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„Iteration Trees and Genericity Iterations“

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Abstract in English

This thesis is a careful and detailed introduction to the basic concepts of inner model theory. We start with developing the construction of ultrapowers and iteration trees. The first main result is the $(\omega_1 + 1)$-iterability of nice iteration trees and weak iterability. We then discuss genericity iterations and some useful variants. They are used in the last main result which is that $L(\mathbb{R})$ is a model of the Axiom of Determinacy.

Abstract auf Deutsch

Diese Arbeit ist eine sorgfältige und detailreiche Einführung in die Grundkonzepte der inneren Modelltheorie. Wir beginnen mit der Entwicklung der Konstruktion von Ultrapotenzen und Iterationsbäumen. Das erste Hauptresultat ist die (ω_1+1)-Iterierbarkeit von schönen Iterationsbäumen und die schwache Iterierbarkeit. Anschließend diskutieren wir Generizitätsiterationen und nützliche Varianten davon. Diese werden im letzen Hauptresultat verwendet. Dort zeigen wir, dass $L(\mathbb{R})$ ein Modell des Axioms der Determiniertheit ist.

Contents

1 Introduction

This thesis is a report of my first steps in inner model theory. It can be a good source for someone who is interested in inner model theory and wants to start with basic concepts. I included a lot of details, so it is a slow start into the topic.

Organization of the thesis. Section 2 is about introducing the tools. We start with ultrafilters and how to build ultrapowers from ultrafilters in chapter 2.1. We can take an ultrapower of an ultrapower, which leads to linear iterations. Those are discussed in chapter 2.2. Extenders are introduced in chapter 2.3. They are a generalization of ultrafilters and can also be used to build ultrapowers. We can form linear iterations using extenders, see chapter 2.4. Chapter 2.5 consists of three parts. In the first part, we define iteration trees and discuss some basic facts. The second part is about $(\omega + 1)$-iterability of nice iteration trees. The third part is about weak iterability. The last chapter 2.6 introduces Woodin cardinals and its characterization in terms of extenders. In section 3 we use iteration trees. Chapter 3.1 is about a method called genericity iteration. We use this method in chapter 3.2, where we prove that $L(\mathbb{R})$ is a model of the Axiom of Determinacy.

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Conventions and Notation. Every elementary embedding that appears is not the identity map. Let ZFC^- be the theory obtained by removing the Axiom of Power Set from ZFC and strengthening the Axiom of Replacement to the Axiom of Collection[1]. M, N are always models of ZFC^- which are transitive

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[1] Read more about why it is important to strengthen Replacement in [GHJ15].

and therefore wellfounded, i.e. there is no strictly $\in$-decreasing sequence in V consisting of elements of M. κ is always a cardinal.

2 Ultrapowers and Iteration Trees

This chapter is based on Steel's paper *An Introduction to Iterated Ultrapowers* [Ste15]. I also used Jech's book *Set Theory* [Jec03] and Kanamori's book *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings* [Kan08] as references. Kasum's Master's Thesis *Projective Determinacy* [Kas21] was very helpful for figuring out details.

2.1 Ultrapowers from Ultrafilters

Definition 2.1.1. $U \subseteq \mathcal{P}(\kappa)^M$ is a **filter on κ for M** iff the following hold:

- $\emptyset \notin U$, $\kappa \in U$,
- $A \in U, B \in \mathcal{P}(\kappa)^M, A \subseteq B \Rightarrow B \in U$ and
- $F_0, F_1 \in U \Rightarrow F_0 \cap F_1 \in U$.

U is an **ultrafilter on κ for M** iff it additionally satisfies

- $A \in \mathcal{P}(\kappa)^M \Rightarrow A \in U$ or $\kappa \setminus A \in U$.

We say that a property holds for **U-almost every α** (**U-a.e. α** for short) iff the set of $\alpha < \kappa$ which have the property is in U. The **critical point of U** is $crit(U) := \kappa$.

Definition 2.1.2. Let U be an ultrafilter on κ for M.

- U is **nonprincipal** iff U doesn't contain singletons.
- U is **M-normal** iff for each function $f \in M$ with $dom(f) = \kappa$ and $f(\alpha) < \alpha$ for U-almost every α, there is some $\beta < \kappa$ such that $f(\alpha) = \beta$ for U-almost every α.
- U is **M-κ-complete** iff for each sequence $\langle A_\alpha \mid a < \beta \rangle \in M$ of length $\beta < \kappa$, where $A_\alpha \in U$ for each $\alpha < \beta$, we have $\bigcap_{\alpha<\beta} A_\alpha \in U$.

From now on, let U be a nonprincipal ultrafilter on κ for M.

Lemma 2.1.3. *Let $j : M \to N$ be an elementary embedding. Let $\kappa := crit(j)$, i.e. κ is the least ordinal α such that $j(\alpha) \neq \alpha$. Define $U_j \subseteq \mathcal{P}(\kappa)^M$ by setting*

$$A \in U_j \text{ iff } \kappa \in j(A)$$

for each $A \in \mathcal{P}(\kappa)^M$. Then U_j is a nonprincipal ultrafilter on κ for M which is M-κ-complete and M-normal. We call U_j the ***ultrafilter derived from*** *j.*

Proof. U_j is an ultrafilter: $j(\emptyset) = \emptyset \not\ni \kappa$, so $\emptyset \notin U_j$ and $\kappa = crit(j)$, so $\kappa \in j(\kappa)$ hence $\kappa \in U_j$. If $A \in U_j$ and $A \subseteq B \in \mathcal{P}(\kappa)^M$, then $\kappa \in j(A) \subseteq j(B)$ and thus $B \in U_j$. We also have $j(A_0) \cap j(A_1) = j(A_0 \cap A_1)$, so $A_0, A_1 \in U_j$ implies that $A_0 \cap A_1 \in U_j$. Finally U_j is an ultrafilter since $\kappa \notin j(A)$ implies $\kappa \in j(\kappa) \setminus j(A) = j(\kappa \setminus A)$.

U_j is nonprincipal: For every $\alpha < \kappa$ we have $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\} \not\ni \kappa$.

U_j is M-κ-complete: Let $\langle A_\alpha \mid \alpha < \beta \rangle \in M$ with $\beta < \kappa$ and $A_\alpha \in U_j$ for each $\alpha < \beta$. Then

$$j(\bigcap_{\alpha<\beta} A_\alpha) = \bigcap_{\alpha<\beta} j(A_\alpha)$$

since $\beta < crit(j)$ and we have $\kappa \in j(A_\alpha)$ for each $\alpha < \beta$. So

$$\kappa \in j(\bigcap_{\alpha<\beta} A_\alpha) \text{ and } \bigcap_{\alpha<\beta} A_\alpha \in U_j.$$

U_j is M-normal: Let $f \in M$ with $dom(f) = \kappa$ and

$$A := \{\alpha < \kappa \mid f(\alpha) < \alpha\} \in U_j.$$

Set $\beta := j(f)(\kappa)$. Then $\beta < \kappa$ since

$$\kappa \in j(A) = \{\alpha < j(\kappa) \mid j(f)(\alpha) < \alpha\}.$$

Also

$$\kappa \in \{\alpha < j(\kappa) \mid j(f)(\alpha) = \beta\} = j(\{\alpha < \kappa \mid f(\alpha) = \beta\}).$$

Therefore $\{\alpha < \kappa \mid f(\alpha) = \beta\} \in U_j$. □

Definition 2.1.4. Define an equivalence relation on the functions of M with domain κ by:

$$f \sim g \text{ iff } \{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in U$$

and denote the equivalence class of f by $[f]$ or $[f]_U$ if we want to emphasize the ultrafilter. We define another relation on the equivalence classes:

$$[f] \tilde{\in} [g] \text{ iff } \{\alpha < \kappa \mid f(\alpha) \in g(\alpha)\} \in U.$$

Now we set

$$Ult(M, U) := (\{[f] \mid f \text{ function in } M, dom(f) = \kappa\}, \tilde{\in})$$

and define the map

$$i_U^M : M \to Ult(M, U), x \mapsto [const_x],$$

where $const_x : \kappa \to M, \alpha \mapsto x$. We collapse the wellfounded part of $Ult(M,U)$. So if $(Ult(M,U), \tilde{\in})$ is wellfounded then this yields a transitive model. We denote it by $(Ult(M,U), \in)$ and call it the **ultrapower of M by U**.

Remark 2.1.5. $Ult(M,U)$ has the same cardinality as M. In particular, if M is countable then $Ult(M,U)$ is also countable.

Lemma 2.1.6. *If U is M-κ-complete then U does not contain bounded subsets of κ. In particular, the interval $(\beta,\kappa) := \{\alpha < \kappa \mid \beta < \alpha\} \in U$ for each $\beta < \kappa$.*

Proof. It suffices to show that $\beta \notin U$ for each $\beta < \kappa$. Fix some $\beta < \kappa$. Note that $\kappa \setminus \{\alpha\} \in U$ for each $\alpha < \beta$ since U is nonprincipal. Therefore

$$\kappa \setminus \beta = \kappa \setminus \bigcup_{\alpha<\beta} \{\alpha\} = \bigcap_{\alpha<\beta} \kappa \setminus \{\alpha\} \in U$$

by M-κ-completeness. □

Lemma 2.1.7. *(Properties of $Ult(M,U)$) Assume that $(Ult(M,U), \tilde{\in})$ is wellfounded. Let f, f_α be functions in M with domain κ for each $\alpha < \kappa$. Denote the identity function on κ by id and $i := i^M_U$.*

(i). Łoś Theorem: Let $\varphi(v_0, \ldots, v_n)$ be an $\mathcal{L}_\in$-formula and $n < \omega$. Then

$$\begin{aligned} &Ult(M,U) \models \varphi([f_0], \ldots, [f_n]) \text{ iff} \\ &M \models \varphi(f_0(\alpha), \ldots, f_n(\alpha)) \text{ for } U\text{-a.e. } \alpha. \end{aligned}$$

(ii). i is elementary. In particular $Ult(M,U) \models ZFC^-$.

(iii). If U is M-κ-complete then $\kappa = crit(i)$.

(iv). $[f] = i(f)([id])$.

(v). Let $A \in \mathcal{P}(\kappa)^M$. Then $A \in U$ iff $[id] \in i(A)$.

(vi). If U is M-normal and M-κ-complete then $[id] = \kappa$.

(vii). If $\langle f_\alpha \mid \alpha < \kappa \rangle \in M$ then $\langle [f_\alpha] \mid \alpha < \kappa \rangle \in Ult(M,U)$.

