x\restriction y(n)} : n < \omega\rangle$ forms a tower, we may pick $\{A^y_n : n < \omega\}$ witnessing that the tower is countably incomplete, so that each $A^y_n \in \mu_{x\restriction y(n)}$. For each $k < \omega$, let n_k be the unique $n \leqslant k$ such that $Z^n \in \mu_{x\restriction k}$. Let

$$B_k = \bigcap\{A^y_k : \langle \mu_{x\restriction y(n)} : n < \omega\rangle \text{ is a tower}, y(n_k) = k\}$$

Each $A^y_k \in \mu_{x\restriction k}$ with $A^y_k \subseteq Z^{n_k}$ since $y(n_k) = k$. Since each $\mu_{x\restriction y(n)}$ is either principal or has measurable completeness, and the intersection is over a set of size $\leqslant 2^{\aleph_0}$, each $B_k \in \mu_{x\restriction k}$. Define now a relation R on $\omega \times Z^{<\omega}$ via

$$(k, s)R(\ell, t) \Leftrightarrow k > \ell, s \in B_k, t \in B_\ell, s \supset t.$$

Then R is a well-founded relation, since if there are $\{(y(k), s_k) : k < \omega\}$ with each $(y(k+1), s_{k+1})R(y(k), s_k)$, then putting $f = \bigcup_{k<\omega} s_k : \omega \to Z$, we see that each $f\restriction y(k) \in B_k \subseteq A^y_{n_k}$, which is a contradiction by our choice of the A^y_n's. Let $\rho : \omega \times Z^{<\omega} \to \theta$ be the corresponding rank function for R, and put $f_k(u) = \rho(k, u)$ for $u \in Z^{n_k}$. Let $\alpha_k = [f_k]_{\mu_{x\restriction k}}$ and $f(k) = \alpha_k$. By Łos' theorem and definition of α_k, if $\mu_{x\restriction i}$ projects to $\mu_{x\restriction k}$, then

$$\begin{aligned} j_{x\restriction k, x\restriction i}(\alpha_j) > \alpha_i &\Leftrightarrow \{u \in Z^{n_i} : f_j^{x\restriction k, x\restriction i}(u) > f_i(u)\} \in \mu_{x\restriction i} \\ &\Leftrightarrow \{u : f_k(u\restriction n_k) > f_i(u)\} \in \mu_{x\restriction i} \\ &\Leftrightarrow \{u : \rho(k, u\restriction n_k) > \rho(i, u)\} \in \mu_{x\restriction i} \\ &\Leftrightarrow \{u : (i, u)R(k, u\restriction n_k)\} \in \mu_{x\restriction i}. \end{aligned}$$

But the last line is true for every $u \in B_i \in \mu_{x\restriction i}$. Thus $(x, f) \in [ms(\mu, \theta)]$, as desired. □

We may now prove the theorem.

Proof. (of Theorem 1.19) Let G be $< \kappa$-generic over V. By the previous proposition, each μ_s extends to a κ-complete μ^*_s in $V[G]$; moreover, for each $f : Z^{|s|} \to V$ with $f \in V[G]$ we can find $g \in V$ with $[f]_{\mu^*_s} = [g]_{\mu^*_s}$ using the same proposition. In particular, for $x, \langle i_n : n < \omega\rangle \in V[G]$ if $j : V \to \varinjlim \mathrm{Ult}(V, \mu_{x\restriction i_n})$ is the direct limit on V using μ[1] and $j^* : V[G] \to \varinjlim \mathrm{Ult}(V[G], \mu^*_{x\restriction i_n})$ is the

[1]Note that this is a direct limit taken in $V[G]$, external to V (although the functions we use are still in V)

one on $V[G]$ using μ^*, then

$$\begin{aligned}
&\varinjlim \mathrm{Ult}(V[G], \mu^*_{x\restriction i_n}) \text{ is ill-founded}\\
\Leftrightarrow &\exists\{f_n \in {}^{Z^n}\mathrm{OR} : n < \omega\} \subseteq V[G],\\
&\{u : f_n(u\restriction(n+1)) > f_{n+1}(u)\} \in \mu^*_{x\restriction i_n}\\
\Leftrightarrow &\exists\{f_n \in {}^{Z^n}\mathrm{OR} : n < \omega\} \subseteq V,\\
&\{u : f_n(u\restriction(n+1)) > f_{n+1}(u)\} \in \mu_{x\restriction i_n}\\
\Leftrightarrow &\varinjlim \mathrm{Ult}(V, \mu_{x\restriction i_n}) \text{ is ill-founded.}
\end{aligned}$$

Also, $(x, f) \in [ms(\mu, \theta)]^{V[G]}$ precisely when it witnesses the direct limits are ill-founded, so that $(\mathrm{proj}_0[ms(\mu, \theta)^V] = Y^\omega - S_{\mu*} = \mathrm{proj}_0[ms(\mu^*, \theta)])^{V[G]}$. Thus it suffices to see that $(\mathrm{proj}_0[T] = S_{\mu*})^{V[G]}$.

Note that if $x \in \mathrm{proj}_0[T]^{V[G]}$ then $x \notin \mathrm{proj}_0[ms(\mu, \theta)^V]^{V[G]}$, as otherwise the tree of attempts to find (x, f_0, f_1) with (x, f_0) branch through T and (x, f_1) through $ms(\mu, \theta)^V$ would be ill-founded in $V[G]$, thus in V by absoluteness, which is impossible since they are disjoint in V, so $x \in S_{\mu*}$. If $x \in S_{\mu*}$, then the tower $\langle \mu_{x\restriction i_n} : n < \omega\rangle$ is countably complete. Since $T_{x\restriction n} \in \mu_{x\restriction i_n} \subseteq \mu^*_{x\restriction i_n}$ for each $n < \omega$, we can fix $f \in V[G]$ with $f\restriction n \in T_{x\restriction n}$, so that $(x, f) \in [T]^{V[G]}$, meaning that $x \in \mathrm{proj}_0[T]^{V[G]}$. Hence $(\mathrm{proj}_0[T] = S_{\mu*})^{V[G]}$. □

Thus every κ-weakly homogeneous set is κ-universally Baire. Conversely, Martin and Steel showed the following.

Theorem 1.22. *Let δ be Woodin and let μ be a δ^+-complete weak homogeneity system over Y with $|Y| < \delta$. For sufficiently large θ, $ms(\mu, \theta)$ is $< \delta$-homogeneous.*

The following theorem is originally due to Woodin using Stationary Tower forcing. In Section 3 we shall give a stationary-tower-free proof of it.

Theorem 1.23. *Suppose δ is Woodin and $A \in \mathrm{UB}_{\delta^+}$. Then for all $\kappa < \delta$, A is κ-weakly homogeneous.*

Combining the results above, we obtain the corollary.

Corollary 1.24. $\mathrm{Hom}_{<\lambda} = \mathrm{UB}_\lambda = \{A \subseteq \mathbb{R} : A$ *is* $< \lambda$*-weakly homogeneous*$\}$.

1.2.3 Games and Borel Codes

The statement of the derived model theorem is that AD^+ holds in the symmetric extension, so we will need to explain what AD^+ means and record a few basic tools that help us prove it.

Definition 1.25. Given a set X and $A \subseteq {}^\omega X$, the *Gale-Stewart game* $G(X, A)$ on A consists of runs in the following form

$$\begin{array}{lcccc} \text{I} & x(0) & & x(2) & \dots \\ \text{II} & & x(1) & & \dots \end{array}$$

where player I makes the first move, then player II makes the second move, and so on, until a run $x \in {}^\omega X$ is formed; I wins if $x \in A$, and II wins otherwise. We write $G(A)$ when the ambient set X is clear from context, and say that A is the payoff set. A strategy for I is a function $\sigma : \bigcup_{n<\omega} X^{2n} \to X$, a strategy for II is a function $\tau : \bigcup_{n<\omega} X^{2n+1} \to X$. A strategy is winning if the corresponding player wins every run of $G(A)$ by following the strategy, in which case we say $G(A)$ is determined. Axiom of Determinacy (AD) is the statement that for every $A \subseteq \mathbb{R}$, $G(A)$ is determined.

Definition 1.26. A binary relation $\preceq$ on X is a *prewellordering* if it's reflexive, transitive, total, and well-founded. To each such relation and $x \in X$ we let $[x]_\preceq = \{y \in X : x \preceq y \preceq x\}$ be its $\preceq$-class and $E = \{[x]_\preceq : x \in X\}$, so that $\preceq$ induces a well-order on E. Let $\mathrm{lh}(\preceq) = \mathrm{ot}(E, \preceq) = \lambda$ so that $(E, \preceq)$ is isomorphic to $(\lambda, <)$ via a unique f. Let $\mathrm{rk}(x) = f([x]_\preceq)$ be the rank of x.

Definition 1.27. $\Theta = \sup\{\mathrm{lh}(\preceq) : \preceq \text{ is prewellordering on } \mathbb{R}\}$

Definition 1.28. Y^ω is endowed with the (metrizable) topology with basic open neighborhood consisting of $\mathcal{N}_s = \{x \in Y^\omega : s \subset x\}$. Given $f : Y^{<\omega} \to Z^{<\omega}$ with $s \subseteq t \Rightarrow f(s) \subseteq f(t)$ and $\sup\{|f(x \restriction n)| : n < \omega\} = \omega$, we say it is a *continuous representation*, with the induced continuous function $\varphi_f : x \mapsto \bigcup_{n<\omega} f(x \restriction n)$. We say φ is a *Lipschitz representation* if $|f(s)| = |s|$.

Remark 1.29. If $M \subseteq N$ are transitive models of enough set theory and $f \in M$, then f being a continuous/Lipschitz representation is absolute, with $(\varphi_f)^M \subseteq (\varphi_f)^N$.

Definition 1.30. Given sets $A, B \subseteq \mathbb{R}$, the *Wadge game* $W(A, B)$ has the same configuration as above, except that player II can choose not to play in some turns. Player II wins just in case

1. x_{II} is played infinitely many times
2. $x_I \in A \Leftrightarrow x_{II} \in B$

We say A is Wadge reducible to B and write $A \leqslant_W B$ when player II has a winning strategy in $W(A, B)$

Remark 1.31. Here are a few basic observations:

1. Every winning strategy for II induces a continuous representation f with $\varphi_f^{-1}[B] = A$, and every such continuous representation induces a winning strategy for II, so that $A \leqslant_W B$ if and only if A is a continuous preimage of B.

2. By using the trick of filling in with principal measures in the previous proposition, one can similarly show that being κ-(weakly)-homogeneous is closed under continuous reduction. That is, if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $B \subseteq \mathbb{R}$ is κ-(weakly)-homogeneous, then so is $f^{-1}[B]$. Thus Hom_κ is closed under Wadge reducibility.

3. If player I has a winning strategy in $W(A, B)$ then player II has a winning strategy in $W(B, \mathbb{R} - A)$.

4. For any A, B we can code $W(A, B)$ by an equivalent game $G(C)$ of the form

$$\begin{aligned} x \in C \Leftrightarrow \exists x_I, x_{II}, y \in \mathbb{R}, x = x_I * (x_{II} * y) \\ \wedge ((x_I \in A \wedge x_{II} \notin B) \vee (x_I \notin A \wedge x_{II} \in B) \\ \vee x_{II} \text{ has finite support}) \end{aligned}$$

   So that if $A, B \in \mathrm{Hom}_{\gamma^+}$ for some Woodin γ then $C \in \mathrm{Hom}_{<\gamma}$

Besides playing games using members of ω, we may also be interested in playing with members of larger ordinals. If $\delta > \omega$ and $A \subseteq {}^\omega\delta$ is any payoff set, then the game played on A can be undetermined. However, we can look at the games that are very definable from subsets of $\mathbb{R}$:

Definition 1.32. For $\delta < \Theta$ and continuous $f : \delta^\omega \to \mathbb{R}$ and $B \subseteq \mathbb{R}$, the game $G(f, B)$ is defined by player I and II taking turns to play $\alpha_i < \delta$, producing some $\alpha \in \delta^\omega$, and I wins precisely when $f(\alpha) \in B$. Ordinal Determinacy is the statement that $G(f, B)$ is determined for all such δ, f, B.