Proof. (i). We show this by induction on the complexity of φ. If $\varphi = (v_0 \circ v_1)$ with variables v_0 and v_1 and $\circ \in \{=, \in\}$ then

$$Ult(M,U) \models \varphi([f_0],[f_1]) \text{ iff } \{\alpha < \kappa \mid M \models f_0(\alpha) \circ f_1(\alpha)\} \in U$$

by definition of $Ult(M,U)$. The cases $\varphi = \neg\psi, \psi_1 \wedge \psi_2, \psi_1 \vee \psi_2$ follow from the ultrafilterness of U. We have to be a little bit more careful in the case

$\varphi = \exists y\, \psi(y)$. We abbreviate $[f_0], \ldots, [f_n]$ by $[\vec{f}]$ and $f_0(\alpha), \ldots, f_n(\alpha)$ by $f(\vec{\alpha})$. We have that

$$\begin{aligned}
&Ult(M, U) \models \exists y\, \psi([\vec{f}], y)\\
&\Leftrightarrow \text{ there is } g \in M \text{ such that } Ult(M, U) \models \psi([\vec{f}], [g])\\
&\Leftrightarrow \text{ there is } g \in M \text{ such that } \{\alpha < \kappa \mid M \models \psi(f(\vec{\alpha}), g(\alpha))\} \in U\\
&\Leftrightarrow \{\alpha < \kappa \mid M \models \exists y\, \psi(f(\vec{\alpha}), y)\} \in U.
\end{aligned}$$

Here, the forward direction in the last implication is obvious. For the other direction, we need to find a function $g \in M$ with $dom(g) = \kappa$ such that $g(\alpha)$ is a witness for $M \models \exists y\, \varphi(f(\vec{\alpha}), y)$ for U-almost every α. This is the point where we use the Axiom of Collection and the Axiom of Choice in M. In order to see that the witnesses for $M \models \exists y\, \varphi(f(\vec{\alpha}), y)$ form a set, we use the Axiom of Collection[2]. Then we can use the Axiom of Choice to find a g with the desired property.

(ii). Let $a_1, \ldots, a_n \in M$ and suppose that $M \models \varphi(a_0, \ldots, a_n)$ for some $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_n)$ with variables $v_0, \ldots, v_n$. Then

$$\{\alpha < \kappa \mid M \models \varphi(const_{a_0}(\alpha), \ldots, const_{a_n}(\alpha))\} = \kappa \in U$$

and by Łoś Theorem

$$Ult(M, U) \models \varphi(i(a_0), \ldots, i(a_n))$$

since $i(a_k) = [const_{a_k}]$ for each $k \leq n$.

(iii). We claim that $i(\alpha) = \alpha$ for each $\alpha < \kappa$. Suppose not and fix $\gamma < \kappa$ minimal with $\gamma < i(\gamma)$. Let f be such that $\gamma = [f]$. Then

$$A := \{\alpha < \kappa \mid f(\alpha) < \gamma\} = \{\alpha < \kappa \mid f(\alpha) < const_\gamma(\alpha)\} \in U.$$

This implies that

$$\bigcap_{\beta < \gamma} \kappa \setminus \{\alpha < \kappa \mid f(\alpha) = \beta\} = \kappa \setminus A \notin U.$$

So by M-κ-completeness, there is some $\beta < \gamma$ with $\{\alpha < \kappa \mid f(\alpha) = \beta\} \in U$. Hence $\gamma = [f] = i(\beta) = \beta$ by the minimality of γ and this is a contradiction.

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[2] Here, we need really need the Axiom of Collection rather than the Axiom of Replacement. Read more about this issue in [GHJ15].

For each $\alpha < \kappa$ we have

$$\{x < \kappa \mid const_\alpha(x) < id(x)\} = (\alpha, \kappa) \in U$$

by Lemma 2.1.6. So $\alpha = i(\alpha) < [id]$ for each $\alpha < \kappa$ and therefore $\kappa \leq [id]$. Since $\{\alpha < \kappa \mid id(\alpha) < const_\kappa(\alpha)\} = \kappa \in U$, $[id] < i(\kappa)$. Hence $\kappa < i(\kappa)$.

(iv). $i(f)([id]) = [const_f]([id]) = [f \circ id] = [f]$.

(v). We have

$$A = \{\alpha < \kappa \mid \alpha \in A\} = \{\alpha < \kappa \mid id(\alpha) \in const_A(\alpha)\}.$$

So by Łoś Theorem,

$$A \in U \Leftrightarrow Ult(M, U) \models [id] \in [const_A] = i(A).$$

(vi). "$\supseteq$" Let $\beta < \kappa$. From (iii), we know that $\beta = i(\beta) = [const_\beta]$. So

$$\{\alpha < \kappa \mid const_\beta(\alpha) \in id(\alpha)\} = \{\alpha < \kappa \mid \beta < \alpha\} = (\beta, \kappa) \in U$$

by 2.1.6. Then $\beta \in [id]$.
"$\subseteq$" Let $[f] \in [id]$. Then $\{\alpha < \kappa \mid f(\alpha) < id(\alpha)\} \in U$. Since we assume that U is M-normal, there is $\beta < \kappa$ such that

$$U \ni \{\alpha < \kappa \mid f(\alpha) = \beta\} = \{\alpha < \kappa \mid f(\alpha) = const_\beta(\alpha)\}.$$

So by Łoś Theorem, we have $[f] = i(\beta) = \beta \in \kappa$.

(vii). We have

$$\langle i(f_\alpha) \mid \alpha < \kappa\rangle = i(\langle f_\alpha \mid \alpha < \kappa\rangle) \upharpoonright \kappa \in Ult(M, U)$$

and $[id] \in Ult(M, U)$. So by (iv), we have

$$\langle [f_\alpha] \mid \alpha < \kappa\rangle = \langle i(f_\alpha)([id]) \mid \alpha < \kappa\rangle \in Ult(M, U).$$

□

Corollary 2.1.8. *If U is M-normal and M-κ-complete then the ultrafilter derived from i^M_U is U itself.*

Proof. Let $A \in \mathcal{P}(\kappa)^M$. We saw that $crit(i^M_U) = [id]$ in Lemma 2.1.7 and by definition $i_U^M(A) = [const_A]$. Therefore

$$A \in U_{i_U^M} \text{ iff } [id] \in [const_A] \text{ iff } \{\alpha < \kappa \mid id(\alpha) \in const_A(\alpha)\} \in U$$

and $\{\alpha < \kappa \mid id(\alpha) \in const_A(\alpha)\} = A$. ☐

Let $j : M \to N$ be elementary. We cannot expect to get j back from U_j, but we can capture part of the information.

Lemma 2.1.9. *Let $j : M \to N$ be an elementary embedding with $crit(j) = \kappa$. Then $k : Ult(M, U_j) \to N, [f] \mapsto j(f)(\kappa)$ is an elementary embedding such that*

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[dr, "i"'] & N \\ & Ult(M, U_j) \arrow[u, "k"'] \end{tikzcd}$$

commutes for $i := i_{U_j}^M$ and $k \restriction \mathcal{P}(\kappa)^{Ult(M,U_j)} = id$.

Proof. k is welldefined: Let $f, g \in M$ with domain κ. Then

$$\begin{aligned} [f] = [g] &\Leftrightarrow \{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in U_j \\ &\Leftrightarrow \kappa \in j(\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\}) = \{\alpha < j(\kappa) \mid j(f)(\alpha) = j(g)(\alpha)\} \\ &\Leftrightarrow j(f)(\kappa) = j(g)(\kappa). \end{aligned}$$

So k does not depend on the representatives of the equivalence classes.
k is elementary: Let φ be a formula and assume that φ has only one free variable for simplicity. Let $[f] \in Ult(M, U_j)$. Then

$$\begin{aligned} &Ult(M, U_j) \models \varphi([f]) \\ &\Leftrightarrow \{\alpha < \kappa \mid M \models \varphi(f(\alpha))\} \in U_j \\ &\Leftrightarrow \kappa \in j(\{\alpha < \kappa \mid M \models \varphi(f(\alpha))\}) = \{\alpha < j(\kappa) \mid N \models \varphi(j(f)(\alpha))\} \\ &\Leftrightarrow N \models \varphi(j(f)(\kappa)) \Leftrightarrow N \models \varphi(k([f])). \end{aligned}$$

The diagram commutes: For $a \in M$, we have that

$$k(i(a)) = k([const_a]) = j(const_a)(\kappa) = const_{j(a)}(\kappa) = j(a).$$

$k \restriction \mathcal{P}(\kappa)^{Ult(M,U_j)} = id$: U_j is M-κ-complete by Lemma 2.1.3 hence $crit(i) = \kappa$ by Lemma 2.1.7. $crit(j) = \kappa$ so $crit(k) \geq \kappa$. U_j is M-normal therefore

$$k(\kappa) = k([id]) = j(id)(\kappa) = \kappa.$$

Let $A \in \mathcal{P}(\kappa)^{Ult(M,U_j)}$. Then

$$Ult(M,U_j) \models A \subseteq \kappa \text{ iff } N \models k(A) \subseteq k(\kappa) = \kappa$$

by the elementarity of k. Now $\alpha \in A$ iff $\alpha = k(\alpha) \in k(A)$ for each $\alpha < \kappa$. Hence $k(A) = A$. □

Corollary 2.1.10. *Let $j : M \to N$ be elementary with $\kappa := crit(j)$. Then $Ult(M,U_j)$ is wellfounded.*

Proof. Follows directly from the existence of the elementary embedding

$$k : Ult(M,U_j) \to N$$

which we discussed in Lemma 2.1.9 and the wellfoundedness of N. □

Definition 2.1.11. U is **ω_1-complete** iff $A_n \in U$ for each $n < \omega$ implies $\bigcap_{n<\omega} A_n \neq \emptyset$.

Remark 2.1.12. Note that ω_1-completeness is the same as V-ω_1-completeness. Clearly V-ω_1-completeness implies ω_1-completeness. Assume that U is ω_1-complete. Let $\langle A_\alpha \mid \alpha < \beta \rangle \in V$, where $\beta < \omega_1$, and $A_\alpha \in U$ for each $\alpha < \beta$. Then there is a surjection $\varphi : \omega \to \beta$. Set $A'_n := A_{\varphi(n)}$ for each $n < \omega$. We have that $\bigcap_{\alpha<\beta} A_\alpha = \bigcap_{n<\omega} A_{\varphi(n)} = \bigcap_{n<\omega} A'_n \neq \emptyset$ since we assumed that U is ω_1-complete.