The notion of being ∞-Borel is a generalization of the usual Borel sets:

Definition 1.33. Given ordinals $\gamma, \delta \geqslant \omega$, $\mathcal{L}^0_{\gamma,\delta}$ is the language with 0-ary predicate P_α for each $\alpha < \delta$ with negation, as well as disjunction and conjunction indexed by members of γ. Each sentence $\varphi \in \mathcal{L}^0_{\gamma,\delta}$ is associated with a code $S \subseteq \mathcal{P}(\gamma)$ for $\delta \leqslant \gamma = |\gamma|$:

- $\varphi_{\{p(0,\alpha)\}} = P_\alpha$

- $\varphi_{\{p(1,\alpha):\alpha\in S\}} = \neg\varphi_S$
- $\varphi_{\{p(2+\alpha,\beta):\alpha\in X,\beta\in S_\alpha\}} = \bigwedge_{\alpha\in X}\varphi_{S_\alpha}$

And $\mathcal{L}^0_{\infty,\delta} = \bigcup_{\gamma\in\mathrm{OR}}\mathcal{L}^0_{\gamma,\delta}$. For $x \in 2^\delta$, define

- $x \models P_\alpha \Leftrightarrow x(\alpha) = 1$
- $x \models \neg\varphi \Leftrightarrow \neg(x \models \varphi)$
- $x \models \bigwedge_{\alpha\in X}\varphi_\alpha \Leftrightarrow \forall\alpha \in X, x \models \varphi_\alpha$

Let $A_\varphi = \{x \in 2^\delta : x \models \varphi\}$. $B \subseteq 2^\delta$ is ∞-Borel if and only if $B = A_\varphi$ for some $\varphi \in \mathcal{L}^0_{\infty,\delta}$. We write $\mathcal{L}^0_\gamma = \mathcal{L}^0_{\gamma,\omega}$ and $\mathcal{L}^0_\infty = \mathcal{L}^0_{\infty,\omega}$.

We also need a localization of the length of ∞-Borel codes for some certain nice set $B \subseteq {}^\omega 2$.

Theorem 1.34. *Assume Lipschitz determinacy. Suppose B is ∞-Borel, and $\delta_B = \sup\{\mathrm{lh}(R) : R$ is prewellorder Wadge reducible to $B\}$. Then B has a code $\varphi \in \mathcal{L}^0_{\delta_B}$.*

Remark 1.35. $\delta_B < \Theta$, as we have a surjection $\sigma \mapsto \mathrm{lh}(\sigma^{-1}[B])$ for every $\sigma \in \mathbb{R}$ coding a continuous reduction with preimage a prewellorder.

Theorem 1.36. *Assume B is Suslin. Then there is an ∞-Borel code for B.*

Definition 1.37. Let $\{\varphi_n : n < \omega\}$ be an effective enumeration of formulae quantifying over $\mathbb{N}$ and $\mathbb{R}$, with extra predicate symbol $\dot{Q}$ for members of $\mathbb{R}$. Let $U^n(Q) \subseteq \mathbb{R}^{n+1}$ be the set defined by

$$(n \frown x_0, ..., x_n) \in U^n(Q) \Leftrightarrow \varphi_n \in \Sigma^1_1(\dot{Q}) \wedge \varphi_n(x_0, ..., x_n)$$

Remark 1.38. $U^1_1(Q)$ is a $\Sigma^1_1(Q)$ set which is $\utilde{\Sigma}^1_1(Q)$-universal. Moreover, note that $U^1_1(Q)$ is definable over the structure (HC, Q) uniformly in Q. Note that this construction also works generally for $Q \subseteq \mathbb{R}^m$ or any list $Q_0, ..., Q_n$ of predicates.

Theorem 1.39 (Coding Lemma)**.** *Assume ZF+AD. For any $X \subseteq \mathbb{R}$, prewellorder $\leq$ on X, and $Z \subseteq X \times {}^\omega\omega$, there's $x \in \mathbb{R}$ such that*

1. $U^2_x(\leq) \subseteq Z$
2. $\forall y \in X$,
$$U^2_x(\leq) \cap ([y]_\leq \times \mathbb{R}) \neq \varnothing \Leftrightarrow Z \cap ([y]_\leq \times \mathbb{R}) \neq \varnothing$$

In the proof of the coding lemma, we build a real game which is $\Delta^1_2(\leq, X, Z)$ and argue that if the theorem fails then the game is not determined; using the same proof, we see the following holds:

Theorem 1.40 (Generalized Coding Lemma)**.** *Let $X \subseteq \mathbb{R}$, $\leq$ prewellorder on X, and $Z \subseteq X \times {}^\omega\omega$; assume $\Delta^1_2(\leq, X, Z)$ determinacy. Then there's $x \in \mathbb{R}$ such that*

1. $U^2_x(\leq) \subseteq Z$
2. $\forall y \in X$,
$$U^2_x(\leq) \cap ([y]_\leq \times \mathbb{R}) \neq \varnothing \Leftrightarrow Z \cap ([y]_\leq \times \mathbb{R}) \neq \varnothing$$

The proof of following observation uses the kind of games mentioned before, which is sketched in [9]

Theorem 1.41. *Let λ be a limit of Woodins. Then there is $\kappa < \lambda$ with* $\mathrm{Hom}_\kappa = \mathrm{Hom}_{<\lambda}$.

Proof. Note that whenever $A \in \mathrm{Hom}_\beta$, the same witness also shows $A \in \mathrm{Hom}_\alpha$ for $\alpha < \beta$, so if the statement were false then fixing Woodin γ_0 below λ, for each $n < \omega$ there would be Woodins

$$\gamma_{n+1} > \delta_{4n+3} > \delta_{4n+2} > \delta_{4n+1} > \delta_{4n} > \gamma_n$$

below λ and $A_n \in \mathrm{Hom}_{\delta^+_{4n+1}} - \mathrm{Hom}_{\delta_{4n+2}}$; then $\mathbb{R} - A_n$ is γ^+_n-homogeneous, so $A_n, \mathbb{R} - A_n \in \mathrm{Hom}_{\gamma^+_n} - \mathrm{Hom}_{\gamma^+_{n+1}}$ (if we have $\mathbb{R} - A_n \in \mathrm{Hom}_{\gamma^+_{n+1}}$ then $A_n \in \mathrm{Hom}_{<\delta_{4n+3}} \subseteq \mathrm{Hom}_{\delta_{4n+2}}$). Then $A_n, \mathbb{R} - A_n \nleq_W A_{n+1}$ for each n as otherwise $A_n \in \mathrm{Hom}_{\gamma_{n+1}}$ by our previous remark. But $A_{n+1}, \mathbb{R} - A_{n+1}, A_n$ are all γ_n homogeneous, so the Wadge games on $W(A_n, A_{n+1}), W(A_n, \mathbb{R} - A_{n+1})$ are determined. So player I has winning strategy σ^1_n in $W(A_n, A_{n+1})$ and winning strategy σ^0_n in $W(A_n, \mathbb{R} - A_{n+1})$. The following argument is due to Martin; a visual presentation can be found in [2]:

For each $n < \omega$ we form a two-player game, where player I follows $\sigma^{x(n)}_n$ in the n^{th} game to play $\langle y(0), y(2), ...\rangle = f_n(x)$. Let player II respond to $y(2i)$ by playing $y(2i+1) = f_{n+1}(x)(i)$ (i.e. the action that player I has just taken in the $n + 1^{st}$ game). Note that for each $n, \ell < \omega$, $f_n(x){\restriction}\ell$ depends only on $x{\restriction}(n+\ell)$, so that each $f_n : {}^\omega 2 \to {}^\omega\omega$ is Lipschitz. Let $X = f_0^{-1}[A_0]$. Since A_0 is γ_0-universally Baire, so is X, so X has the Baire property. Write $A^0_n = {}^\omega\omega - A_n, A^1_n = A_n$, and note $f_n(x) \in A_n \Leftrightarrow f_{n+1}(x) \in A^{x(n)+1}_{n+1}$, since $\sigma^{x(n)}_n$ wins for player I in the game $W(A_n, A_{n+1}^{x(n)})$. An easy induction shows that

$$f_0(x) \in A_0 \Leftrightarrow f_n(x) \in A_n^{\sum_{i\leqslant n}(x(i)+1)}$$

The above equivalence implies that if x, y differs at exactly one $k < \omega$, then $x \in X \Leftrightarrow y \notin X$: note that $f_\ell(x), f_\ell(y)$ depends only on $x(m), y(m)$ for $m \geqslant \ell$; in particular, $f_{k+1}(x) = f_{k+1}(y)$, so

$$\begin{aligned}
f_0(x) \in A_0 &\Leftrightarrow f_k(x) \in A_k^{\sum_{i\leqslant k}(x(i)+1) \bmod 2} \\
&\Leftrightarrow f_{k+1}(x) \in A_{k+1}^{x(k+1)+x(k)+\sum_{i<k}(x(i)+1) \bmod 2} \\
&\Leftrightarrow f_{k+1}(y) \in A_{k+1}^{y(k+1)+x(k)+\sum_{i<k}(y(i)+1) \bmod 2} && (x(i) = y(i)) \\
&\Leftrightarrow f_{k+1}(y) \notin A_{k+1}^{\sum_{i\leqslant k+1}(y(i)+1) \bmod 2} && (x(k) \neq y(k)) \\
&\Leftrightarrow f_0(y) \notin A_0
\end{aligned}$$

Since X has the Baire property, we may fix $s \in 2^{<\omega}$ so that X is meager or comeager in $\mathcal{N}_s$. Let $\ell = |s|$ and let $g : {}^\omega 2 \to {}^\omega 2$ be the homeomorphism given by

$$g(x)(i) = \begin{cases} x(i) \Leftrightarrow i \neq \ell + 1 \\ 1 - x(i) \text{ otherwise} \end{cases}$$

Then $g[\mathcal{N}_s] = \mathcal{N}_s$, but $g[X \cap \mathcal{N}_s] = \mathcal{N}_s - X$ is comeager or meager in $\mathcal{N}_s$, which is a contradiction. Consequently, this process terminates; that is, there is $\kappa < \lambda$ with $\mathrm{Hom}_\kappa = \mathrm{Hom}_{<\lambda}$ □

Definition 1.42. For a set $X \neq \varnothing$, DC_X is the statement that for every total relation $R \subseteq X^2$, there is $f : \omega \to X$ with $f(n)Rf(n+1)$ for every $n < \omega$. DC is the statement that $\forall X, \mathrm{DC}_X$ holds.

Definition 1.43. AD^+ is the conjunction of $\mathrm{AD}+\mathrm{DC}_\mathbb{R}+$Ordinal Determinacy+ "every set of reals has an ∞-Borel code".

1.3 Derived Model Theorem

Finally, we are ready to define the derived model and state our main theorem.

Definition 1.44. Let $\mathbb{P}$ be the proper class $\prod_{\alpha\in\mathrm{OR}} \mathrm{Coll}(\omega, \alpha)$ and $\mathbb{P}^{<\alpha} = \prod_{\beta<\alpha} \mathrm{Coll}(\omega, \beta), \mathbb{P}^{[\alpha,\beta)} = \mathbb{P}^{<\beta} - \mathbb{P}^{<\alpha}$ and so on. For $D \subseteq \mathbb{P}^{<\beta}$ and $\alpha \leqslant \beta$, also put $D{\restriction}\alpha = \{p{\restriction}\alpha : p \in D\}$.

Definition 1.45. If λ is a limit of Woodin cardinals and G is V-generic for $\mathrm{Coll}(\omega, <\lambda)$, then we write

$$\mathbb{R}^* = \bigcup_{\alpha<\lambda} \mathbb{R}^{V[G\upharpoonright\alpha]}$$

For each $\alpha < \lambda$, λ is still a limit of Woodin cardinals, and for $A \in \mathrm{Hom}_{<\lambda}^{V[G\upharpoonright\alpha]}$ we fix λ-absolutely complementing trees T, S with $\mathrm{proj}_0[T]^{V[G\upharpoonright\alpha]} = A$ and put

$$A^* = \mathrm{proj}_0[T]^{V[G]} \cap \mathbb{R}^*$$

$$\mathrm{Hom}^* = \{A^* : \exists\alpha < \lambda, A \in \mathrm{Hom}_{<\lambda}^{V[G\upharpoonright\alpha]}\}$$

In such a scenario, we call $L(\mathbb{R}^*, \mathrm{Hom}^*)$ "the" derived model, although it depends on the choice of G. In the definition above, one might be worried about whether Hom^* depends on the choice of tree representation for A; the following observation, essentially the proof of Mostowski Σ_1^1-absoluteness, shows that it does not.

Proposition 1.46. *Let $M \subseteq N$ be transitive models of enough set theory and $T, U \in M$ be trees on $Y \times Z$; then $\mathrm{proj}_0[T]^M \cap \mathrm{proj}_0[U]^M = \varnothing \Leftrightarrow \mathrm{proj}_0[T]^N \cap \mathrm{proj}_0[U]^N = \varnothing$.*

Proof. The backward direction is obvious; for the forward direction, let $S = \{(s, u_0, u_1) : (s, u_0) \in T, (s, u_1) \in U\}$; then $[S]^M = \varnothing$ since an infinite branch $(x, f_0, f_1) \in [S]^M$ would witness $x \in \mathrm{proj}_0[T]^M \cap \mathrm{proj}_0[U]^M$. Let $\rho : S \to \mathrm{OR}^M$ be the corresponding rank function; if $\mathrm{proj}_0[T]^N \cap \mathrm{proj}_0[U]^N \neq \varnothing$ then we can find $(x, f_0, f_1) \in [S]^N$ as above, so that $\langle \rho((x, f_0, f_1)\upharpoonright n) : n < \omega\rangle$ would be an infinite decreasing sequence of ordinals, which is impossible. ☐

Corollary 1.47. *If $(T_0, U_0), (T_1, U_1)$ are κ-absolute complements with $\mathrm{proj}_0[T_0] = \mathrm{proj}_0[T_1] \subseteq Y^\omega$ and G is $<\kappa$-generic, then $\mathrm{proj}_0[T_0]^{V[G]} = \mathrm{proj}_0[T_1]^{V[G]}$.*

Proof. If $x \in \mathrm{proj}_0[T_0]^{V[G]}$ but $x \notin \mathrm{proj}_0[T_1]^{V[G]}$ then $x \in \mathrm{proj}_0[U_1]$ since $(\mathrm{proj}_0[T_1] \cup \mathrm{proj}_0[U_1] = Y^\omega)^{V[G]}$, so $(\mathrm{proj}_0[T_0] \cap \mathrm{proj}_0[U_1] \neq \varnothing)^{V[G]}$. By the previous proposition, the same holds in V, which is impossible since $\mathrm{proj}_0[U_0]^V = \mathrm{proj}_0[U_1]^V$; the other direction holds similarly. ☐

Another useful observation about the derived model is that it is a symmetric extension; more formally:

Proposition 1.48. *Let λ be a limit of Woodins and $\mathbb{P} = \mathrm{Coll}(\omega, <\lambda)$. Then there are names $\tau_{\mathbb{R}}, \tau_H$ such that $\mathbb{P}$ is a $\{\tau_{\mathbb{R}}, \tau_H\}$-weakly homogeneous forcing;*

that is,

$$\forall p, q \in \mathbb{P} \exists \pi \in Aut(\mathbb{P})(\pi(p) || q \wedge \pi(\tau_{\mathbb{R}}) = \tau_{\mathbb{R}} \wedge \pi(\tau_H) = \tau_H).$$

Proof. (sketch) The construction of $\tau_{\mathbb{R}}$ is just like in other forcing extensions such as the Solovay model (the construction itself does not require λ to be regular). For τ_H, note that if $A \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]}$ for some potential generic G and $\alpha < \lambda$, then we can pick the tree T projecting to A as the Martin-Solovay tree $ms(\nu, \theta)$ for some $\nu = \langle \nu_s : s \in \omega^{<\omega} \rangle \in V[G\restriction\alpha]$, where the completeness of μ is some large enough $\gamma \in (\alpha, \lambda)$. But since $G\restriction\alpha$ is $< \gamma$-generic, we can pick ν to be a measure on some κ such that for each $s \in \omega^{<\omega}$, $\nu_s = \mu_s^*$ for some $\mu_s \in V$. One can define a name τ_{μ_s} for each such measure $\mu_s \in V$ so that the image of τ_{μ_s} under the forcing map is μ_s^* for any V-generic $H \subseteq \mathrm{Coll}(\omega, < \alpha)$, so the only thing that is really "added" in $V[G\restriction\alpha]$ is the indexing $s \mapsto \mu_s^*$, which is a countable sequence whose image has canonical names in V. Now the construction is similar to the one for $\mathbb{R}^*$. □

The derived model theorem due to Woodin, is the following:

Theorem 1.49 (Derived Model Theorem)**.** *Let λ be a limit of Woodin and G be V-generic for* $\mathrm{Coll}(\omega, < \lambda)$*. Then*

1. *AD^+ holds in $L(\mathbb{R}^*, \mathrm{Hom}^*)$*
2. *$A \in \mathrm{Hom}^* \Leftrightarrow A$ is Suslin and co-Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$*

2 Neeman's Iteration

A tool critical to the stationary-tower-free proof of the theorem is Neeman's iteration. However, since the version of the result that we need is not technically what is stated in [6], we will give a brief sketch of the main result.

Let M be a transitive model of enough set theory, $\mathcal{E}, \delta \in M$ are such that $M \models \mathcal{E}$ is a set of extenders witnessing δ is Woodin. Let $\dot{A} \in M$ be a name for a set of reals for $\mathrm{Coll}(\omega, \delta)$.

Definition 2.1. A run of the game $\hat{G}(\dot{A})$ is played as follows, where each $x_n \in \omega$, T_n is a finite iteration tree on M with $i < j \Rightarrow T_i \subset T_j$ using only extenders from $\mathcal{E}$, h_n is a function with $\mathrm{dom}(h_n) = \mathrm{lh}(T_n), i < j \Rightarrow h_i \subset h_j$. The game ends after ω turns with $x \in \mathbb{R}, T := \bigcup_{n<\omega} T_n$, and for a branch b through T, put $h(b) = \bigcup_{n \in b} h(n)$. Player I wins if and only if for every branch b of T (in V) such that M_b is well-founded, $h(b)$ is M_b-generic for $\mathrm{Coll}(\omega, i_b(\delta))$ and $x \in i_b(\dot{A})_{h(b)}$.

$$\begin{array}{llllll} \text{I} & x(0), T_0 & & x(2), T_2 & \dots \\ \text{II} & & x(1), h_0, T_1 & & \dots \end{array}$$

Remark 2.2. If k is M-generic for some $\mathbb{P} \in (H_\kappa)^M$ with $\kappa < \delta$ and $\mathrm{crit}(E) \geqslant \kappa$ for each $E \in \mathcal{E}$, then k is also M_b-generic and $h(b)$ would be $M_b[k]$-generic. For a name $\dot{A} \in M[k]$ we can define the game $\hat{G}(\dot{A})$ exactly the same way, except that we require $h(b)$ be $M_b[k]$-generic for $\mathrm{Coll}(\omega, i_b(\delta))$ and $x \in \tilde{i}_b(\dot{A})_{h(b)}$, where $\tilde{i}_b : M[k] \to M_b[k]$ naturally extends i_b.

Theorem 2.3. *Let $M, \mathcal{E}, \delta, \dot{A}, k$ be as above. Suppose there is $g \subseteq \mathrm{Coll}(\omega, \delta)$ which is $M[k]$-generic. Exactly one of the following holds*

1. *I has a winning strategy $\sigma \in M[k][g]$ against all plays $x_{odd} \in V$.*
2. *II has a winning strategy $\tau \in M[k][g]$ in the standard game $G(\dot{A}_g)$ against all $x_{even} \in M[k][g']$, for any $g' \subseteq \mathrm{Coll}(\omega, \delta)$ which is M-generic.*

The proof of the (superficially stronger) theorem above is essentially the same as in [6]. In there, one defines an open game $G^*(\dot{A})$ in M, in which player I plays a tree of potential names $\dot{y}^n$ for the real, a tree of conditions extending their predecessors and meeting some dense set, and some *type* (an element of V_δ^M which, in some sense, witnesses the reflection phenomenon of δ in M given by its Woodinness there); player II in turn produces the odd digits of the real and some type. One then picks a winning strategy σ^* in M and translate it to a strategy $\sigma \in M[g]$. Here the proof is the same, with the trivial modification that we define the game in $M[k]$ and that for an extender $E \in \mathcal{E}$, using E^* in $M[k]$ and E in M essentially gives us the same conclusion.

Corollary 2.4. *Suppose M is a countable model of enough set theory and $M \models \delta$ is Woodin. Suppose k is $< \delta$-generic over M; then for every $y \in V$ there is a well-founded iteration $i_b : M \to M_b$ and $g \in V$ which is $M_b[k]$-generic for* $\mathrm{Coll}(\omega, i_b(\delta))$ *with $y \in M_b[k][g]$.*

Proof. The existence of a generic for the theorem above is for free since M and therefore $M[k]$ is countable. Putting $\dot{A} = \dot{\mathbb{R}}$ to be the canonical $\mathrm{Coll}(\omega, \delta)$-name for the set of reals in $M[k]$, we easily see that I has winning strategy σ. For any $y \in V$, let II play $x_{2n+1} = y_n$ to form an iteration tree T on M of length ω, and [5] guarantees the existence of a well-founded branch b; since σ is winning, $h(b) = g$ is $M_b[k]$-generic; $x \in \mathbb{R}^{M_b[k][g]}$ and we can easily read off $y = x_I \in M_b[k][g]$. □

3 Reflection

We are now in the position to present a complete, stationary-tower-free proof of the Derived Model Theorem. First we give an encoding necessary for the proof of Windzus' lemma:

Let M be a countable transitive model of enough ZFC to prove Łos' theorem for the relevant extenders. We will encode the collection of iteration trees on M as follows: let Y be the countable set of tuples $(n, \mathcal{T})$ where $\mathcal{T} = (<_T, \langle E_i^{\mathcal{T}} : i < n+1 \rangle)$ is a finite iteration tree on M. Then $(b, \tilde{\mathcal{T}}) \in {}^\omega Y$ encodes an iteration tree $\mathcal{T}$ of length $\omega+1$ precisely when b_n is the n^{th} node of $[0, \omega]_T$ and $\tilde{\mathcal{T}}_i = \mathcal{T} \upharpoonright (b_i + 1)$. Note that the set of encodings is a closed subset of the Polish space ${}^\omega Y$.

Lemma 3.1 (Windzus). *Let $\pi : M \to V_\theta$ be elementary with $\operatorname{ran}(\pi) \subseteq V_\alpha$ for some $\alpha < \operatorname{cof}(\theta)$ and $2^{\aleph_0} < \operatorname{cof}(\theta)$, $\mu \in \mathrm{OR}^M$. Let*

$$\begin{aligned} W = &\{(b, \tilde{\mathcal{T}}) : (b, \tilde{\mathcal{T}}) \textit{ codes a } 2^{\aleph_0}\textit{-closed iteration tree } \mathcal{T} \textit{ on } M, \\ &\operatorname{lh}(\mathcal{T}) = \omega + 1, \operatorname{crit}(\mathcal{T}) > \mu, \mathcal{M}_\omega^{\pi[\mathcal{T}]} \textit{ is well-founded}\}. \end{aligned}$$

If there is a measurable cardinal $\kappa > \pi(\mu)$, then W is $\pi(\mu)^+$-homogeneous Suslin.