Lemma 2.1.13. *If U is ω_1-complete then $Ult(M,U)$ is wellfounded.*

Proof. Suppose that $Ult(M,U)$ is not wellfounded. Then there are $[f_n] \in Ult(M,U)$ for each $n < \omega$ such that $[f_n] \ni [f_{n+1}]$ for each $n < \omega$. So we have that $A_n := \{\alpha < \kappa \mid f_n(\alpha) \ni f_{n+1}(\alpha)\} \in U$ for each $n < \omega$. By the ω_1-completeness there is an $\alpha \in \bigcap_{n<\omega} A_n$. But then

$$f_0(\alpha) \ni \dots \ni f_n(\alpha) \ni f_{n+1}(\alpha) \ni \dots$$

contradicts the Axiom of Foundation in V. □

Lemma 2.1.14. *Assume $U \in M$. Then the following are equivalent:*

(i). U is ω_1-complete and

(ii). for every countable, transitive N and every elementary map $\pi : N \to M$ with $\pi(W) = U$ there is an elementary embedding $\sigma : Ult(N,W) \to M$

such that

$$\begin{tikzcd}
M & \\
N \arrow[u, "\pi"] \arrow[r, "i_W^N"'] & Ult(N,W) \arrow[ul, "\sigma"']
\end{tikzcd}$$

*commutes. The map σ is called **π-realization**.*

Proof. "$(i) \Rightarrow (ii)$": Fix some N, W and π as in the statement. Note that i_W^N is meaningful because W is a nonprincipal ultrafilter on N by the elementarity of π. Set $\kappa' := crit(W)$. Then $\kappa = \pi(\kappa')$. $\bigcap_{B \in W} \pi(B)$ is a countable intersection since N is transitive and countable. So by the ω_1-completeness of U, there is some $\gamma \in \bigcap_{B \in W} \pi(B)$. Define

$$\sigma : Ult(N,W) \to M, [f] \mapsto \pi(f)(\gamma).$$

This is welldefined since $[f] = [g]$ implies

$$B := \{\alpha < \kappa' \mid f(\alpha) = g(\alpha)\} \in W.$$

Hence $\pi(f)(\gamma) = \pi(g)(\gamma)$ because $\gamma \in \pi(B)$. In order to show that σ is elementary, let $[f_1], \dots, [f_n] \in Ult(N,W)$ and let φ be an $\mathcal{L}_\in$-formula such that

$$Ult(N,W) \models \varphi([f_1], \dots, [f_n]).$$

$$\text{Then } B := \{\alpha < \kappa' \mid N \models \varphi(f_1(\alpha), \dots, f_n(\alpha))\} \in W.$$

Since $\gamma \in \pi(B)$ and $\sigma([f_l]) = \pi(f_l)(\gamma)$ for $l = 1, \dots, n$, we have that

$$M \models \varphi(\sigma([f_1]), \dots, \sigma([f_n])).$$

The diagram commutes since every $x \in N$ satisfies

$$\sigma(i_W^N(x)) = \sigma([const_x]) = \pi(const_x)(\gamma) = \pi(x).$$

"$(ii) \Rightarrow (i)$": Let $A_n \in U$ for every $n < \omega$. Set

$$N := mos(Hull_M(\{U\} \cup \{A_n \mid n < \omega\}))$$

and let $\pi : N \to M$ be the anti-collapse embedding[3][4]. Find $W, \kappa', B_n \in N$ such

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[3] $Hull_M(\{U\} \cup \{A_n \mid n < \omega\})$ is the Skolem Hull of $\{U\} \cup \{A_n \mid n < \omega\}$ in M which is a countable elementary submodel of M.

[4] *mos* stands for Mostowski collapse which turns a wellfounded model into an isomorphic

that $\pi(W) = U$, $\pi(\kappa') = \kappa$ and $\pi(B_n) = A_n$ for each $n < \omega$. Then N and W satisfy the conditions of (ii). So there is an elementary map

$$\sigma : Ult(N, W) \to M \text{ such that } \sigma \circ i^N_W = \pi.$$

We claim that $\sigma([id]) \in \pi(B)$ for every $B \in W$. Then, in particular,

$$\sigma([id]) \in \pi(B_n) = A_n \text{ for each } n < \omega$$

and therefore $\bigcap_{n<\omega} A_n \neq \emptyset$. Let χ_B be the characteristic function of B as a subset of κ' in N and $\chi_{\pi(B)}$ the one for $\pi(B) \subseteq \kappa$ in M. Observe that $\chi_{\pi(B)} = \pi(\chi_B) = \sigma(i^N_W(\chi_B))$ since π is elementary and the diagram commutes. Now we have

$$\begin{aligned}\chi_{\pi(B)}(\sigma([id])) &= \sigma(i^N_W(\chi_B))(\sigma([id])) \text{ by the observation}\\ &= \sigma\big(i^N_W(\chi_B)([id])\big) \text{ since } \sigma \text{ is elementary}\\ &= \sigma([\chi_B]) \text{ by 2.1.7(iv)}\\ &= \sigma([const_1]) \text{ since } [\chi_B] = [const_1] \text{ is witnessed by } B \in W\\ &= \sigma(i^N_W(1)) = \pi(1) = 1 \text{ since the diagram commutes.}\end{aligned}$$

This proves the claim and concludes the proof. ☐

A proof of the next lemma can be found in Chapter 10 in [Jec03].

Lemma. *Every measurable cardinal is inaccessible.*

Lemma 2.1.15. *Let U be M-κ-complete. Then $U \notin Ult(M, U)$.*

Proof. U witnesses that κ is measurable in M. Since $i := i^M_U$ is elementary we have that $i(\kappa)$ is measurable and therefore inaccessible in $Ult(M, U)$. If $U \in Ult(M, U)$ then also the function

$$\Phi : \kappa^\kappa \to Ult(M, U), f \mapsto [f]$$

is an element of $Ult(M, U)$. We claim that $im(\Phi) = i(\kappa)$.
"$\subseteq$": For $f \in \kappa^\kappa$ we have that

$$\{\alpha < \kappa \mid f(\alpha) < const_\kappa(\alpha)\} = \{\alpha < \kappa \mid f(\alpha) < \kappa\} = \kappa \in U$$

so $[f] \in i(\kappa)$.
"$\supseteq$": Let $[g] \in i(\kappa)$. Then $A := \{\alpha < \kappa \mid g(\alpha) \in const_\kappa(\alpha)\} \in U$. Define $f \in \kappa^\kappa$

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transitive model.

by

$$f(\alpha) := \begin{cases} g(\alpha), & \alpha \in A \\ 0, & \alpha \in \ \kappa \setminus A. \end{cases}$$

Then $[g] = [f]$ since $\{\alpha < \kappa \mid g(\alpha) = f(\alpha)\} \supseteq A \in U$. So $[g] \in im(\Phi)$.

But now $Ult(M, U) \models \kappa^\kappa \geq im(\Phi) = i(\kappa) > \kappa$, which is a contradiction to $i(\kappa)$ being inaccessible in $Ult(M, U)$. □

2.2 Iterated Ultrapowers

In Chapter 2.1 we discussed how to extend a model by an ultrafilter. Since this extension is a model itself, we can use the same method again and so on. But we saw in the first chapter that the ultrafilter U is never in $Ult(M, U)$, so we can't use the same ultrafilter again.

Definition 2.2.1. Let $Y \subseteq M$. We say that (M, Y) is an **amenable structure** iff $x \cap Y \in M$ for each $x \in M$. An M**-nuf** U is a nonprincipal ultrafilter for M which is M-normal and M-$crit(U)$-complete. A pair $(M, \mathcal{U})$ is called a **good pair** iff $(M, \mathcal{U})$ is an amenable structure and every element of $\mathcal{U}$ is an M-nuf.

Definition 2.2.2. Let U be an M-nuf and let (M, Y) be an amenable structure. The **ultrapower of** (M, Y) **by** U is defined as

$$Ult((M, Y), U) := (Ult(M, U), Y_U),$$

$$\text{where } [f]_U \in Y_U \text{ iff } \{\alpha < \kappa \mid f(\alpha) \in Y\} \in U$$

for each function $f \in M$ with $dom(f) = \kappa$.

Remark 2.2.3. Note that asking for $\{\alpha < \kappa \mid f(\alpha) \in Y\} \in U$ in Definition 2.2.2 makes sense since

$$\{\alpha < \kappa \mid f(\alpha) \in Y\} = f^{-1}[ran(f) \cap Y].$$

$f^{-1}[ran(f) \cap Y] \in \mathcal{P}(\kappa)^M$ since $f \in M$ and $ran(f) \cap Y \in M$ because (M, Y) is an amenable structure.

Lemma 2.2.4. *Let U be an M-nuf and let (M, Y) be an amenable structure. Assume that $(Ult(M, U), \tilde{\in})$ is wellfounded. Then $Ult((M, Y), U) = (Ult(M, U), Y_U)$ is an amenable structure.*

Proof. Fix some $x \in Ult(M, U)$. We need to show that $x \cap Y_U \in Ult(M, U)$. Set $i := i_U^M$.

We claim that there is a transitive $y \in M$ such that $x \in i(y)$. Pick any $f \in M$ with $[f] = x$ and set $y := trcl(\{ran(f)\})$. This is transitive by definition and

$y \in M$. We have that $\{\alpha < \kappa \mid M \models f(\alpha) \in y\} = \kappa \in U$. So by Łoś Theorem, $Ult(M, U) \models [f] \in [const_y] = i(y)$.

(M, Y) is amenable by the assumption, hence there is some $u \in M$ such that $u = y \cap Y$. This implies that

$$i(u) = i(y) \cap Y$$

because $i : (M, Y) \to (Ult(M, U), Y_U)$ is elementary. Note that $i(y)$ is transitive since y is transitive, so $x \in i(y)$ implies that $x \subseteq i(y)$. We compute that

$$x \cap Y_U = \big(x \cap i(y)\big) \cap Y_U = x \cap \big(i(y) \cap Y_U\big) = x \cap i(u).$$

Hence $x \cap Y_U \in Ult(M, U)$. $\square$

From now on, let $(M, \mathcal{U})$ be a good pair.

Lemma 2.2.5. *Let U be an M-nuf and assume that $(Ult(M, U), \tilde{\in})$ is well-founded. Then $Ult((M, \mathcal{U}), U)$ is a good pair.*

Proof. We know that $Ult((M, \mathcal{U}), U)$ is an amenable structure from Lemma 2.2.4. We need to show that every element of $\mathcal{U}_U$ is an $Ult(M, U)$-nuf. Let $[W]_U \in \mathcal{U}_U$. Then $\{\alpha < \kappa \mid W(\alpha) \in \mathcal{U}\} \in U$. Since every element of $\mathcal{U}$ is an M-nuf, we have that $W(\alpha)$ is an M-nuf for U-a.e. α. Then Łoś Theorem implies that $[W]_U$ is an $Ult(M, U)$-nuf. $\square$

Definition 2.2.6. Let β be some ordinal. We call $I = \langle U_\alpha \mid \alpha < \beta \rangle$ a **linear iteration of** $(M, \mathcal{U})$ **of length** β iff there are $\langle M_\alpha, \mathcal{U}_\alpha \mid \alpha < \beta \rangle$ and elementary embeddings $\langle i_{\alpha,\gamma} : M_\alpha \to M_\gamma \mid \alpha < \gamma < \beta \rangle$ such that

(i). $M_0 = M$ and $\mathcal{U}_0 = \mathcal{U}$,

(ii). for each $\alpha < \beta$:
M_α is a transitive model of ZFC^- and $U_\alpha \in \mathcal{U}_\alpha$,

(iii). for successors $\alpha + 1 < \beta$:
$(M_{\alpha+1}, \mathcal{U}_{\alpha+1}) = Ult((M_\alpha, \mathcal{U}_\alpha), U_\alpha)$,
$i_{\alpha,\alpha+1} = i^{M_\alpha}_{U_\alpha}$ and
$i_{\gamma,\alpha+1} = i_{\alpha,\alpha+1} \circ i_{\gamma,\alpha}$ for each $\gamma < \alpha$,

(iv). for limit ordinals $\lambda < \beta$:
$(M_\lambda, \mathcal{U}_\lambda)$ is the direct limit of $\langle (M_\alpha, \mathcal{U}_\alpha), i_{\alpha,\gamma} \mid \alpha < \gamma < \lambda \rangle$ and $i_{\alpha,\lambda}$ are the direct limit embeddings.

We also write $U^I_\alpha, M^I_\alpha, i^I_{\alpha,\gamma}$ and κ^I_α for $U_\alpha, M_\alpha, i_{\alpha,\gamma}$ and $crit(U_\alpha)$. We associate the "last model" M^I_∞. If β is a limit ordinal, then

M^I_∞ is defined as the direct limit of $\langle M^I_\alpha, i^I_{\alpha,\gamma} \mid \alpha < \gamma < \beta\rangle$
with direct limit embeddings $i^I_{\alpha,\infty}$ for each $\alpha < \beta$.