Proof. Let $X_0 = \pi[M]$ and, for each $n < \omega$, let X_{n+1} be a Skolem closure in V_θ of $X_n \cup {}^\omega X_n$ of size $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. Let $X = \bigcup_{n<\omega} X_n$; a simple elementary chain argument shows $\pi[M] \prec X \prec V_\theta$ with $|X| = 2^{\aleph_0}$, and moreover ${}^\omega X \subseteq X$. Let $\psi : N \to X$ be the inverse of the collapse and let $\sigma = \psi^{-1} \circ \pi : M \to N$. Note that ${}^\omega N \subseteq N$ as well. Moreover, if $\mathcal{T}$ is an iteration tree on M with $\operatorname{lh}(\mathcal{T}) = \omega + 1$, then $\pi[\mathcal{T}] \in X$ and $\sigma[\mathcal{T}] \in N$, so the iterations $\mathcal{M}_\omega^{\pi[\mathcal{T}]}, \mathcal{M}_\omega^{\sigma[\mathcal{T}]}$ can be formed inside X, N respectively and moreover, by absoluteness of well-foundedness (between V, V_θ and N, V) and elementarity (between V_θ, X and X, N),

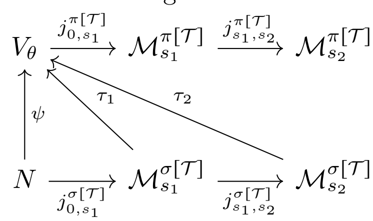
$$\mathcal{M}_\omega^{\pi[\mathcal{T}]} \text{ is well-founded} \Leftrightarrow \mathcal{M}_\omega^{\sigma[\mathcal{T}]} \text{ is well-founded}$$

We now define a tree U via

$$\begin{aligned} (\langle (b_i, \tilde{\mathcal{T}}_i) : i < n \rangle, \langle \tau_i : i < n \rangle) \in U \Leftrightarrow &(b, \tilde{\mathcal{T}}) \text{ codes a } 2^{\aleph_0}\text{-closed iteration tree } \mathcal{T} \text{ on } M, \\ &\operatorname{dom}(<_T) = s_{n-1} + 1, \operatorname{crit}(\mathcal{T}) > \mu, \\ &\forall i < j < n, \tau_j : \mathcal{M}_{s_j}^{\sigma[\mathcal{T}]} \to V_\theta \text{ is elementary}, \\ &\tau_j \circ j_{s_i, s_i}^{\pi[\mathcal{T}]} = \tau_i, \tau_0 = \psi \end{aligned}$$

U is obviously a tree, and $U \in V_\theta$ since θ is sufficiently large; in fact, since θ

is sufficiently large, we can find $Y \in V_\theta$ such that $Y \prec V_\theta$ and $\mathrm{ran}(\tau_i) \subseteq Y$ for each such τ_i, so U is definable from M, N, Y. Each stage of U is approximating a realization map that makes the diagram commutes:



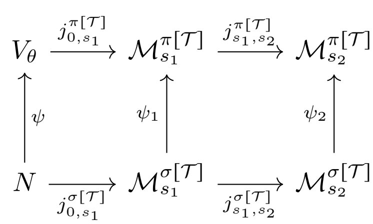
If $\vec{\tau}$ is a witness to $(b, \tilde{\mathcal{T}}) \in \mathrm{proj}_0[U]$ encoding the iteration tree $\mathcal{T}$, then we can use the universal property of the direct limit on the embeddings

$$\langle \tau_n : m \leqslant_{\mathcal{T}} n <_{\mathcal{T}} \omega \rangle$$

to get an embedding $\tau_\omega : \mathcal{M}^{\sigma[T]}_\omega \to V_\theta$, so that $\mathcal{M}^{\sigma[T]}_\omega$ is well-founded. But then $\mathcal{M}^{\pi[T]}_\omega$ is also well-founded, so $(b, \tilde{\mathcal{T}}) \in W$. Suppose now that $(b, \tilde{\mathcal{T}})$ codes a $2^{\aleph_0}$-closed $\mathcal{T}$ on M with $\mathcal{M}^{\sigma[T]}_\omega$ well-founded. For each $n < \omega$, define the copying map $\psi^{\mathcal{T}}_n : \mathcal{M}^{\sigma[\mathcal{T}]}_{s_n} \to \mathcal{M}^{\pi[\mathcal{T}]}_{s_n}$ via $\psi^{\mathcal{T}}_n = \psi$ and

$$\psi^{\mathcal{T}}_n([a, f]^E_{\mathcal{M}^{\sigma[\mathcal{T}]}_{s_n}}) = [\psi^{\mathcal{T}}_n(a), \psi^{\mathcal{T}}_n(f)]^F_{\mathcal{M}^{\pi[\mathcal{T}]}_{s_n}}$$

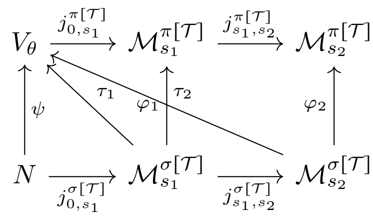
where $F = \psi^{\mathcal{T}}_n[E]$. It's not hard to check that each $\psi^{\mathcal{T}}_n$ is well-defined and elementary:



Also, by induction each $\mathcal{M}^{\sigma[\mathcal{T}]}_{s_n} \in \mathcal{M}^{\pi[\mathcal{T}]}_{s_n}$, and $\mathcal{M}^{\pi[\mathcal{T}]}_{s_n} \models |\mathcal{M}^{\sigma[\mathcal{T}]}_{s_n}| = 2^{\aleph_0}$, so $\psi_n \in \mathcal{M}^{\pi[\mathcal{T}]}_{s_n}$ by $2^{\aleph_0}$-closedness of $\mathcal{T}$. Let $\Psi_n = j^{\pi[\mathcal{T}]}_{s_n,\omega}(\psi^{\mathcal{T}}_n)$, so for each $n < \omega$,

$$\mathcal{M}^{\pi[\mathcal{T}]}_\omega \models \Psi_n : j^{\pi[\mathcal{T}]}_{s_n,\omega}(\mathcal{M}^{\sigma[\mathcal{T}]}_{s_n}) \to V$$

so $\langle \Psi_n : n < \omega \rangle$ is a branch through $j^{\pi[\mathcal{T}]}_{0,\omega}(U)_{(x,\tilde{\mathcal{T}})}$ in V. But this means $j^{\pi[\mathcal{T}]}_{0,\omega}(U)_{(b,\tilde{\mathcal{T}})} = j^{\pi[\mathcal{T}]}_{0,\omega}(U)_{j^{\pi[\mathcal{T}]}_{0,\omega}((b,\tilde{\mathcal{T}}))}$ is ill-founded, so it must be ill-founded in $\mathcal{M}^{\pi[\mathcal{T}]}_\omega$ by absoluteness of ill-foundedness. By elementarity, $U_{(b,\tilde{\mathcal{T}})}$ is ill-founded in V_θ, so $(b, \tilde{\mathcal{T}}) \in \mathrm{proj}_0[U]$. Thus $W = \mathrm{proj}_0[U]$.



To obtain a homogeneity system for U, let $(b, \tilde{\mathcal{T}})$ be a node on the first coordinate of U. The system of measure $\nu_{(b,\tilde{\mathcal{T}})}$ on $U_{(b,\tilde{\mathcal{T}})}$ is given by

$$A \in \nu_{(b,\tilde{\mathcal{T}})} \Leftrightarrow \langle j^{\pi[\mathcal{T}]}_{b_i,b_{n-1}}(\psi^{\mathcal{T}}_i) : i < n\rangle \in j^{\pi[\mathcal{T}]}_{0,b_{n-1}}(A)$$

It's obviously an ultrafilter. It concentrates on $U_{(s,\tilde{\mathcal{T}})}$ as each $j^{\pi[T]}_{b_i,b_{n-1}}(\psi^{\mathcal{T}}_i)$ embeds $j^{\pi[\mathcal{T}]}_{b_i,b_{n-1}}(\mathcal{M}^{\sigma[\mathcal{T}]}_{b_i})$ into $\mathcal{M}^{\pi[\mathcal{T}]}_{b_{n-1}}$, so $\langle j^{\pi[\mathcal{T}]}_{b_i,b_{n-1}}(\psi^{\mathcal{T}}_i) : i < n\rangle \in j^{\pi[\mathcal{T}]}_{0,b_{n-1}}(U_{(b,\tilde{\mathcal{T}})})$. The completeness of $\nu_{(b,\tilde{\mathcal{T}})}$ follows from the fact that $\mathrm{crit}(\pi[\mathcal{T}]) > \pi(\mu)$, so

$$\langle j^{\pi[\mathcal{T}]}_{b_i,b_{n-1}}(\psi^{\mathcal{T}}_i) : i < n\rangle \in \bigcap_{\alpha<\pi(\mu)} j^{\pi[\mathcal{T}]}_{0,b_{n-1}}(A_\alpha) = j^{\pi[\mathcal{T}]}_{0,b_{n-1}}\left(\bigcap_{\alpha<\pi(\mu)} A_\alpha\right)$$

whenever each $A_\alpha \in \nu_{(b,\tilde{\mathcal{T}})}$. We finally check that it is a homogeneity system for U; that is,

$$(b, \tilde{\mathcal{T}}) \in \mathrm{proj}_0[U] \Leftrightarrow \langle \nu_{(b,\tilde{\mathcal{T}})\restriction n} : n < \omega\rangle \text{ is countably complete}$$

The backward direction is true in general. Suppose $(b, \tilde{\mathcal{T}}) \in \mathrm{proj}_0[U]$ so that $\mathcal{M}^{\pi[\mathcal{T}]}_\omega$ is well-founded. Suppose $\langle A_n : n < \omega\rangle$ is such that each $A_n \in \nu_{(b,\tilde{\mathcal{T}})}$. Note that by applying $j^{\pi[\mathcal{T}]}_{b_{n-1},\omega}$,

$$\begin{aligned} A_n \in \nu_{(b,\tilde{\mathcal{T}})} &\Leftrightarrow \langle j^{\pi[\mathcal{T}]}_{b_i,b_{n-1}}(\psi^{\mathcal{T}}_i) : i < n\rangle \in j^{\pi[\mathcal{T}]}_{0,b_{n-1}}(A_n) \\ &\Leftrightarrow \langle j^{\pi[\mathcal{T}]}_{b_i,\omega}(\psi^{\mathcal{T}}_i) : i < n\rangle \in j^{\pi[\mathcal{T}]}_{0,\omega}(A_n) \end{aligned}$$

so $\langle j^{\pi[\mathcal{T}]}_{b_n,\omega}(\psi^{\mathcal{T}}_n) : n < \omega\rangle$ is a thread through each $j^{\pi[\mathcal{T}]}_{0,\omega}(A_n)$. By elementarity, there is a thread through each A_n, as desired. □

As promised in Section 1.2, we can use the above lemma to shift from universally Baire to weakly homogeneous:

Corollary 3.2. *Let δ be Woodin and $A \in \mathrm{UB}_{\delta^+}$. For every $\gamma < \delta$, A is γ-weakly* *homogeneous.*

Proof. Let T, U be δ^+-absolutely complementing trees with $\text{proj}_0[T] = A$. Let $\theta > \delta^+$ be large enough to satisfy the requirements of Windzus' lemma with $T, U \in V_\theta$. Let $\gamma < \delta$. Let $X \prec V_\theta$ be countable elementary with $T, U, \gamma \in X$, $\pi : M \to X$ the inverse of the Mostowski collapse with

$$\pi(\bar{\gamma}) = \gamma, \pi(\bar{\delta}) = \delta, \pi(\bar{T}) = T, \pi(\bar{U}) = U$$

Let W be the corresponding set in Windzus' lemma with $\mu = \bar{\gamma}$.