If $\beta = \beta' + 1$, then $M^I_\infty := M^I_{\beta'}$.

Remark 2.2.7.

- $(M,\mathcal{U})$ is a good pair and $U_0 \in \mathcal{U}$. In particular, U_0 is an M-nuf. Lemma 2.2.5 yields that $(M_1,\mathcal{U}_1)$ is a good pair. Therefore $U_1 \in \mathcal{U}_1$ is an M_1-nuf and $(M_2,\mathcal{U}_2) = Ult((M_1,\mathcal{U}_1), U_1)$ is welldefined. The same applies to every successor step.
- We will sometimes suppress the $\mathcal{U}_\alpha$'s in favor of the readability.
- The models $\langle M_\alpha \mid \alpha < \beta\rangle$ and elementary embeddings $\langle i_{\alpha,\gamma} \mid \alpha < \gamma < \beta\rangle$ from Definition 2.2.6 are unique. Therefore we call them **the models and the elementary embeddings of** I.
- If $\beta \leq \omega_1$, then every M_α with $\alpha < \beta$ has the same cardinality as M.

Definition 2.2.8. Let θ be an ordinal. We call $(M,\mathcal{U})$ **<θ-linearly iterable**[5] iff M^I_∞ is wellfounded for every linear iteration I of $(M,\mathcal{U})$ of length less than θ. If $(M,\mathcal{U})$ is <θ-linearly iterable for every ordinal θ then we call $(M,\mathcal{U})$ **linearly iterable**.

The goal for the rest of this chapter is the following theorem:

Theorem 2.2.9. *If every $U \in \mathcal{U}$ is ω_1-complete then $(M,\mathcal{U})$ is linearly iterable.*

The proof uses a characterization of linear iterability which is discussed in Lemma 2.2.12. The following lemma is important for the proof of Lemma 2.2.12. It shows how to pull back linear iterability through an elementary embedding.

Definition 2.2.10. Let $(M,\mathcal{U})$ and $(N,\mathcal{W})$ be amenable structures. We write **"$\pi : (N,\mathcal{W}) \to (M,\mathcal{U})$ is an elementary embedding "** iff $\pi : N \to M$ is an elementary embedding and in addition $x \in \mathcal{W} \Leftrightarrow \pi(x) \in \mathcal{U}$ for each $x \in N$.

Lemma 2.2.11. *(Pull Back Linear Iterability) Let $(M,\mathcal{U})$ be <θ-linearly iterable for some ordinal θ. Let $(N,\mathcal{W})$ be a good pair and let $\pi : (N,\mathcal{W}) \to (M,\mathcal{U})$ be an elementary map. Then $(N,\mathcal{W})$ is <θ-linearly iterable, too.*

Proof. Let $\beta < \theta$ and fix some linear iteration $J = \langle W_\alpha \mid \alpha < \beta\rangle$ of $(N,\mathcal{W})$ with models $\langle N_\alpha \mid \alpha < \beta\rangle$ and embeddings $\langle j_{\alpha,\gamma} \mid \alpha < \gamma < \beta\rangle$. We recursively

---

[5] Usually this is called θ-linearly iterable but I prefer this version because it is less ambiguous.

construct good pairs $(M_\alpha,\mathcal{U}_\alpha)$ and elementary maps $\pi_\alpha : (N_\alpha, \mathcal{W}_\alpha) \to (M_\alpha,\mathcal{U}_\alpha)$ for each $\alpha < \beta$ such that

- $I := \langle U_\alpha \mid \alpha < \beta \rangle$ is a linear iteration of $(M,\mathcal{U})$, where $U_\alpha := \pi_\alpha(W_\alpha)$ for each $\alpha < \beta$, and

- $\pi_{\alpha+1} \circ j_{\alpha,\alpha+1} = i_{\alpha,\alpha+1} \circ \pi_\alpha$ for each α with $\alpha + 1 < \beta$.

Assume that we already constructed π_α for each $\alpha < \beta$. We define an elementary map $\pi_\infty : N^J_\infty \to M^I_\infty$. If $\beta = \beta' + 1$ for some ordinal β' then π_∞ is constructed as in the successor step below as $\pi_{\beta'+1}$. If β is a limit ordinal then π_∞ is constructed as in the limit step below. In both cases π_∞ is an elementary map. We know that M_∞ is wellfounded because $(M,\mathcal{U})$ is $<\theta$-linearly iterable. Hence N_∞ is wellfounded, too. We have the following diagram:

$$\begin{tikzcd}
M \arrow[r, "i_{0,1}"] & M_1 \arrow[r, "i_{1,2}"] & \dots \arrow[r] & M_\alpha \arrow[r, "i_{\alpha,\alpha+1}"] & M_{\alpha+1} \arrow[r] & \dots \arrow[r] & M_\infty \\
N \arrow[u, "\pi"] \arrow[r, "j_{0,1}"'] & N_1 \arrow[u, "\pi_1"] \arrow[r, "j_{1,2}"'] & \dots \arrow[r] & N_\alpha \arrow[u, "\pi_\alpha"] \arrow[r, "j_{\alpha,\alpha+1}"'] & N_{\alpha+1} \arrow[u, "\pi_{\alpha+1}"] \arrow[r] & \dots \arrow[r] & N_\infty \arrow[u, "\pi_\infty"]
\end{tikzcd}$$

Let's build the recursion.

$\alpha = 0$:

$$\text{Set } (M_0,\mathcal{U}_0) := (M,\mathcal{U})$$

$$\text{and } \pi_0 := \pi.$$

$\alpha + 1 < \beta$:

$$\text{Set } (M_{\alpha+1},\mathcal{U}_{\alpha+1}) := Ult((M_\alpha,\mathcal{U}_\alpha), U_\alpha)$$

$$\text{and define } \pi_{\alpha+1} : N_{\alpha+1} \to M_{\alpha+1} \text{ by } [f]_{W_\alpha} \mapsto [\pi_\alpha(f)]_{U_\alpha}.$$

We need to show that this is welldefined. Pick $f, g \in N_\alpha$ such that $[f]_{W_\alpha} = [g]_{W_\alpha}$, i.e. $\{x < \kappa^J_\alpha \mid f(x) = g(x)\} \in W_\alpha$. The elementarity of π_α implies that

$$\begin{aligned}
&\{x < \kappa^I_\alpha \mid \pi_\alpha(f)(x) = \pi_\alpha(g)(x)\} \\
&= \pi_\alpha\big(\{x < \kappa^J_\alpha \mid f(x) = g(x)\}\big) \in \pi_\alpha(W_\alpha) = U_\alpha.
\end{aligned}$$

Hence $[\pi_\alpha(f)]_{U_\alpha} = [\pi_\alpha(g)]_{U_\alpha}$. The elementarity of $\pi_{\alpha+1}$ follows from Łoś Theorem and the elementarity of π_α. We claim that $\pi_{\alpha+1} : (N_{\alpha+1}, \mathcal{W}_{\alpha+1}) \to$

$(M_{\alpha+1},\mathcal{U}_{\alpha+1})$ is elementary. In order to show that, pick $[g]_{W_\alpha} \in N_{\alpha+1}$. Then

$$\begin{aligned}
[g]_{W_\alpha} \in \mathcal{W}_{\alpha+1} = (\mathcal{W}_\alpha)_{W_\alpha} & \\
&\Leftrightarrow \{x < \kappa^J_\alpha \mid g(x) \in \mathcal{W}_\alpha\} \in W_\alpha \\
&\Leftrightarrow \pi_\alpha(\{x < \kappa^J_\alpha \mid g(x) \in \mathcal{W}_\alpha\}) \in \pi_\alpha(W_\alpha) \\
&\Leftrightarrow \{x < \kappa^I_\alpha \mid \pi_\alpha(g)(x) \in \mathcal{U}_\alpha\} \in U_\alpha \\
&\Leftrightarrow \pi_{\alpha+1}([g]_{W_\alpha}) = [\pi_\alpha(g)]_{U_\alpha} \in (\mathcal{U}_\alpha)_{U_\alpha} = \mathcal{U}_{\alpha+1}.
\end{aligned}$$

The definition of $\pi_{\alpha+1}$ directly implies that $\pi_{\alpha+1} \circ j_{\alpha,\alpha+1} = i_{\alpha,\alpha+1} \circ \pi_\alpha$.
$\underline{\gamma \text{ limit ordinal:}}$ Let $(M_\gamma,\mathcal{U}_\gamma)$ be the direct limit of $\langle (M_\alpha,\mathcal{U}_\alpha) \mid \alpha < \gamma\rangle$. For $x \in N_\gamma$, there is some $\alpha < \gamma$ and $x' \in N_\alpha$ such that $j_{\alpha,\gamma}(x') = x$. Set

$$\pi_\gamma(x) := i_{\alpha,\gamma}(\pi_\alpha(x')).$$

It is easy to check that $(M_\gamma,\mathcal{U}_\gamma)$ that $\pi_\gamma : (N_\gamma,\mathcal{W}_\gamma) \to (M_\gamma,\mathcal{U}_\gamma)$ is an elementary map which commutes with the $j's$ and $i's$. ☐

Lemma 2.2.12. *(Characterization of Linear Iterability) The following are equivalent*

- *(i). $(M,\mathcal{U})$ is $<\omega_1$-linearly iterable.*
- *(ii). If $(N,\mathcal{W})$ is a good pair, N is countable and transitive and $\pi : (N,\mathcal{W}) \to (M,\mathcal{U})$ is an elementary map. Then $(N,\mathcal{W})$ is $<\omega_1$-linearly iterable.*
- *(iii). $(M,\mathcal{U})$ is linearly iterable.*

Proof. "$(i) \Rightarrow (ii)$" is a special case of Lemma 2.2.11.
"$(ii) \Rightarrow (iii)$" Suppose that $(M,\mathcal{U})$ is not linearly iterable. Fix some linear iteration I such that M^I_∞ is illfounded. By the Reflection Principle, there is an ordinal θ such that $V_\theta \models$ "M^I_∞ is illfounded". Set

$$H := mos(Hull_{V_\theta}(\{M,\mathcal{U},I\}))$$

and let $\sigma : H \to V_\theta$ be the anti-collapsing embedding. Then there are $N,\mathcal{W},J \in H$ such that $\sigma(N) = M, \sigma(\mathcal{W}) = \mathcal{U}$ and $\sigma(J) = I$. Since H is countable and transitive, we know that N is countable and transitive, J is of length $< \omega_1$ and $\pi := \sigma \restriction (N,\mathcal{W}) : (N,\mathcal{W}) \to (M,\mathcal{U})$ is a welldefined elementary embedding. So by the assumption, $(N,\mathcal{W})$ is $<\omega_1$-linearly iterable and N^J_∞ is wellfounded. But on the other hand, the elementarity of σ implies

$$H \models \text{“}N^J_\infty \text{ is illfounded”}.$$

"N^J_∞ is illfounded" is a Σ_1 statement and therefore upwards absolute. Hence we have that N^J_∞ is illfounded which is a contradiction to the above.
"$(iii) \Rightarrow (i)$" follows from the definition. $\square$

We have all the ingredients to prove Theorem 2.2.9.

Proof of Theorem 2.2.9. Fix some good pair $(N, \mathcal{W})$ with N countable and transitive and an elementary map $\pi : (N, \mathcal{W}) \to (M, \mathcal{U})$. By Lemma 2.2.12, it is enough to show that $(N, \mathcal{W})$ is $<\!\omega_1$-linearly iterable. Fix a countable linear iteration $J = \langle W_\alpha \mid \alpha < \beta \rangle$ for $\beta < \omega_1$ on $(N, \mathcal{W})$ with associated models $\langle N_\alpha \mid \alpha < \beta \rangle$ and associated embeddings $\langle j_{\alpha,\gamma} \mid \alpha < \gamma < \beta \rangle$. We need to show that N_∞ is wellfounded. We recursively define elementary embeddings

$$\langle \pi_\alpha : (N_\alpha, \mathcal{W}_\alpha) \to (M, \mathcal{U}) \mid \alpha < \beta \rangle$$

and $\pi_\infty : N_\infty \to M$ such that the following diagram commutes:

$$\begin{tikzcd}
M & & & & & & \\
N \arrow[u, "\pi"] \arrow[r, "j_{0,1}"'] & N_1 \arrow[ul, "\pi_1"] \arrow[r] & \dots \arrow[r] & N_\alpha \arrow[ulll, "\pi_\alpha"] \arrow[r, "j_{\alpha,\alpha+1}"'] & N_{\alpha+1} \arrow[ullll, "\pi_{\alpha+1}"] \arrow[r] & \dots \arrow[r] & N_\infty \arrow[ullllll, "\pi_\infty"]
\end{tikzcd}$$

Then N_∞ embeds into the wellfounded model M. Hence N_∞ is wellfounded.
$\underline{\alpha = 0}$: Set $\pi_0 := \pi$.
$\underline{\alpha \to \alpha + 1}$: N_α is countable by Remark 2.2.7. $W_\alpha \in \mathcal{W}_\alpha$ hence $\pi_\alpha(W_\alpha) \in \mathcal{U}$, so by assumption, $\pi_\alpha(W_\alpha)$ is ω_1-complete. Set $\pi_{\alpha+1}$ to be the π_α-realization described in Lemma 2.1.14, i.e.

$$\pi_{\alpha+1} : Ult(N_\alpha, W_\alpha) \to M, [f]^{N_\alpha}_{W_\alpha} \mapsto \pi_\alpha(f)(\gamma),$$

where γ is any element of $\bigcap_{B \in W_\alpha} \pi_\alpha(B)$. We claim that

$$\pi_{\alpha+1} : (N_{\alpha+1}, \mathcal{W}_{\alpha+1}) \to (M, \mathcal{U}) \text{ is elementary.}$$

We have that

$$
\begin{aligned}
[f]^{N_\alpha}_{W_\alpha} &\in \mathcal{W}_{\alpha+1} = (\mathcal{W}_\alpha)_{W_\alpha} \\
&\Leftrightarrow B := \{x < crit(W_\alpha) \mid f(x) \in \mathcal{W}_\alpha\} \in W_\alpha \\
&\Leftrightarrow \gamma \in \pi_\alpha(B) = \{x < \pi_\alpha(crit(W_\alpha)) \mid \pi_\alpha(f)(x) \in \mathcal{U}\} \\
&\Leftrightarrow \pi_{\alpha+1}([f]^{N_\alpha}_{W_\alpha}) = \pi_\alpha(f)(\gamma) \in \mathcal{U}.
\end{aligned}
$$

γ limit ordinal: We define $\pi_\gamma : N_\gamma \to M$ as follows. For $x \in N_\gamma$, there is some $\alpha < \gamma$ and $y \in N_\alpha$ such that $x = j_{\alpha,\gamma}(y)$. Set $\pi_\gamma(x) := \pi_\alpha(y)$. It is easy to check that $\pi_\gamma : (N_\gamma, \mathcal{W}_\gamma) \to (M, \mathcal{U})$ is elementary.

If $\beta = \beta' + 1$ is a successor ordinal, then define π_∞ as in the successor case. Otherwise β is a limit ordinal and we define π_∞ as in the limit case. □

2.3 Extenders and Ultrapowers from Extenders

We want to generalize the idea of extending a model via an ultrafilter from Chapter 2.1. We are going to use extenders which allow us to extend a model by a lot of ultrafilters at the same time. From now on, let λ be an ordinal.

Definition 2.3.1. For a set of ordinals a, we denote the i-th smallest element of a by a_i. Let $n < m$, $a \in [\lambda]^n, b \in [\lambda]^m$ with $b = \{b_1, \dots, b_m\}$ and $a = \{b_{i_1}, \dots, b_{i_n}\}$. For $X \subseteq [\kappa]^n$, set

$$X^{ab} := \{u \in [\kappa]^m \mid \{u_{i_1}, \dots, u_{i_n}\} \in X\} \subseteq [\kappa]^m.$$

If $f : [\kappa]^n \to M$, set

$$f^{ab} : [\kappa]^m \to M, u \mapsto f(\{u_{i_1}, \dots, u_{i_n}\}).$$

Definition 2.3.2. ((Pre-)Extender) A set $E \subseteq [\lambda]^{<\omega} \times \mathcal{P}([\kappa]^{<\omega})$ is called a (κ, λ)**-pre-extender over** M (or M**-pre-extender**) iff for every $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$

(i). The set $E_a := \{X \in \mathcal{P}([\kappa]^{<\omega}) \mid (a, X) \in E\}$ is an M-κ-complete ultrafilter on $[\kappa]^{|a|}$ for M.

(ii). (Compatibility) If $X \in M$ then $X \in E_a$ iff $X^{ab} \in E_b$.

(iii). (M-normality) If $f \in M$ with $dom(f) = [\kappa]^{|a|}$ and $f(u) < u_i$ for E_a-a.e. u. Then there is $\xi < a_i$ such that $f^{a, a\cup\{\xi\}}(u) = u_k$ for $E_{a\cup\{\xi\}}$-a.e. u, where k is such that $\xi = (a \cup \{\xi\})_k$.

We call $\kappa =: crit(E)$ the **critical point of** E and $\lambda =: lh(E)$ the **length of** E.

Remark 2.3.3. Note that the definition of an M-pre-extender only depends on $\mathcal{P}(\kappa)^M$. If $Q \models ZFC^-$ is transitive and $\mathcal{P}(\kappa)^Q = \mathcal{P}(\kappa)^M$. Then E is a Q-pre-extender iff E is an M-pre-extender.

Lemma 2.3.4. *Let $j : M \to N$ be an elementary embedding with $crit(j) = \kappa$ and $\lambda \leq j(\kappa)$. Set*

$$E(j, \lambda) := \{\langle a, X\rangle \mid a \in [\lambda]^{<\omega}, X \in [\kappa]^{|a|}, a \in j(X)\}.$$

*This is a (κ, λ)-pre-extender over M which we call the (κ, λ)****-pre-extender derived from*** *j.*

Proof. Let $a, b \in [\lambda]^{<\omega}$ with $a = \{b_{i_1}, \ldots, b_{i_n}\}$. $E(j, \lambda)_a$ is an M-κ-complete ultrafilter on $[\kappa]^{|a|}$ for M by the same arguments as in Lemma 2.1.3. In order to check that $E(j, \lambda)$ is compatible, let $X \in M$. Note that

$$j(X^{ab}) = \{u \in [j(\kappa)]^{|b|} \mid \{u_{i_1}, \ldots, u_{i_n}\} \in j(X)\}.$$

$$\begin{aligned} \text{So } X \in E(j,\lambda)_a &\Leftrightarrow a \in j(X) \\ &\Leftrightarrow b \in j(X^{ab}) \\ &\Leftrightarrow X^{ab} \in E(j,\lambda)_b. \end{aligned}$$

For M-normality, let $f \in M$ with $dom(f) = [\kappa]^{|a|}$ and $f(u) < u_i$ for E_a-a.e. u, i.e.

$$a \in j(\{u \in [\kappa]^{|a|} \mid f(u) < u_i\}) = \{u \in [j(\kappa)]^{|a|} \mid j(f)(u) < u_i\}.$$

Hence $j(f)(a) < a_i$. Set $\xi := j(f)(a)$ and k such that $\xi = (a \cup \{\xi\})_k$. Then

$$j(f^{a, a\cup\{\xi\}})(a \cup \{\xi\}) = j(f)^{a, a\cup\{\xi\}}(a \cup \{\xi\}) = j(f)(a) = \xi = (a \cup \{\xi\})_k.$$

Therefore $f^{a, a\cup\{\xi\}}(u) = u_k$ for $E_{a\cup\{\xi\}}$-a.e. u. □

Definition 2.3.5. Let E be a (κ, λ)-pre-extender over M. For functions $f, g \in M$ with $dom(f) = [\kappa]^{|a|}$ and $dom(g) = [\kappa]^{|b|}$, where $a, b \in [\lambda]^{<\omega}$, we define an equivalence relation by

$$\langle a, f\rangle \sim \langle b, g\rangle \text{ iff } f^{a, a\cup b}(u) = g^{b, a\cup b}(u) \text{ for } E_{a\cup b}\text{-a.e. } u$$

Denote the equivalence class of $\langle a, f\rangle$ by $[a, f]$ or $[a, f]^M_E$. We also define the relation $\tilde{\in}$ by

$$[a, f]\tilde{\in}[b, g] \text{ iff } f^{a, a\cup b}(u) \in g^{b, a\cup b}(u) \text{ for } E_{a\cup b}\text{-a.e. } u.$$

$$\text{Let } Ult(M,E) := (\{[a,f] \mid a \in [\lambda]^{<\omega}, f \text{ function in } M, dom(f) = [\kappa]^{|a|}\}, \tilde{\in})$$

$$\text{and } i_E^M : M \to Ult(M,E), x \mapsto [\{0\}, const_x^1]$$

where $const_x^n : [\kappa]^n \to M, u \mapsto x$ for each $n < \omega$. Again $(Ult(M,E), \tilde{\in})$ does not have to be wellfounded, but we collapse its wellfounded part. If $(Ult(M,E), \tilde{\in})$ is wellfounded then this yields a transitive model and we denote it by $(Ult(M,E), \in)$ and call it the **ultrapower of M by E.**

Lemma 2.3.6. *(Properties of $Ult(M,E)$) Let E be a (κ,λ)-pre-extender over M. Fix some $n < \omega$. Let $a, a^k \in [\lambda]^{<\omega}$ and f, f_k be functions in M with $dom(f) = [\kappa]^{|a|}$ and $dom(f_k) = [\kappa]^{|a^k|}$ for each $k \leq n$. Denote the identity function on $[\kappa]^n$ by id^n for each $n < \omega$ and $i := i_E^M$. Then the following properties hold:*

(i). Łoś Theorem: Let $\varphi(v_0, \dots, v_n)$ be a formula, $n < \omega$ and $b := \bigcup_{i \leq n} a^i$. Then

$$Ult(M,E) \models \varphi([a^0, f_0], \dots, [a^n, f_n]) \textit{ iff}$$
$$M \models \varphi(f_0^{a^0,b}(u), \dots, f_n^{a^n,b}(u)) \textit{ for } E_b\textit{-a.e. } u.$$

(ii). i is elementary. In particular, $Ult(M,E) \models ZFC^-$.

(iii). $\kappa = crit(i)$.