Claim 3. *For every $x \in \mathbb{R}$, the following are equivalent:*

1. $x \in A$.
2. $$\exists g \exists (b, f) \in W, (b, f) \textit{ codes } \mathcal{T}, g \textit{ is } M_\omega^{\mathcal{T}}\textit{-generic for } \text{Coll}(\omega, j_{0,\omega}^{\mathcal{T}}(\bar{\delta})),$$
$$x \in M_\omega^{\mathcal{T}}[g], M_\omega^{\mathcal{T}}[g] \models x \in \textit{proj}_0[j_{0,\omega}^{\mathcal{T}}(\bar{T})].$$

Proof of claim: Suppose $x \in A = \text{proj}_0[T]$. Let $\mathcal{T}, h$ be the iteration tree from Neeman's iteration with

$$\mathcal{E} = \{E \in M : M \models \text{Ult}(V, E) \text{ is } 2^{\aleph_0}\text{-closed, crit}(E) > \mu\}$$

resulted from a run with player II playing x; by [5] there is a well-founded branch b through $\pi[\mathcal{T}]$, so $(b, \tilde{\mathcal{T}}) \in W$. b is also a well-founded branch through $\mathcal{T}$. Let $g = h(b)$ be the corresponding $M_\omega^{\mathcal{T}}$-generic filter so that $x \in M_\omega^{\mathcal{T}}[g]$. Since $\mathcal{M}_b^{\pi[\mathcal{T}]}$ is well-founded, the argument before gives a realization $\sigma : M_\omega^{\mathcal{T}} \to V_\theta$ with $\sigma \circ j_{0,\omega}^{\mathcal{T}} = \pi$. By elementarity,

$$M_\omega^{\mathcal{T}} \models j_{0,\omega}^{\mathcal{T}}(\bar{T}), j_{0,\omega}^{\mathcal{T}}(\bar{U}) \text{ are } j_{0,\omega}^{\mathcal{T}}(\bar{\delta})^+\text{-absolutely complementing}$$

so either $x \in \text{proj}_0[j_{0,\omega}^{\mathcal{T}}(\bar{T})] \cap M_\omega^{\mathcal{T}}[g]$ or $x \in \text{proj}_0[j_{0,\omega}^{\mathcal{T}}(\bar{U})] \cap M_\omega^{\mathcal{T}}[g]$. But if $x \in \text{proj}_0[j_{0,\omega}^{\mathcal{T}}(\bar{U})] \cap M_\omega^{\mathcal{T}}[g]$ then there is f such that $(x, f) \in [j_{0,\omega}^{\mathcal{T}}(\bar{U})]$, so $(x \restriction n, f \restriction n) \in j_{0,\omega}^{\mathcal{T}}(\bar{U})$ for each $n < \omega$. By elementarity, $(x \restriction n, \sigma(f \restriction n)) \in U$ and $\sigma(f \restriction n) \subseteq \sigma(f \restriction m)$ for every $n \leqslant m$, so if we put $\sigma(f) = \bigcup_{n<\omega} \sigma(f \restriction n)$, then $(x, \sigma(f)) \in [U]$, so $x \in \text{proj}_0[U]$, which is a contradiction. So $x \in \text{proj}_0[j_{0,\omega}^{\mathcal{T}}(\bar{T})] \cap M_\omega^{\mathcal{T}}[g]$.

Conversely, if the second clause is satisfied, then again there is an elementary embedding $\sigma : M_\omega^{\mathcal{T}} \to V_\theta$ with $\sigma \circ j_{0,\omega}^{\mathcal{T}} = \pi$. Pick $f \in M_\omega^{\mathcal{T}}[g]$ with $(x, f) \in$

$[j_{0,\omega}^{\mathcal{T}}(\bar{T})]$. Let $\sigma(f)$ be defined as before, we have $(x, \sigma(f)) \in [\sigma(j_{0,\omega}^{\mathcal{T}}(\bar{T}))] = [T]$, so $x \in \mathrm{proj}_0[T] = A$. △

Now, note that the existential quantifier of countable triples $(g, b, \tilde{\mathcal{T}})$ is a real existential quantifier, and the condition $(b, f) \in W$ is γ^+-homogeneous by Windzus' lemma, and the remaining part of clause 2 in the claim above is arithmetic in the parameter coding M, so A is the real projection of a γ^+-homogeneous set, so A is γ^+-weakly homogeneous. □

We also have the following lemma.

Lemma 3.3 (Flipping Function). *Let δ be Woodin, $Y \subseteq meas_{\delta^+}(Z^{<\omega})$ with $|Y| < \delta$. Then for any $\gamma < \delta$ there is W and $R \subseteq meas_\gamma(W^{<\omega})$ and Lipschitz representation $f : TW_Y^{<\omega} \to TW_R^{<\omega}$ such that for any $< \gamma$-generic extension $V[G]$, if*

$$\tilde{f} : \mu^* \mapsto f(\mu)^*$$

then

$$\mu \text{ is well-founded} \Leftrightarrow \varphi_{\tilde{f}}(\mu) \text{ is ill-founded}$$

In this scenario, we call such $\varphi_{\tilde{f}}$ a flipping function.

Proof. We may as well assume that there is measurable $\kappa > \delta^+$, as otherwise Y consists of only principal ultrafilters, and for $\gamma < \delta$ we can just find $\gamma_0 \in [\gamma, \delta)$ measurable and use the U^n in the preliminaries section to ill-found every tower.

Work in V, for each $\mu = \langle \mu_i : i < \omega \rangle \in \mathrm{TW}_Y$ which is not countably complete, pick $\langle A_i^\mu : i < \omega \rangle$ such that each $A_i^\mu \in \mu_i$ but there's no $f \in Z^\omega$ with $f \restriction i \in \mu_i$. For $\nu = \langle \nu_i : i < n \rangle \in \mathrm{TW}_Y^{<\omega}$, put

$$B_i^\nu = \bigcap_{\mu \in \mathrm{TW}_Y, \nu \subset \mu} A_i^\mu$$

Define a tree T on $Y \times Z$ via

$$(\nu, t) \in T \Leftrightarrow \forall i < |t|, t \restriction i \in B_i^\nu$$

Then we have

$$\begin{aligned} \mu \in \mathrm{proj}_0[T] &\Leftrightarrow \exists f \forall i, f \restriction i \in B_i^{\mu \restriction i} \\ &\Leftrightarrow \forall \langle A_i : i < \omega \rangle \exists f \forall i, f \restriction i \in A_i && (\text{as } B_i^{\mu \restriction i} \subseteq A_i) \\ &\Leftrightarrow \mu \text{ is well-founded.} \end{aligned}$$

For $\nu = \langle \nu_i : i < n \rangle \in \mathrm{TW}_Y^{<\omega}$, put $\mu_\nu = \nu_{n-1}$ (if $\nu \in Y^{<\omega} - \mathrm{TW}_Y^{<\omega}$ then use the U^n as in the proof of preliminaries that ill-founds the final tower). By the exact same calculation as above, each $\mu_\nu \in \text{meas}_{\delta^+}(T_\nu)$ and μ is a homogeneity system for T, so T is δ^+-homogeneous with $|Y| < \delta$, so for $\theta = |Y|^+ < \delta$ we have $S = ms(\mu, \theta)$ is γ-homogeneous. Let $f_S = \langle f_S(\nu) : \nu \in Y^{<\omega} \rangle$ be a γ-homogeneity system for S and let $f(\langle \nu_i : i < n \rangle) = \langle f_S(\nu_i) : i < n \rangle$. Recall that every γ-homogeneous S is γ-universally Baire, and this f extends to $V[G]$, which is as desired. □

Using the lemma above, Steel proves the following theorem.

Theorem 3.4 (Derived Model Reflection)**.** *Let λ be a limit of Woodin, G be V-generic for $\mathbb{P}^{<\lambda}$. Let $L(\mathbb{R}^*, \text{Hom}^*) = M^*$ and $\text{HC}^* = (\text{HC})^{M^*}$ be the set of hereditarily countable sets in M^*. Suppose φ is a $\{\in, \dot{A}, \dot{B}\}$-sentence, $A \in \text{Hom}_{<\lambda}^{V[G^{<\alpha}]}$, and*

$$\exists B \subseteq \mathbb{R}^*, B \in M^* \wedge (\text{HC}^*, A^*, B) \models \varphi$$

Then

$$\exists B \in \text{Hom}_{<\lambda}^{V[G^{<\alpha}]}, (\text{HC}^{V[G^{<\alpha}]}, A, B) \models \varphi$$

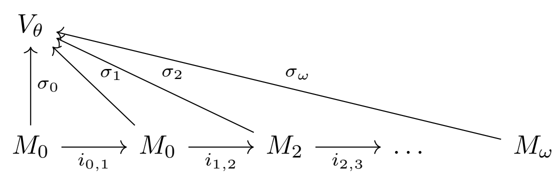
Proof. By working in $V[G^{<\alpha}]$, we may as well assume $\alpha = 0$, so $A \in \text{Hom}_{<\lambda}^V$. Since M^* is a symmetric extension, we have that

$$\Vdash_{\mathbb{P}^{<\lambda}} \exists B \subseteq \mathbb{R}^*, B \in M^* \wedge (\text{HC}^{M^*}, A^*, B) \models \varphi$$

Work in V, we first prove a weaker version of the theorem.

Lemma 3.5. *For some $B \in L(\mathbb{R}, \text{Hom}_{<\lambda})$, $(\text{HC}, A, B) \models \varphi$*

Proof of lemma: Since $A \in \text{Hom}_{<\lambda}$, A is λ-universally Baire. Fix λ-absolutely complementing (T, S) with $\text{proj}_0[T] = A$. Let $\theta > \lambda$ be sufficiently large so that $T, S \in V_\theta$ and $X \prec V_\theta$ a countable elementary substructure with $T, S, \lambda \in X$. Let $\sigma_0 = \sigma : M_0 \to V_\theta$ be the inverse collapse embedding with $\sigma[M] = X, \sigma(\bar{S}, \bar{T}, \bar{\lambda}) = (S, T, \lambda)$. Picking θ to be large enough, we can make M, V_θ to satisfy the hypotheses of Windzus' lemma. Put $\delta_0 = 0$. Let H be V-generic for $\text{Coll}(\omega, \mathbb{R})$, with enumeration $\{x_n : 1 < n < \omega\}$ of $\mathbb{R}^V$. For each $n < \omega$, work in V, and let $\lambda_n = i_{0,n}(\bar{\lambda})$, pick in V an increasing sequence $\langle \eta_k^n : k < \omega \rangle$ cofinal in λ_n with $\eta_k^{n+1} \geqslant \eta_k^n$, and δ'_{n+1} be Woodin in M_n with $\max\{\delta_n, \eta_n^n\} < \delta'_{n+1} < \lambda_n$. Let $\mathcal{T}_n$ be the iteration tree on M_n of length ω so that if b is a well-founded branch then there's $h_{n+1} \subseteq \text{Coll}(\omega, i_b(\delta'_{n+1}))$ generic over $\mathcal{M}_b^{\mathcal{T}_n}[g_n]$ with $x \in \mathcal{M}_b^{\mathcal{T}_n}[g_n][h_{n+1}]$, given by Corollary 2.4, with each extender having critical point $> \delta_n$. Since M_n is countable and $\sigma_n : M_n \to V_\theta$ is elementary, Theorem 3.12 of [5] yields a well-founded branch b_n through $\mathcal{T}_n$ along with a realization map $\sigma_n : \mathcal{M}_{b_n}^{\mathcal{T}_n} \to V_\theta$ so that $\sigma_{n+1} = \sigma_n \circ j_{b_n}^{\mathcal{T}_n}$. Put $M_{n+1} = \mathcal{M}_{b_n}^{\mathcal{T}_n} \in V$ where M_{n+1} is countable in V, and $i_{m,n+1} = j_{b_n}^{\mathcal{T}_n} \circ i_{m,n} : M_m \to M_{n+1} \in V$ for $m \leqslant n+1$, and $\delta_{n+1} = i_{n,n+1}(\delta'_{n+1})$. Also, $M_n[g_n][h_n]$ is still countable, so in V we can fix $h \subseteq \mathbb{P}^{[\delta_n+1,\delta_{n+1})}$ which is $M_n[g_n][h_n]$-generic. Put $g_{n+1} = g_n \cup h \cup h_{n+1}$. In $V[H]$, let M_ω be the direct limit of the system $\langle (M_n, i_{n,m}) : n \leqslant m < \omega \rangle$. Since we have a direct limit embedding $\sigma_\omega : M_\omega \to V_\theta$ via the embeddings $\langle \sigma_n : n < \omega \rangle$, M_ω is well-founded, and we have the commutative diagram:



We can use the g_n's to build a M_ω-generic $g \subseteq \mathbb{P}^{<\lambda_\omega}$ such that for each $0 < n < \omega$, $g \restriction (\delta_n + 1) \in V$ and $g_n \in M_n[g \restriction (\delta_n + 1)]$ (which makes sense since $\mathrm{crit}(i_{n,\omega}) > \delta_n$ so that $g \restriction (\delta_n + 1)$ is M_n-generic). Concretely, for each $k < n+1$, we can fix a V-enumeration $\{D_i^k : i < \omega\}$ of open dense subsets of $\mathbb{P}^{<\lambda_k}$ in $M_k[g_k]$; then $i_{k,n}(D_i^k) \restriction (\delta_n + 1)$ meets g_n at some $p_i^k \restriction (\delta_n + 1)$ for each $k, i < n+1$ by genericity, and we can modify g_{n+1} in a finitary way so that g_{n+1} meets each of $i_{n,n+1}(p_i^k) \restriction (\delta_{n+1} + 1)$ and extends g_n, since $i_{n,n+1}(p_i^k \restriction (\delta_n + 1)) = p_i^k \restriction (\delta_n + 1)$, and the forcing extension would be equivalent; note that $D' \in M_\omega \cap \mathcal{P}(\mathbb{P}^{<\lambda_\omega})$ is dense if and only if $D' = i_{n,\omega}(D_i^n)$ for some dense $D_i^n \in M_n \cap \mathcal{P}(\mathbb{P}^{\leqslant \lambda_n})$. Also, $\sup\{\delta_n : n < \omega\} = \sup\{\lambda_n : n < \omega\} = \lambda_\omega$, where the first equality is because if $\alpha < \lambda_n$ then $\alpha < \eta_m^n$ for some m, so $\alpha < \eta_m^m < \delta_m$, and the second equality is by definition of the direct limit embedding.