(iv). Let $\varepsilon : [\kappa]^1 \to \kappa, \{\beta\} \mapsto \beta$ and $\varepsilon_{n,i} : [\kappa]^n \to \kappa, u \mapsto u_i$ for each $i \leq n < \omega$. Then every β in the wellfounded part of $Ult(M,E)$ satisfies

  (a) $[\{\beta\}, \varepsilon] = \beta$ if $\beta < \lambda$,

  (b) $[a, \varepsilon_{|a|,i}] = a_i$ for $0 < i \leq |a|$ and

  (c) $a = [a, id^{|a|}]$.

(v). $[a,f] = i(f)(a)$.

Proof. (i). Works exactly the same as in Lemma 2.1.7.

(ii). Works exactly the same as in Lemma 2.1.7.

(iii). Works exactly the same as in Lemma 2.1.7.

(iv). (a) Define a function

$$F : \lambda \to Ult(M,E)$$
$$\beta \mapsto [\{\beta\}, \varepsilon].$$

This is an order preserving map. For $\beta_1 < \beta_2 < \lambda$, we have $\varepsilon^{\{\beta_i\},\{\beta_1,\beta_2\}}(u) = u_i$ for $i = 1, 2$ and every $u \in [\kappa]^2$. $u_1 < u_2$ holds by definition. By Łoś Theorem, $F(\beta_1) = [\{\beta_1\}, \varepsilon] \in [\{\beta_2\}, \varepsilon] = F(\beta_2)$. We claim that for every $\beta < \lambda$ and $[b, f] \in F(\beta)$ there is $\xi < \beta$ such that $[b, f] = F(\xi)$. We may assume that $\beta \in b$, say $\beta = b_i$. By definition, $\{u \in [\kappa]^{|b|} \mid f(u) < u_i\} = \{u \in [\kappa]^{|b|} \mid f(u) \in \varepsilon^{\{\beta\},b}(u)\} \in E_b$. By the M-normality of E, there is some $\xi < b_i = \beta$ such that $\{u \in [\kappa]^{|b\cup\{\xi\}|} \mid f^{b\cup\{\xi\}}(u) = u_k\} \in E_{b\cup\{\xi\}}$, where k is such that $(b \cup \{\xi\})_k = \xi$. Therefore $[b, f] = [\{\xi\}, \varepsilon] = F(\xi)$. The fact that F is order preserving and the claim imply that $F(\beta)$ is an ordinal for each $\beta < \lambda$. The claim also implies that $im(F)$ is an initial segment of the ordinals hence $im(F)$ is an ordinal itself. This shows that F is an order preserving bijection between λ and some ordinal. Hence $F = id_\lambda$.

(b) Fix $i \leq |a|$. We have $\varepsilon_{|a|,i} = \varepsilon^{\{a_i\},a}$. So by Łoś Theorem, $[a, \varepsilon_{|a|,i}] = [a, \varepsilon^{\{a_i\},a}] = [\{a_i\}, \varepsilon] = a_i$ by (iv)(a).

(c) We have $id^{|a|}(u) = \{u_1, \ldots, u_{|a|}\} = \{\varepsilon_{|a|,1}(u), \ldots, \varepsilon_{|a|,|a|}(u)\}$ for every $u \in [\kappa]^{|a|}$. Again by Łoś Theorem,

$$[a, id^{|a|}] = \{[a, \varepsilon_{|a|,1}], \ldots, [a, \varepsilon_{|a|,|a|}]\} = \{a_1, \ldots, a_{|a|}\} = a$$

using (iv)(b).

(v). $i(f)(a) = [\{0\}, const^1_f](a) = [a, const^{|a|}_f](a) = [a, const^{|a|}_f]([a, id^{|a|}]) = [a, f \circ id^{|a|}] = [a, f]$.

□

We can perfectly recover the extender from the ultrapower embedding.

Lemma 2.3.7. *Let E be a (κ, λ)-pre-extender over M. Then $E(i^M_E, \lambda) = E$.*

Proof. Let $a \in [\lambda]^{<\omega}$ and $X \in [\kappa]^{|a|}$.

$$\begin{aligned}
&\langle a, X\rangle \in E(i^M_E, \lambda)\\
&\Leftrightarrow a \in i^M_U(X) = [\{0\}, const^1_X] \text{ by the definition of } E(i^M_E, \lambda)\\
&\Leftrightarrow a \in [a, const^{|a|}_X] \text{ since } [\{0\}, const^1_X] = [a, const^{|a|}_X]\\
&\Leftrightarrow [a, id^{|a|}] \in [a, const^{|a|}_X] \text{ by Lemma 2.3.6(iv)}\\
&\Leftrightarrow \{u \in [\kappa]^{|a|} \mid id^{|a|}(u) \in const^{|a|}_X(u)\} \in E_a\\
&\Leftrightarrow X = \{u \in [\kappa]^{|a|} \mid u \in X\} \in E_a \Leftrightarrow \langle a, X\rangle \in E.
\end{aligned}$$

□

It doesn't work exactly as good in the other direction, but if we have the derived extender then we can recover at least a part of j.

Lemma 2.3.8. *(Recovering j) Let $j : M \to N$ be an elementary embedding with $crit(j) = \kappa$ and $\lambda \leq j(\kappa)$. Then $k : Ult(M, E(j, \lambda)) \to N, [a, f] \mapsto j(f)(a)$ is an elementary embedding with $crit(k) \geq \lambda$ and*

$$\begin{tikzcd}
M \arrow[r, "j"] \arrow[dr, "i"'] & N \\
& Ult(M, E(j, \lambda)) \arrow[u, "k"']
\end{tikzcd}$$

commutes for $i := i^M_{E(j,\lambda)}$.

Proof. This proof is exactly the same as the proof of Lemma 2.1.9. We use Lemma 2.3.6(iv)(a) to compute that $crit(k) \geq \lambda$. ☐

In Chapter 2.2, we iteratively used ultrafilters and we want to do the analogous construction with pre-extenders. So far, we established that $Ult(M, E)$ is a model of ZFC^-. But we assumed that M is a wellfounded model. So if we want to extend $Ult(M, E)$ again then we need to make sure that $Ult(M, E)$ is also wellfounded. We simply add this as a condition to the definition of being a pre-extender.

Definition 2.3.9. Let E be a (κ, λ)-pre-extender over M. If $Ult(M, E)$ is wellfounded then we call E a **(κ, λ)-extender over M** (or **M-extender**).

Remark 2.3.10. Note that being an M-extender does not only depend on $\mathcal{P}(\kappa)^M$. If $Q \models ZFC^-$ with $\mathcal{P}(\kappa)^Q = \mathcal{P}(\kappa)^M$ and E is an M-extender, then we know that E is a Q-pre-extender but it is not necessarily a Q-extender.

Our definition of extenders is not a first order property. But analogous to the ultrafilter case there is a first order property called ω_1-completeness, which implies wellfoundedness of ultrapowers.

Definition 2.3.11. Let E be a (κ, λ)-pre-extender over M. E is **ω_1-complete** iff for every $\langle a^n \in [\lambda]^{<\omega} \mid n < \omega \rangle$ and $\langle X_n \in E_{a^n} \mid n < \omega \rangle$ there is an order-preserving map $\Phi : \bigcup_{n<\omega} a^n \to \kappa$ such that $\Phi"[a^n] \in X_n$ for each $n < \omega$.

Corollary 2.3.12. *Let E be a (κ, λ)-pre-extender over M and assume that E is ω_1-complete. Then E_a is an ω_1-complete ultrafilter for every $a \in [\lambda]^{<\omega}$.*

Proof. Let $A_n \in E_a$ and set $a^n := a$ for each $n < \omega$. Since E is ω_1-complete, there is a map $\Phi : \bigcup_{n<\omega} a^n \to \kappa$ such that $\Phi"[a^n] \in A_n$ for each $n < \omega$. We have that $\bigcup_{n<\omega} a^n = a$ and thus $\Phi"[a] \in \bigcap_{n<\omega} A_n$. ☐

Lemma 2.3.13. *Let E be an M-pre-extender. If E is ω_1-complete then $Ult(M, E)$ is wellfounded.*

Proof. Suppose that $Ult(M, E)$ is not wellfounded. Then there are $[a^n, f_n] \in Ult(M, E)$ such that

$$[a^{n+1}, f_{n+1}] \in [a^n, f_n] \text{ for each } n < \omega.$$

We can assume that $a^n \subseteq a^{n+1}$ for each $n < \omega$. Then

$$X_{n+1} := \{u \in [\kappa]^{|a^{n+1}|} \mid f_{n+1}(u) \in f_n^{a^n, a^{n+1}}(u)\} \in E_{a^{n+1}}.$$

Set $X_0 := [\kappa]^{|a^0|} \in E_{a^0}$. E is ω_1-complete by assumption, so there is an order-preserving map

$$\Phi : \bigcup_{n<\omega} a^n \to \kappa \text{ such that } \Phi\text{"}[a^n] \in X_n$$

for each $n < \omega$. We set $y_n := f_n\big(\Phi\text{"}[a^n]\big)$ for each $n < \omega$ and claim that $\langle y_n \mid n < \omega \rangle$ is an infinite descending chain in V. Fix some $n < \omega$. Note that

$$y_n = f_n^{a^n, a^{n+1}}\big(\Phi\text{"}[a^{n+1}]\big)$$

since Φ is order-preserving. By the definition of X_{n+1} and Φ, we have that $y_{n+1} = f_{n+1}(\Phi\text{"}[a^{n+1}]) \in f_n^{a^n, a^{n+1}}(\Phi\text{"}[a^{n+1}]) = y_n$. This shows that $\langle y_n \mid n < \omega \rangle$ is a descending chain which is a contradiction. $\square$

The following Lemma is the analog of Lemma 2.1.14:

Lemma 2.3.14. *Let $E \in M$ be an M-pre-extender. Then the following are equivalent:*

(i). E is ω_1-complete,

(ii). For every N and F with $\pi : N \to M$ elementary, N countable, transitive and $\pi(F) = E$ there is an elementary embedding $\sigma : Ult(N, F) \to M$ such that

$$\begin{tikzcd}
M & \\
N \arrow[u, "\pi"] \arrow[r, "i_F^N"'] & Ult(N,F) \arrow[ul, "\sigma"']
\end{tikzcd}$$

commutes.