The construction gives us such g which is M_ω-generic, since if $D = i_{n,\omega}(D_k^n) \subseteq \mathbb{P}^{<\lambda_\omega}$ is dense then we can find $p_k^n \cap (\omega \times (\delta_n + 1)) \in g_n \cap D_k^n$, and letting m be such that $n, k \leqslant m$ and $\mathrm{ran}(p) \subseteq \delta_m$, we see that $i_{n,m+1}(p) \in g_{m+1} \cap i_{n,m+1}(D_k^n)$, so that $i_{n,\omega}(p) \in g_{m+1} \cap i_{n,\omega}(D_k^n) \subseteq g \cap D$; by relabeling, we may assume $g \restriction \delta_n = g_n$.

Whenever $n \leqslant k$, since $\delta_n < \mathrm{crit}(i_{n,k})$, $g_n = i_{n,k}[g_n]$, and $i_{n,k}$ extends to an embedding $\tilde{i}_{n,k} : M_n[g_n] \to M_k[g_n]$, with $x_n = \tilde{i}_{n,k}(x_n) \in M_k[g_n]$. Similarly, $g_n = i_{n,\omega}[g_n]$ is M_ω-generic and we get extension $\tilde{i}_{n,\omega} : M_n[g_n] \to M_\omega[g_n]$. Put

$$\mathbb{R}_I^* = \bigcup_{n<\omega} \mathbb{R}^{M_\omega[g_n]}$$

$$\mathrm{Hom}^*_I = \bigcup\{\mathrm{proj}_0[T] \cap \mathbb{R}^*_\omega : \exists n < \omega, T \in \mathrm{Hom}^{M_\omega[g_n]}_{<\lambda}\}$$

So $(L(\mathbb{R}^*_I, \mathrm{Hom}^*_I))^{M_\omega[g]}$ is the derived model in $M_\omega[g]$. Note that since $M_\omega[g_n]$ is the extension of a direct limit and $\mathbb{R}^{M_k[g_n]} \subseteq \mathbb{R}^{M_k[g_k]}$ whenever $n \leqslant k$ and that $\tilde{i}_{n,\omega}[\mathbb{R}^{M_n[g_n]}] = \mathbb{R}^{M_n[g_n]}$, we have

$$\bigcup_{n<\omega} \mathbb{R}^{M_\omega[g_n]} = \bigcup_{n<\omega} \bigcup_{n\leqslant k<\omega} \tilde{i}_{k,\omega}[\mathbb{R}^{M_k[g_n]}] = \bigcup_{n<\omega} \mathbb{R}^{M_n[g_n]}$$

It follows that $\mathbb{R}^*_I = \mathbb{R}^V$: if $x \in \mathbb{R}^*_I$ then $x \in M_n[g_n]$ for some $n < \omega$; note that $M_n[g_n] \in V$, so $x \in V$. If $x \in \mathbb{R}^V$ then $x = x_n$ for some $n < \omega$, whence $x \in M_n[g_n]$, thus $x \in M_\omega[g_n]$. It then follows that $(\mathrm{HC}^*)^{M_\omega[g]} = \mathrm{HC}^V$, since each member of HC is coded by a real.

Since $L(\mathbb{R}^*, \mathrm{Hom}^*_{<\lambda})$ is a symmetric extension,

$$(\Vdash_{\mathbb{P}^{<\lambda}} \exists B \subseteq \mathbb{R}^*, B \in M^* \wedge (\mathrm{HC}^*, A^*, B) \models \varphi)^{V_\theta}.$$

By elementarity of σ_ω, we have

$$(\Vdash_{\mathbb{P}^{<\lambda_\omega}} \exists B \subseteq \mathbb{R}^*, B \in M^* \wedge (\mathrm{HC}^*, i_{0,\omega}(\bar{A})^*, B) \models \varphi)^{M_\omega}.$$

So in $M_\omega[g]$, there's $B \in L(\mathbb{R}^V, \mathrm{Hom}^*_I)$ with $(\mathrm{HC}^V, i_{0,\omega}(A)^*, B) \models \varphi$. Since

$$\begin{aligned}\mathrm{proj}_0[i_{0,\omega}(\bar{T})] \cap \mathbb{R}^*_I &= \sigma_\omega(\mathrm{proj}_0[i_{0,\omega}(\bar{T})]) \cap \mathbb{R}^V \\ &= \mathrm{proj}_0[\sigma_\omega \circ i_{0,\omega}(\bar{T})] \cap \mathbb{R}^V \\ &= \mathrm{proj}_0[T] \cap \mathbb{R}^V = A\end{aligned}$$

it follows that $i_{0,\omega}(\bar{A})^* = A$. To prove the lemma, it will suffice to see

Claim 4. $\mathrm{Hom}^*_I \subseteq \mathrm{Hom}^V_{<\lambda}$.

Proof of claim: Let $C \in \mathrm{Hom}^*_I$ so that $C = C^*_0$ for some $< \lambda_\omega$-homogeneously Suslin C_0 in $M_\omega[g_k]$, so $C_0 = i_{k,\omega}(C_1)$ for some $< \lambda_k$-homogeneously Suslin $C_1 \in M_k[g_k]$. Then C_1 is $< \lambda_k$-universally Baire in $M_k[g_k]$. Fix absolutely complementing $(T, W) \in M_k[g_k]$ with $C = \mathrm{proj}_0[i_{k,\omega}(T)] \cap \mathbb{R}^V$. In $M_k[g_k]$, let $\delta(\eta)$ be the least Woodin in the interval (η, λ_k) and let $\rho = \delta(\delta_k)$.

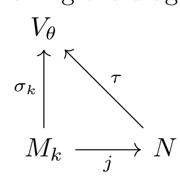
For each $\eta \in [\rho, \lambda_k)$, by previous results on small forcing, we may fix in $M_k[g_k]$ a system $\mu^*_\eta = \langle(\mu^\eta_u)^* : u \in (\omega \times Z)^{<\omega}\rangle$ which is $\delta(\eta)^+$-complete homogeneity system in $M_k[g_k]$ with $S_{\mu^*_\eta} = \mathrm{proj}_0[T]$ in $M_k[g_k]$ and each $\mu^\eta_u \in M_k$. We may thus let $\nu^\eta_u = \sigma_k(\mu^\eta_u)$, so that by elementarity, ν_η is a $\sigma_k(\delta(\eta))^+$-complete homogeneity system in V (since incompatibility between measures is determined at finite steps). Fix $\xi_k < \lambda_k$ with $\mathrm{Hom}^{M_k}_{<\lambda_k} = \mathrm{Hom}^{M_k}_{\xi_k}$, it suffices to see that $S_{\nu_\eta} = \mathrm{proj}_0[i_{k,\omega}(T)] \cap \mathbb{R}^V = C$ for $\eta \geqslant \xi_k$, for then we know $C \in V$ and its $\sigma_k(\delta(\eta))^+$ homogeneous Suslin is witnessed by S_{ν_η}.

Subclaim. *If $\rho < \eta < \gamma < \lambda_k$, then $S_{\nu_\eta} = S_{\nu_\gamma}$.*

Proof of subclaim: For each η and each $u \in \omega^{<\omega}$, we can fix a name $\dot{\mu}_{\eta,u} \in M_k$ for $(\mu^\eta_u)^*$. Work in M_k, we may pick a $\delta(\eta)^+$-complete measure $\mu_{\eta,u,p} = \mu$ for each condition $p \in \mathrm{Coll}(\omega, \leqslant \delta_k), \eta, u$ such that $p \Vdash \check{\mu}^* = \dot{\mu}_{\eta,u}$ and let

$$Y_\eta = \{\mu_{\eta,u,p} : u \in \omega^{<\omega}, p \in \mathrm{Coll}(\omega, \leqslant \delta_k)\}$$

so that $|Y_\eta| \leqslant \delta_k$. Lemma 2.2 gives a flipping function $f_\eta : \mathrm{TW}_{Y_\eta} \to \mathrm{TW}_{Z_\eta}$. Now, for each $x \in \mathbb{R}^V$, we may use the previous argument to get an iteration $j : M_k \to N$ with $x \in N[g_k][h]$ for some $N[g_k]$-generic h on $\mathrm{Coll}(\omega, j(\rho))$, and a realization map $\tau : N \to V_\theta$ making the diagram commute:



Note that j extends to $\tilde{j} : M_k[g_k] \to N[g_k]$, and $N[g_k] \models j(S_{\mu^*_\eta}) = j(S_{\mu^*_\gamma})$ and is $j(\rho)^+$-absolutely complemented. Thus in $N[g_k][h]$, if $j(\mu_\eta)^*_x$ is well-founded then $j(\mu_\gamma)^*_x$ must also be well-founded as well. Similarly, if $(j(f(\mu_\eta))^*_x$ is well-founded then $(j(f(\mu_\eta)))^*_x$ is well-founded. Thus it suffices to show that $(\nu_\eta)_x$ is well-founded if and only if $j(\mu_\gamma)^*_x$ is well-founded and similarly for γ. If $j(\mu_\eta)^*_x$ is ill-founded, then we have an $\langle [x\restriction n, y_n]_\mu : n < \omega\rangle$ with $x_n \subset x_{n+1}$ and $\{u : y_n(u) > y_{n+1}(u)\} \in j(\mu^\eta_{x\restriction n})^*$. As both g, h are $< j(\rho)^+$-generic (thus below the completeness of $j(\mu)$), we can make it so that each $y_n \in N$ with $\{u : y_n(u) > y_{n+1}(u)\} \in j(\mu^\eta_{x\restriction n})$, even though the sequence $\langle y_n : n < \omega\rangle$ may not be in N. Applying τ yields $\{u : \tau(y_n)(u) > \tau(y_n)(u)\} \in \sigma_k(\mu^\eta_{x\restriction n})$ for each $n < \omega$, so $(\nu_\eta)_x$ is ill-founded as witnessed by $\langle [x\restriction n, \tau(y_n)] : n < \omega\rangle$. If $j(\mu_\eta)^*_x$ is well-founded, then $(j(f(\mu_\eta)))^*_x$ is ill-founded; an entirely symmetric argument shows that $\sigma_k(f)(\nu_\eta)_x = \sigma_k(\mu_\eta)_x$ is ill-founded, so $(\nu_\eta)_x$ is well-founded by elementarity. Hence the subclaim is proved. ▲

We proceed to showing that $S_{\nu_\eta} \cap \mathbb{R}^V = \mathrm{proj}_0[i_{k,\omega}(T)] \cap \mathbb{R}^V$. Let $x \in \mathbb{R}^V$ be arbitrary. Suppose $x \in \mathrm{proj}_0[i_{k,\omega}(T)]$, so that $x \in \mathrm{proj}_0[i_{k,n}(T)]$ for some $n > k$. Let

$$\langle \tau_\eta : \xi_n < \eta < \lambda_n\rangle = i_{k,n}(\langle \mu_\eta : \xi_k < \eta < \lambda_k\rangle)$$

$$\langle h_\eta : \xi_n < \eta < \lambda_n\rangle = i_{k,n}(\langle f_\eta : \xi_k < \eta < \lambda_k\rangle)$$

and put $(\rho_\eta)_u = \sigma_n(\tau_\eta)_u$. Replacing (M_k, ρ_η, ν) with $(M_n, i_{k,n}(\rho), \rho_\eta)$ in the proof of the subclaim above, we get that $S_{\rho_\eta} = S_{\rho_\gamma}$ for $i_{k,n}(\rho) < \eta < \gamma < \lambda_n$. Note that $\sigma_n \circ i_{k,n} = \sigma_k$, so $\rho_{i_{k,n}(\eta)} = i_{k,n}(\mu_\eta) = \nu_\eta$, and it suffices to see that $x \in S_{\rho_{i_{k,n}(\eta)}}$ for any (every) η. But by elementarity of $\tilde{i}_{k,n}$, since

$$M_k[g_k] \models \forall x \in \text{proj}_0[T], (\mu_\eta)_x \text{ is well-founded}$$

we have

$$M_n[g_k] \models \forall x \in \text{proj}_0[\tilde{i}_{k,n}(T)], \tau_{i_{k,n}(\eta)} \text{ is well-founded.}$$

The calculation in subclaim above gives us that $(\rho_{i_{k,n}(\eta)})_x$ is well-founded by replacing σ_k with σ_n. A symmetric argument shows that if $x \notin \text{proj}_0(i_{k,n}[T])$ then $(\rho_{i_{k,n}(\eta)})_x$ is ill-founded. This completes the proof of the claim, thus the lemma. △

▲

By the statement above, there is $B \in L(\mathbb{R}^V, \text{Hom}^V_{<\lambda})$ with $(\text{HC}^V, A, B) \models \varphi$. We'll show that in fact $B \in \text{Hom}^V_{<\lambda}$. Recall that we have $\text{Hom}_{<\lambda}$-determinacy, thus there is a Wadge rank on $\text{Hom}_{<\lambda}$. Let $\text{Hom} \restriction \alpha$ be the collection of $C \in \text{Hom}^V_{<\lambda}$ with rank less than α. Let (α_0, β_0) be the lexicographically least pair such that there's $B \in L_\beta(\mathbb{R}^V, \text{Hom} \restriction \alpha)$ satisfying our requirement. Suppose first that $\text{Hom} \restriction \alpha_0 = \text{Hom}^V_{<\lambda} := \text{Hom}$. Note that in general B is ordinal definable over $L_{\beta_0}(\mathbb{R}^V, \text{Hom}^V)$ from $\{C, \text{Hom}^V\}$ for some $C \in \text{Hom}^V$, where we can easily encode the real parameter into a part of C without losing its homogeneity, and by our case hypothesis, B is ordinal definable over $L_{\beta_0}(\mathbb{R}^V, \text{Hom}^V)$ from $\{A, C\}$. Minimizing the ordinal parameter, we may assume the definition only involves A, C. Fix formula ψ so that

$$x \in B \Leftrightarrow L_{\beta_0}(\mathbb{R}^V, \text{Hom}^V) \models \psi(x, A, C)$$

Let (T_A, S_A) be $<\lambda$-absolutely complementing trees for A with $\text{proj}_0[T] = A$, and similarly for C. Fix sufficiently large θ with $T_A, S_A, T_C, S_C \in V_\theta$, and $\pi : M \cong X \prec V_\theta$ countable elementary with $\pi(\bar{S}_A, \bar{T}_A, \bar{S}_C, \bar{T}_C, \bar{\lambda}) = (S_A, T_A, S_C, T_C \lambda)$ and M transitive. Let $\chi(v_0, v_1, v_2, v_3)$ be the sentence:

1. v_0 is a limit of Woodin cardinals
2. $v_1 \in \mathbb{R}$, and v_2, v_3 are trees on $\omega \times Z$ for some Z
3. (a) If $\mathbb{R}^*, \text{Hom}^*$ are derived from $\text{Coll}(\omega, < v_0)$ then there is $\beta \in \text{OR}$ so that $\exists B \in L_\beta(\mathbb{R}^*, \text{Hom}^*)$ with $(\text{HC}, \text{proj}_0[v_2] \cap \mathbb{R}^*, B) \models \varphi$

(b) Moreover, for the least such β,

$$L_\beta(\mathbb{R}^*, \mathrm{Hom}^*) \models \psi(v_1, \mathrm{proj}_0[v_2], \mathrm{proj}_0[v_3])$$

Let $\mu < \bar{\lambda}$ be so that every $\pi(\mu)$-weakly homogeneous C is $< \lambda$-homogeneous, and let W be the set corresponding to M, V_θ, μ, π as in Windzus' lemma, so it is $\pi(\mu)^+$-homogeneously Suslin. Let $\rho \in [\mu+1, \bar{\lambda})$ be the least such that $M \models \rho$ is Woodin.

Lemma 3.6. *Work in V. For every $x \in \mathbb{R}$, the following are equivalent:*

1. *$x \in B$*
2. *$\exists(b, \tilde{\mathcal{T}}) \in W \exists g \in \mathbb{R}$, g codes a $\mathcal{M}^{\mathcal{T}}_\omega$-generic filter for* $\mathrm{Coll}(\omega, i^{\mathcal{T}}_{0,\omega}(\bar{\lambda}))$ *with $(x, z) \in \mathcal{M}^{\mathcal{T}}_\omega[g]$ and $\mathcal{M}^{\mathcal{T}}_\omega[g] \models \chi(i^{\mathcal{T}}_{0,\omega}(\bar{\lambda}), x, i^{\mathcal{T}}_\omega(\bar{T}_A), i^{\mathcal{T}}_\omega(\bar{T}_C))$*

Proof of lemma: First note that by the same calculation in the proof of 3.2, there is $(b, \tilde{\mathcal{T}}) \in W$ and $i^{\mathcal{T}}_{0,\omega} = i_{0,1} : M_0 \to M_1$ and $g_1 \in V$ which is M_1-generic for $\mathrm{Coll}(\omega, i_{0,1}(\rho))$ with $x \in M_1[g_1]$. By elementarity of the embedding $\tilde{i}_{1,b} : \mathcal{M}^{\mathcal{T}}_1[g_1] \to \mathcal{M}^{\mathcal{T}}_\omega[g_1]$ it suffices to see that for arbitrary such $(b, \tilde{\mathcal{T}})$ with corresponding iteration tree $\mathcal{T}$,

$$x \in B \Leftrightarrow M_1[g_1] \models \chi(i_{0,1}(\bar{\lambda}), x, z, i_{0,1}(\bar{T}))$$

As in the previous lemma, work in $V[H]$ for some $H \subseteq \mathrm{Coll}(\omega, \mathbb{R})$ which is V-generic and obtain $\langle (M_n, g_n, \sigma_n) : n < \omega \rangle$. We obtain the well-founded direct limit M_ω and realization $\sigma_\omega : M_\omega \to V_\theta$. Since $i_{1,\omega}$ extends to an elementary $\tilde{i}_{1,\omega} : M_1[g_1] \to M_\omega[g_1]$ and $\tilde{i}_{0,\omega}(\bar{\lambda}, \bar{T}, x, z) = (i_{1,\omega}(i_{0,1}(\bar{\lambda})), i_{1,\omega}(i_{0,1}(\bar{T})), x, z)$, it suffices to see

$$x \in B \Leftrightarrow M_\omega[g_1] \models \chi(i_{0,\omega}(\lambda_0), x, z, i_{0,\omega}(\bar{T}))$$

Just like before, we have $\mathbb{R}^*_I = \mathbb{R}^V, (\mathrm{HC}^*)^{M_\omega[g]} = \mathrm{HC}, \mathrm{proj}_0[i_{0,\omega}(T)] \cap \mathbb{R}^V = A, \mathrm{Hom}^*_I \subseteq \mathrm{Hom}^V_{<\lambda}$. Let $\gamma = M_\omega \cap \mathrm{OR}$. Since every derived model has $D \subseteq \mathbb{R}^*, D \in L(\mathbb{R}^*, \mathrm{Hom}^*)$ with $(\mathrm{HC}^*, A^*, D) \models \varphi$ (it is forced by the empty condition), so the elementarity of σ_ω yields

$$D \subseteq (\mathbb{R}^*)^{M_\omega[g]} = \mathbb{R}^V, D \in (L(\mathbb{R}^*, \mathrm{Hom}^*))^{M_\omega[g]} = L_\gamma(\mathbb{R}^V, \mathrm{Hom}^*_I)$$

with $(\mathrm{HC}^V, A, D) \models \varphi$. Since $\mathrm{Hom} \restriction \alpha_0 = \mathrm{Hom}$ and Hom^*_I is a Wadge initial segment of Hom (we've seen that Hom is closed under continuous reduction, and the same reason applies to Hom^*), it cannot be the case that $\mathrm{Hom}^*_I \neq \mathrm{Hom}$, since otherwise the Wadge rank restricted to Hom^*_I would have order type $< \alpha_0$, which contradicts the minimality of α_0. We also have that $\beta_0 < \gamma$ since otherwise this would contradict the minimality of β_0 (as there is some other D below level γ also satisfying this). Thus the β mentioned in χ equals β_0 in $M_\omega[g]$. It now follows that

$$\begin{aligned} x \in B &\Leftrightarrow L_{\beta_0}(\mathbb{R}^V, \mathrm{Hom}\restriction\alpha_0) \models \psi(x, A, C) \\ &\Leftrightarrow (L_{\beta_0}(\mathbb{R}^*, \mathrm{Hom}^*_I \models \psi(x, A^*, C^*))^{M_\omega[g]} \\ &\Leftrightarrow M_\omega[g_1] \models \chi(i_{0,\omega}(\lambda_0), x, i_{0,\omega}(\bar{T}_A), i_{0,\omega}(\bar{T}_C)) \end{aligned}$$

This completes the proof, assuming that $\mathrm{Hom}\restriction\alpha_0 = \mathrm{Hom}$. If $\mathrm{Hom}\restriction\alpha_0 \neq \mathrm{Hom}$, then we fix $D \in \mathrm{Hom}$ of Wadge rank α_0, with λ-absolutely complementing T_D, S_D with $\mathrm{proj}_0[T_D] = D$, and use the parameters (A, C, D) for $C <_W D$ so that

$$x \in B \Leftrightarrow L_{\beta_0}(\mathbb{R}^V, \{X \in \mathrm{Hom}^V_{<\lambda} : X <_W D\}) \models \psi(x, C, D)$$

and then define $\chi(v_0, v_1, v_2, v_3, v_4)$ analogously, modifying clause 3(b) of χ to be

$$L_\beta(\mathbb{R}^*, \{X \in \mathrm{Hom}^* : X <_W \mathrm{proj}_0[v_4]\}) \models \psi(v_1, \mathrm{proj}_0[v_3], \mathrm{proj}_0[v_4]).$$

As argued before, we have $i_{0,\omega}(\bar{T}_D)$ projects to D in the derived model of $M_\omega[g]$. This completes the proof of the lemma. ▲

Finally, by Windzus' lemma, W is $\pi(\mu)^+$-homogeneously Suslin, and the existence of a g coding $\mathcal{M}^{\mathcal{T}}_\omega$-generic filter is a real projective quantification in the parameter encoding M, and the satisfaction relation is arithmetic in the parameters. Then B is $\pi(\mu)^+$-weakly homogeneous, thus $< \lambda$-homogeneous, whence the theorem follows. □

Corollary 3.7. *Let G be V-generic for* $\mathrm{Coll}(\omega, < \lambda)$ *where λ is a limit of Woodin cardinals; let $\alpha < \lambda$ and $A \in \mathrm{Hom}^{V[G\restriction\alpha]}_{<\lambda}$. Then* $(\mathrm{HC}^{V[G\restriction\alpha]}, A) \prec (\mathrm{HC}^*_G, A^*)$

Proof. Note that each $\Sigma_n^{\mathrm{HC}^{V[G\restriction\alpha]}}$-statement is equivalent to a $\Sigma^1_{n+1}(z)$ statement for some $z \in \mathbb{R}$ in $V[G\restriction\alpha]$, and $\{z\} \in \mathrm{Hom}^{V[G\restriction\alpha]}_{<\lambda}$, whence the theorem follows by an induction on formula complexity. □

We now have all the tools we need to prove the Derived Model Theorem, following the sketch in [9]. Note that below we are proving (2) before proving (1), which differs from the order of [9] for a technical reason.