Proof. "$(i) \Rightarrow (ii)$": Fix N, F and π as in the statement. Then F is a (κ', λ')-pre-extender, where $\pi(\kappa') = \kappa$ and $\pi(\lambda') = \lambda$. We define $X_b := \bigcap_{Y \in F_b} \pi(Y)$ for every $b \in [\lambda']^{<\omega}$. Since N is countable, Corollary 2.3.12 implies that $X_b \in E_{\pi(b)}$. Enumerate $[\lambda']^{<\omega} = \langle c^n \mid n < \omega \rangle$. We can do this because $\lambda' \in N$ and therefore λ' is countable. E is ω_1-complete by assumption, i.e. there is an order-preserving map $\Phi : \bigcup_{n<\omega} \pi(c^n) \to \kappa$ such that $\Phi"[\pi(c^n)] \in X_{c^n}$. Set

$$\sigma([b,g]^N_F) := \pi(g)(\Phi"[\pi(b)]).$$

Note that $\pi(b) \in [\pi(\lambda')]^{<\omega}$ and thus $\Phi"[\pi(b)] \in \kappa$. $\pi(g) \in M$ is a function with domain $\pi(\kappa') = \kappa$. Therefore $\pi(g)(\Phi"[\pi(b)]) \in M$. In order to show that the map is welldefined and elementary, let φ be a formula and let $[b^0, g_0], \dots, [b^k, g_k] \in Ult(N, F)$ such that

$$Ult(N,F) \models \varphi([b^0, g_0], \dots, [b^k, g_k]).$$

Set $b := b^0 \cup \dots \cup b^k$. By Łoś Theorem, we have that

$$Y := \{u \in [\kappa']^{|b|} \mid N \models \varphi\big(g_0^{b^0,b}(u), \dots, g_k^{b^k,b}(u)\big)\} \in F_b.$$

The construction of X_b and Φ imply that $\Phi"[\pi(b)] \in X_b \subseteq \pi(Y)$ and thus $\Phi"[\pi(b)] \in \pi(Y)$. We have that

$$\pi(Y) = \{u \in [\kappa]^{|b|} \mid M \models \varphi\big(\pi(g_0)^{\pi(b^0),\pi(b)}(u), \dots, \pi(g_k)^{\pi(b^k),\pi(b)}(u)\big)\}.$$

Note that

$$\pi(g_l)^{\pi(b^l),\pi(b)}(\Phi"[\pi(b)]) = \pi(g_l)(\Phi"[\pi(b^l)]) = \sigma([b^l, g_l])$$

for $l \leq k$ because Φ is order-preserving. Hence

$$M \models \varphi\big(\sigma([b^0, g_0]), \dots, \sigma([b^k, g_k])\big).$$

"*(ii)* $\Rightarrow$ *(i)*": Fix $\langle a^n \in [\lambda]^{<\omega} \mid n < \omega \rangle$ and $\langle X_n \in E_{a^n} \mid n < \omega \rangle$. Set $N := mos(Hull_M(\{E\} \cup \{X_n \mid n < \omega\})$ and let $\pi : N \to M$ be the anti-collapsing embedding such that $\pi(F) = E$, $\pi(Y_n) = X_n$ and $\pi(b^n) = a^n$. For every $\alpha \in a^n$, there is $\beta \in b^n$ such that $\pi(\beta) = \alpha$. Set $\Phi(\alpha) := \sigma(\beta) = \sigma([\{\beta\}, \varepsilon]^N_F)$, where the second equality holds by Lemma 2.3.6(iv)(a). Ψ is order-preserving because π and σ are elementary. In order to show that $\Phi"[a^n] \in X_n$, we compute $\chi_{X_n}(\Phi"[a^n])$, where χ_{X_n} is the characteristic function of X_n as a subset of $[\kappa]^{|a^n|}$.

$$\begin{aligned}
\chi_{X_n}(\Phi\text{"}[a^n]) &= \pi(\chi_{Y_n})(\Phi\text{"}[\pi(b^n)]), \text{ since } a^n = \pi(b^n) \text{ and } X_n = \pi(Y_n) \\
&= \sigma\big([\{0\}, const^1_{\chi_{Y_n}}]\big)(\sigma(b^n)), \text{ because } \pi = \sigma \circ i^N_F \\
&= \sigma\big([b^n, const^{|b^n|}_{\chi_{Y_n}}]\big)\big(\sigma([b^n, id^{|b^n|}])\big), \text{ by Lemma 2.3.6}(iv)(c) \\
&= \sigma\big([b^n, const^{|b^n|}_{\chi_{Y_n}}]([b^n, id^{|b^n|}])\big), \text{ follows from the elementarity of } \sigma \\
&= \sigma([b^n, \chi_{Y_n}]) \\
&= \sigma([b^n, const_1^{|b^n|}]), \text{ since } Y_n \in F_{b^n} \\
&= \sigma([\{0\}, const^1_1]) = \sigma(i^N_F(1)) = 1.
\end{aligned}$$

□

Corollary 2.3.15. *If $j : M \to N$ and N is wellfounded then $E(j, \lambda)$ is an M-extender.*

Proof. In Lemma 2.3.8, we showed that $Ult(M, E(j, \lambda))$ embeds into the wellfounded model N. In particular, $Ult(M, E(j, \lambda))$ is wellfounded. □

Definition 2.3.16. Let Q and Q' be any models and θ an ordinal. We say that

- Q and Q' **agree up to** θ iff $Q \cap V_\theta = Q' \cap V_\theta$
- Q and Q' **agree well beyond** θ iff they have the same first inaccessible cardinal μ above θ and agree up to μ.

Let $j : Q \to P$ and $j' : Q' \to P'$ be elementary embeddings for some models P and P'. We say that

- j and j' **agree up to** θ iff Q and Q' agree up to θ,

  $$j \restriction (Q \cap V_\theta) = j' \restriction (Q' \cap V_\theta) \text{ and } j(\theta) = j'(\theta).$$

- j and j' **agree well beyond** θ iff Q and Q' agree well beyond θ and

  $$j \restriction (Q \cap V_\mu) = j' \restriction (Q' \cap V_\mu),$$

  where μ is the first inaccessible above θ in both Q and Q'.

Proposition 2.3.17. *Let E be a (κ, λ)-pre-extender over M and suppose that M and N agree up to $\kappa + 1$. Then*

(i). E is an N-pre-extender,

(ii). i^M_E and i^N_E agree up to $\kappa + 1$ and

(iii). $Ult(M,E)$ and $Ult(N,E)$ agree up to $i_E^M(\kappa)+1$.

Proof. (i). M and N agree up to $\kappa+1$ so in particular $\mathcal{P}^M(\kappa) = \mathcal{P}^N(\kappa)$.

(ii). Let $x \in M \cap V_{\kappa+1}$. Then $const_x^1 \in M \cap V_{\kappa+1} = N \cap V_{\kappa+1}$ and therefore

$$i_E^M(x) = [\{0\}, const_x^1]_E^M = [\{0\}, const_x^1]_E^N = i_E^N(x).$$

In particular, $i_E^M(\kappa) = i_E^N(\kappa)$ and thus $i_E^M(\kappa+1) = i_E^N(\kappa+1)$.

(iii). Set $\kappa' := i_E^M(\kappa)$. It is enough to show that

$$Ult(M,E) \text{ and } Ult(M \cap V_{\kappa+1}, E) \text{ agree up to } \kappa'+1.$$

Note that E is an $(M \cap V_{\kappa+1})$-pre-extender by (i). On the one hand,

$$Ult(M \cap V_{\kappa+1}, E) \cap V_{\kappa'+1} \subseteq Ult(M,E) \cap V_{\kappa'+1}$$

by construction. On the other hand, let $[a,f]_E^M \in Ult(M,E) \cap V_{\kappa'+1}$. Then

$$Ult(M,E) \models [a,f]_E^M \in V_{i_E^M(\kappa)+1}.$$

Łoś Theorem implies that

$$A_f := \{u \in [\kappa]^{|a|} \mid M \models f(u) \in V_{\kappa+1}\} \in E_a.$$

Define $g : [\kappa]^{|a|} \to V_{\kappa+1}$ by

$$g(u) := \begin{cases} f(u), & u \in A_f \\ 0, & \text{otherwise.} \end{cases}$$

Then $[a,g]_E^M = [a,f]_E^M$ and $g \in M \cap V_{\kappa+1}$, so

$$[a,f]_E^M = [a,g]_E^{M \cap V_{\kappa+1}} \in Ult(M \cap V_{\kappa+1}, E).$$

□

Definition 2.3.18. Let E be an M-extender. The **strength of E in M, $str^M(E)$,** is the largest ordinal α such that $M \cap V_\alpha \subseteq Ult(M,E)$. We say that E is **nice in M** iff $str^M(E) = lh(E)$ is inaccessible in M.

Proposition 2.3.19. *Let $E \in M$ be a (κ,λ)-extender over M. Then*

(i). $\kappa+1 \leq str^M(E)$,

(ii). $E \notin Ult(M,E)$ and

(iii). $str^M(E) \leq \lambda$.

Proof. (i). We know that $i^M_E \restriction (M \cap V_\kappa) = id$ because $\kappa = crit(i^M_E)$. Therefore every $A \in M \cap V_{\kappa+1}$, i.e. $A \subseteq V_\kappa$, satisfies $A = i^M_E(A) \cap V_\kappa \in Ult(M, E) \cap V_{\kappa+1}$. So $M \cap V_{\kappa+1} \subseteq Ult(M, E)$ and $str^M(E) \geq \kappa + 1$.

(ii). In the ultrafilter-case, we used the fact that the image of κ is inaccessible in the ultrapower. We will also use it here. We write κ^+ for $(\kappa^+)^M$. Note that we can code functions from κ to κ^+ in M as elements of $V_{\kappa+1}$. Since $E \in M$, we have that $Ult(M, E) \subseteq M$. Therefore (i) implies that $M \cap V_{\kappa+1} = Ult(M, E) \cap V_{\kappa+1}$ and thus

$$({}^{\kappa}(\kappa^+))^{Ult(M,E)} = ({}^{\kappa}(\kappa^+))^M.$$

If we assume that $E \in Ult(M, E)$ then

$$i^M_E(\kappa^+) \text{ is the order type of } \{[a, f] \mid a \in [\lambda]^{<\omega}, f \in {}^{\kappa}(\kappa^+)\}.$$

$\kappa^+ \leq i^M_E(\kappa)$ because $\kappa < i^M_E(\kappa)$. So inside $Ult(M, E)$, we have

$$|i^M_E(\kappa^+)| \leq |\lambda| \cdot |{}^{\kappa}(\kappa^+)| \leq |i^M_E(\kappa)|$$

which is a contradiction.

(iii). We have that $E \in M \cap V_{\lambda+1}$ so by (ii) $M \cap V_{\lambda+1} \not\subseteq Ult(M, E)$. Hence $str^M(E) < \lambda + 1$.

□

Lemma 2.3.20. *(Shift Lemma) Let $\pi : M \to N$ and $\sigma : M' \to N'$ be elementary embeddings, where $M', N' \models ZFC^-$. Assume that π and σ agree up to $\kappa + 1$. Let $E \in M'$ be a (κ, λ)-extender over M'. Then*

(i). *$F := \sigma(E)$ is a $(\sigma(\kappa), \sigma(\lambda))$-pre-extender over N.*

(ii). *$\tau : Ult(M, E) \to Ult(N, F)$, $[a, f]^M_E \mapsto [\sigma(a), \pi(f)]^N_F$ is an elementary embedding and $\tau \circ i^M_E = i^N_F \circ \pi$.*

(iii). *If F is an N-extender then E is an M-extender, i.e. if $Ult(N, F)$ is wellfounded then $Ult(M, E)$ is wellfounded.*

(iv). *$\sigma \restriction \lambda = \tau \restriction \lambda$.*

(v). *If π and σ agree well beyond κ and E is nice in M', then σ and τ agree up to λ.*

$$\begin{tikzcd}
N \arrow[rr, bend left, "i_F^N"] & F \in N' & Ult(N,F) \\
M \arrow[u, "\pi"'] \arrow[rr, bend right, "i_E^M"'] & E \in M' \arrow[u, "\sigma"'] & Ult(M,E) \arrow[u, "\tau"']
\end{tikzcd}$$

Proof. First of all we show that N and N' agree up to $\sigma(\kappa)+1$. We compute

$$N \cap V_{\sigma(\kappa)+1} = N \cap V_{\pi(\kappa)+1} = V^N_{\pi(\kappa)+1} = \pi(V^M_{\kappa+1}) = \sigma(V^{M'}_{\kappa+1}) = V^{N'}_{\sigma(\kappa)+1} = N' \cap V_{\sigma(\kappa)+1}$$

where the first equality holds because π is elementary and $\sigma(\kappa) = \pi(\kappa)$. The last one holds because σ is elementary and the middle one because M and M' agree up to $\kappa + 1$.