Theorem 3.8. *Suppose λ is a limit of Woodin cardinals. Let $G \subseteq \mathrm{Coll}(\omega, <\lambda)$ be V-generic, with derived $\mathbb{R}^*, \mathrm{Hom}^*$. The following holds:*

1. $L(\mathbb{R}^*, \mathrm{Hom}^*) \models AD^+$
2. $\mathrm{Hom}^* = \{A \subseteq \mathbb{R}^* : A \text{ is Suslin and co-Suslin in } L(\mathbb{R}^*, \mathrm{Hom}^*)\}$

Proof. The proof of (2) is the same as in [9]. First fix $C \in \mathrm{Hom}^*$, so that $C = A^*$ for some $A \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]}$. By [7], we may let $B \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]}$ code a scale on A. This is definable over $\mathrm{HC}^{V[G\restriction\alpha]}$ using predicates A, B; by Corollary 3.7, B^* codes a scale on $A^* = C$ in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, so C is Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$. Since $\mathbb{R}^* - C \in \mathrm{Hom}^*$, C is also co-Suslin.
Suppose now that $A \in L(\mathbb{R}^*, \mathrm{Hom}^*)$ is Suslin and co-Suslin, witnessed by trees T and U, so that $\mathrm{proj}_0[T] = C, \mathrm{proj}_0[U] = \mathbb{R}^* - C$. Since $T, U \in L(\mathbb{R}^*, \mathrm{Hom}^*)$, there is some $C \in \mathrm{Hom}^*$ such that $T, U \in \mathrm{OD}_C^{L(\mathbb{R}^*, \mathrm{Hom}^*)}$. Let $\alpha < \lambda, B \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]}, S \in V[G\restriction\alpha]$ be such that $C = B^* = \mathrm{proj}_0[S] \cap \mathbb{R}^*$, and write $W = V[G\restriction\alpha]$. Note that then $T, U, C \in \mathrm{OD}_{\{\mathbb{R}^*, \mathrm{Hom}^*, S\}}^{V[G]}$. Let $H \subseteq \mathrm{Coll}(\omega, <\lambda)$ be W-generic with $V[G] = W[H]$, so $T, U \in \mathrm{OD}_{\{\mathbb{R}^*, \mathrm{Hom}^*, S\}}^{W[H]}$ and $T, U \subseteq W$. Note that there are canonical names $\tau_{\mathbb{R}}, \tau_{\mathrm{Hom}}, \check{S} \in W$ for $\mathbb{R}^*, \mathrm{Hom}^*, S$ which are homogeneous for $\mathrm{Coll}(\omega, <\lambda)$, so in fact $T, U \in W$. Since $\mathrm{proj}_0[T] \cap \mathbb{R}^* = \mathbb{R}^* - \mathrm{proj}_0[U]$ and $L(\mathbb{R}^*, \mathrm{Hom}^*)$ is a symmetric extension,

$$\Vdash_{\mathrm{Coll}(\omega,<\lambda)}^{W} \mathrm{proj}_0[\check{T}] \cap \tau_{\mathbb{R}} = \tau_{\mathbb{R}} - \mathrm{proj}_0[\check{U}]$$

In particular, given any $\kappa < \lambda$ and $K_0 \subseteq \mathrm{Coll}(\omega, \kappa)$ is any W-generic filter, then by extending it to $K \subseteq \mathrm{Coll}(\omega, <\lambda)$, we see that

$$\mathrm{proj}_0[T]] \cap {\mathbb{R}^*}^{W[K]} = {\mathbb{R}^*}^{W[K]} - \mathrm{proj}_0[U]$$

so $\mathrm{proj}_0[T]^{W[K_0]} = \mathbb{R}^{W[K_0]} - \mathrm{proj}_0[U]$. Since $\mathrm{Coll}(\omega, \kappa)$ is universal forcing, we see that T, U are $<\lambda$-absolutely complementing, so $C = \mathrm{proj}_0[T] \in \mathrm{Hom}^*$.

We now proceed to the proof of (1). For AD, if$B \in L(\mathbb{R}^*, \mathrm{Hom}^*) \cap \mathcal{P}(\mathbb{R}^*)$ is not determined in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, then

$$(\mathrm{HC}^*, \varnothing, B) \models \forall\sigma, \sigma \text{ is not a winning strategy for } G(B)$$

By the reflection theorem, there is $B_0 \in \mathrm{Hom}_{<\lambda}^V$ such that

$$(\mathrm{HC}^V, \varnothing, B_0) \models \forall\sigma, \sigma \text{ is not a winning strategy for } G(B_0)$$

But we have $\mathrm{Hom}_{<\lambda}$-determinacy, and every winning strategy in V is in HC^V, which is a contradiction. Similarly for $\mathrm{DC}_{\mathbb{R}}$, if $R \in L(\mathbb{R}^*, \mathrm{Hom}^*) \cap \mathcal{P}(\mathbb{R}^2)$ is a total relation but

$$(\mathrm{HC}^*, \varnothing, R) \models \forall f \in \mathbb{R}^\omega \neg(\forall n < \omega, f(n) R f(n+1))$$

then there is a $R_0 \in \mathrm{Hom}_{<\lambda}$ satisfying the same sentence. But every total relation on $\mathbb{R}^2$ has such an f, as choice holds in HC^V. For Ordinal Determinacy, suppose toward the contrary that $\delta < \Theta, f : \delta^\omega \to {}^\omega\omega \in L(\mathbb{R}^*, \mathrm{Hom}^*)$ is continuous, $B \in L(\mathbb{R}^*, \mathrm{Hom}^*) \cap \mathcal{P}(\mathbb{R}^*)$ is such that $f^{-1}[B]$ is not determined in $L(\mathbb{R}^*, \mathrm{Hom}^*)$. Pick a prewellorder $\preceq$ of length δ and let $F \subseteq \mathbb{R}$ recursively code f via $\preceq$. By assumption,

$$\begin{aligned}(\mathrm{HC}^*, B, F, \preceq) \models\ & \preceq \text{ is prewellordering} \\ & \wedge F \text{ codes a continuous function } f : {}^\omega\,\mathrm{lh}(\preceq) \to \mathbb{R} \\ & \wedge \forall x \in \mathbb{R}, U^2_x(\preceq) \text{ does not code a winning strategy for } \tilde{G}(B, F, \preceq)\end{aligned}$$

where $\tilde{G}(B, F, \preceq)$ is the game of playing a sequence of reals with winning condition determined by $B, F, \preceq$ as a game on ${}^\omega\,\mathrm{lh}(\preceq)$. By coding $f : {}^\omega\,\mathrm{lh}(\preceq) \to {}^\omega\omega$ we mean F as a subset of $\mathbb{R} \times \mathbb{R}$ coding a function $f : {}^\omega\mathbb{R} \to \mathbb{R}$ continuous under product topology with $\mathbb{R}$ taken discrete, such that $f(x) = f(y)$ if $x_n \equiv y_n$ according to $\preceq$ for every $n < \omega$. Then there are $B_0, F_0, \preceq_0 \in \mathrm{Hom}^V_{<\lambda}$ such that

$$\begin{aligned}(\mathrm{HC}^V, B_0, F_0, \preceq_0) \models\ & \preceq_0 \text{ is prewellordering} \\ & \wedge F_0 \text{ codes a continuous function } f : {}^\omega\,\mathrm{lh}(\preceq_0) \to {}^\omega\omega \\ & \wedge \forall x \in \mathbb{R}, \\ & \quad U^2_x(\preceq_0) \text{ does not code a winning strategy for } \tilde{G}(B_0, F_0, \preceq_0).\end{aligned}$$

So by 3.7,

$$\begin{aligned}(\mathrm{HC}^*, B_0^*, F_0^*, \preceq_0^*) \models\ & \preceq_0^* \text{ is prewellordering} \\ & \wedge F_0^* \text{ codes a continuous function } f : {}^\omega\,\mathrm{lh}(\preceq_0^*) \to {}^\omega\omega \\ & \wedge \forall x \in \mathbb{R}, \\ & \quad U^2_x(\preceq_0^*) \text{ does not code a winning strategy for } \tilde{G}(B_0^*, F_0^*, \preceq_0^*).\end{aligned}$$

But $B_0^* \in \mathrm{Hom}^*_G$, thus is Suslin and co-Suslin, so $\tilde{G}(B_0^*, F_0^*, \preceq_0^*)$ is determined in $L(\mathbb{R}^*, \mathrm{Hom}^*)$ as given by [3]. Let $Z \subseteq (\mathbb{R}^*)^2$ code a winning strategy for $\tilde{G}(B_0^*, F_0^*, \preceq_0^*)$; by the coding lemma, we can find $x \in \mathbb{R}^*$ such that $U^2_x(\preceq_0^*)$ codes the same strategy as Z. This is a contradiction.

Finally, to see that every subset of ${}^{\omega}2$ is ∞-Borel, suppose $B \in L(\mathbb{R}^*, \mathrm{Hom}^*)$ is a counter-example to this. We may assume that δ_B is closed under ordinal arithmetic, so that every $\varphi \in \mathcal{L}^0_{\delta_B}$ can be recursively encoded by some $S \subseteq \delta_B$. Fix prewellorder $\preceq$ of length δ_B so that

$$(\mathrm{HC}^*, B, \preceq) \models \ \preceq \text{ is a prewellorder} \wedge \mathrm{lh}(\preceq) = \delta_B$$
$$\wedge \forall x \in \mathbb{R}, U^2_x(\preceq) \text{ does not code a } \delta_B\text{-Borel code for } B$$

where by $\mathrm{lh}(\preceq) = \delta_B$ we mean

$\forall x \in \mathbb{R} \exists \sigma, z \in \mathbb{R}(\sigma$ codes a continuous reduction from B to a prewellorder $\sqsubseteq$,
$U^2_z(\preceq, \sqsubseteq)$ codes an embedding from $\sqsubseteq$ to $\{y : y \preceq x\}) \wedge$
$\forall \sigma, z \in \mathbb{R} \neg(\sigma$ codes a continuous reduction from B to a prewellorder $\sqsubseteq$,
$U^2_z(\preceq, \sqsubseteq)$ codes an embedding from $\preceq$ to $\sqsubseteq)$

which is indeed equivalent to $\mathrm{lh}(\preceq) = \delta_B$ in $L(\mathbb{R}^*, \mathrm{Hom}^*)$ as it satisfies AD. Then there are $B_0, \preceq_0 \in \mathrm{Hom}_{<\lambda}$ with

$$(\mathrm{HC}, B_0, \preceq_0) \models \ \preceq_0 \text{ is a prewellorder} \wedge \mathrm{lh}(\preceq_0) = \delta_{B_0}$$
$$\wedge \forall x \in \mathbb{R}, U^2_x(\preceq_0) \text{ does not code a } \delta_{B_0}\text{-Borel code for } B_0.$$

As before,

$$(\mathrm{HC}^*, B^*_0, \preceq^*_0) \models \ \preceq^*_0 \text{ is a prewellorder} \wedge \mathrm{lh}(\preceq^*_0) = \delta_{B^*_0}$$
$$\wedge \forall x \in \mathbb{R}, U^2_x(\preceq^*_0) \text{ does not code a } \delta_{B^*_0}\text{-Borel code for } B^*_0.$$

But B^*_0 is Suslin, so it has a $\delta_{B^*_0}$-Borel code in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, encoded by some $Z \subseteq \mathbb{R}^2$ with $Z \in L(\mathbb{R}^*, \mathrm{Hom}^*)$. Using the Coding Lemma again, we can fix $x \in \mathbb{R}$ so that $U^2_x(\preceq^*_0)$ encodes the same $\delta_{B^*_0}$-Borel code for B^*_0. We reach a contradiction, thus completing the proof. □

Remark 3.9. The proof above also gives a nice consequence: if $B, F, \preceq \in \mathrm{Hom}_{<\lambda}$ code an ordinal game as above, then (in V) there is $x \in \mathbb{R}$ so that $U^2_x(\preceq)$ codes a winning strategy for the game; in particular, the strategy is also in $\mathrm{Hom}_{<\lambda}$, which is closed under projection. The same also holds for ∞-Borel code.

4 Acknowledgment

I would like to thank the ISURF program at Carnegie Mellon University for funding this project. I would also like to thank Professor Sandra Müller for supervising this project, thank Lena Wallner for helping me revise these notes, and thank William Chan and Lukas Koschat for helping clarify parts of the proof. Finally, I would like to thank Professor Ernest Schimmerling for introducing me to this research opportunity.

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