(i). F is an N'-pre-extender by the elementarity of σ. So F is an N-pre-extender by Proposition 2.3.17. κ is definable from E since κ is the critical point of the ultrafilters of E. Therefore $\sigma(\kappa) = crit(F)$. λ is also definable from E since formally $E = \{\langle a, X\rangle \mid a \in [\lambda]^{|a|}, X \in E_a\}$, so $\lambda = \bigcup\{a \mid \exists X\,(a, X) \in E\}$. Hence $\sigma(\lambda) = lh(F)$.

(ii). First we show that formulas true in $Ult(M,E)$ transform to formulas true in $Ult(N,F)$.

Claim. Let $\varphi(x_0, \ldots, x_n)$ be a formula and let $[a^0, f_0]^E_M, \ldots, [a^n, f_n]^E_M \in Ult(M,E)$ such that

$$Ult(M,E) \models \varphi([a^0, f_0]^E_M, \ldots, [a^n, f_n]^E_M).$$

Then

$$Ult(N,F) \models \varphi([\sigma(a^0), \pi(f_0)]^F_N, \ldots, [\sigma(a^n), \pi(f_n)]^F_N).$$

Proof of Claim. For simplicity and readability, we assume that $n = 1$. Set $b := a^0 \cup a^1$ and $X := \{u \in [\kappa]^{|b|} \mid M \models \varphi(f_0^{a^0,b}(u), f_1^{a^1,b}(u))\}$. Then

$$Ult(M,E) \models \varphi([a^0, f_0]^E_M, [a^1, f_1]^E_M) \Leftrightarrow (b, X) \in E.$$

By the elementarity of π,

$$X = \{u \in [\kappa]^{|b|} \mid N \models \varphi\big(\pi(f_0^{a^0,b}(u)), \pi(f_1^{a^1,b}(u))\big)\}.$$

For simplicity, assume that a^0 is an initial segment of b and let $|a^0| = k, |b| = k+m$. Then for any $u \in [\kappa]^{|b|}$, we have

$$\begin{aligned} & \pi(f_0^{a^0,b}(u)) \\ =& \pi(f_0^{a^0,b}(u_1,\dots u_k, u_{k+1},\dots,u_{k+m})) \\ =& \pi(f_0(u_1,\dots u_k)) \\ =& \pi(f_0)(\pi(u_1),\dots \pi(u_k)) \\ =& \pi(f_0)(\sigma(u_1),\dots \sigma(u_k)) \\ =& \pi(f_0)^{\sigma(a^0),\sigma(b)}(\sigma(u_1),\dots \sigma(u_k), \sigma(u_{k+1}),\dots \sigma(u_{k+m})) \\ =& \pi(f_0)^{\sigma(a^0),\sigma(b)}(\sigma(u)). \end{aligned}$$

Therefore

$$X = \{u \in [\kappa]^{|b|} \mid N \models \varphi(\pi(f_0)^{\sigma(a^0),\sigma(b)}(\sigma(u)), \pi(f_1)^{\sigma(a^1),\sigma(b)}(\sigma(u)))\}$$

and

$$\begin{aligned} & \sigma(X) \\ =& \{\sigma(u) \in [\sigma(\kappa)]^{|\sigma(b)|} \mid N \models \varphi(\pi(f_0)^{\sigma(a^0),\sigma(b)}(\sigma(u)), \pi(f_1)^{\sigma(a^1),\sigma(b)}(\sigma(u)))\} \\ =& \{t \in [\sigma(\kappa)]^{|\sigma(b)|} \mid N \models \varphi(\pi(f_0)^{\sigma(a^0),\sigma(b)}(t), \pi(f_1)^{\sigma(a^1),\sigma(b)}(t))\} \end{aligned}$$

Now we have

$$\begin{aligned} & (b, X) \in E \Leftrightarrow (\sigma(b), \sigma(X)) \in F \\ & \Leftrightarrow Ult(N, F) \models \varphi([\sigma(a^0), \pi(f_0)]^F_N, [\sigma(a^1), \pi(f_1)]^F_N)). \end{aligned}$$

■

The claim implies that τ is a welldefined elementary embedding. The diagram is commuting by construction.

(iii). It follows directly from the existence of the elementary embedding τ.

(iv). We want to use Lemma 2.3.6. Note that $Ult(M,E) \cap V_{i_E^M(\kappa)+1}$ is wellfounded because Proposition 2.3.17 implies that

$$Ult(M,E) \text{ and } Ult(M',E) \text{ agree up to } i_E^M(\kappa)+1$$

and $Ult(M',E)$ is wellfounded because E is an M'-extender. The same argument shows that $Ult(N,F) \cap V_{i_F^N(\sigma(\kappa))+1}$ is wellfounded.
Let $\beta < \lambda$. Then β is in the wellfounded part of $Ult(M,E)$, so by Lemma

2.3.6 $\beta = [\{\beta\}, \varepsilon]^M_E$. We compute

$$\tau(\beta) = \tau([\{\beta\}, \varepsilon]^M_E) = [\sigma(\{\beta\}), \pi(\varepsilon)]^N_F = [\{\sigma(\beta)\}, \{\gamma\} \mapsto \gamma]^N_F = \sigma(\beta)$$

because $\sigma(\beta) < \sigma(\lambda)$ is in the wellfounded part of $Ult(N, F)$.

(v). In Proposition 2.3.17, we showed that $Ult(M, E)$ and $Ult(M', E)$ agree up to $i^M_E(\kappa) + 1$. Since E is nice in M', $Ult(M', E)$ and M' agree up to λ. So $Ult(M, E)$ and M' agree up to λ. Now take some $x \in M' \cap V_\lambda = Ult(M, E) \cap V_\lambda$. Since $x \in V_{i^M_E(\kappa)}$, there is some $f : \kappa \to V_\kappa$ in M and some $a \in [\lambda]^{<\omega}$ such that $x = [a, f]^M_E$. Note that the rank of f is below the first inaccessible above κ in M and we assumed that σ and π agree well beyond κ. Therefore $\sigma(f) = \pi(f)$. Proposition 2.3.17 implies that $[a, f]^M_E = [a, f]^{M'}_E$ and $[\sigma(a), \sigma(f)]^N_F = [\sigma(a), \sigma(f)]^{N'}_F$. We compute

$$\begin{aligned}\sigma(x) &= \sigma([a, f]^M_E) = \sigma([a, f]^{M'}_E) = [\sigma(a), \sigma(f)]^{N'}_F \\ &= [\sigma(a), \sigma(f)]^N_F = [\sigma(a), \pi(f)]^N_F = \tau([a, f]^M_E) = \tau(x).\end{aligned}$$

In order to show that $\sigma(\lambda) = \tau(\lambda)$, notice that $\lambda \in Ult(M, E)$ because $i(\kappa) \in Ult(M, E)$ and $\lambda < i(\kappa)$. In particular, $\lambda \in Ult(M, E) \cap V_{i(\kappa)+1}$, so there is some $g : \kappa \to V_{\kappa+1}$ in M and some $b \in [\lambda]^{<\omega}$ such that $\lambda = [b, g]^M_E$. As before, the rank of g is below the first inaccessible above κ in M and therefore $\sigma(g) = \pi(g)$. The same computation as above shows that $\sigma(\lambda) = \tau(\lambda)$.

□

Definition 2.3.21. Let $M, M', N, N', \sigma, \pi$ and E be as in Lemma 2.3.20. We call the $(\sigma(\kappa), \sigma(\lambda))$-extender $\sigma(E)$ over N from Lemma 2.3.20(i) the **shift of** E **to** N **via** $\langle \pi, \sigma \rangle$ and the elementary embedding $\tau : Ult(M, E) \to Ult(N, \sigma(E))$ from Lemma 2.3.20(ii) the **shift map of** $\langle \pi, \sigma \rangle$ **via** E.

2.4 Linear Iterations via Extenders

In Chapter 2.2, we iteratively extended a model by ultrafilters. In this chapter, we are going to do the same with extenders. The following definitions and results are exactly the same as in Chapter 2.2, where we exchange "M-nuf" by "M-extender".

Definition 2.4.1. Let E be any M-pre-extender and let (M, Y) be an amenable structure. The **ultrapower of** (M, Y) **by** E is defined as

$$Ult((M, Y), E) := (Ult(M, E), Y_E),$$

$$\text{where } [a,f]_E \in Y_E \text{ iff } \{u < [\kappa]^{|a|} \mid f(u) \in Y\} \in E_a$$

for every $a \in \lambda^{<\omega}$ and each function $f \in M$ with $dom(f) = [\kappa]^{|a|}$.

Lemma 2.4.2. *Let E be any E-pre-extender and let (M, Y) be an amenable structure. Assume that $(Ult(M, E), \tilde{\in})$ is wellfounded. Then $Ult((M, Y), E)$ is an amenable structure.*

The proof is analogous to the proof of Lemma 2.2.4.

Definition 2.4.3. A pair $(M, \mathcal{E})$ is a **good pair** iff $(M, \mathcal{E})$ is an amenable structure and every $E \in \mathcal{E}$ satisfies $M \models$ "E is an M-pre-extender". It will always be clear from the context whether it is about a good pair in the ultrafilter sense or a good pair in the extender sense.

From now on, let $(M, \mathcal{E})$ be a good pair.

Lemma 2.4.4. *Let E be an M-pre-extender and assume that $(Ult(M, E), \tilde{\in})$ is wellfounded. Then $Ult((M, \mathcal{E}), E)$ is a good pair.*

The proof is analogous to the proof of Lemma 2.2.5.

Definition 2.4.5. Let β be some ordinal. We call $I = \langle E_\alpha \mid \alpha < \beta \rangle$ a **linear iteration of** $(M, \mathcal{E})$ **of length** β iff there are $\langle M_\alpha, \mathcal{E}_\alpha \mid \alpha < \beta \rangle$ and elementary embeddings $\langle i_{\alpha,\gamma} : M_\alpha \to M_\gamma \mid \alpha < \gamma < \beta \rangle$ such that

(i). $M_0 = M$ and $\mathcal{E}_0 = \mathcal{E}$,

(ii). for each $\alpha < \beta$:
M_α is a transitive model of ZFC^- and $E_\alpha \in \mathcal{E}_\alpha$,

(iii). for successors $\alpha + 1 < \beta$:
$(M_{\alpha+1}, \mathcal{E}_{\alpha+1}) = Ult((M_\alpha, \mathcal{E}_\alpha), E_\alpha)$,
$i_{\alpha,\alpha+1} = i^{M_\alpha}_{E_\alpha}$ and
$i_{\gamma,\alpha+1} = i_{\alpha,\alpha+1} \circ i_{\gamma,\alpha}$ for each $\gamma < \alpha$,

(iv). for limit ordinals $\lambda < \beta$:
$(M_\lambda, \mathcal{E}_\lambda)$ is the direct limit of $\langle (M_\alpha, \mathcal{E}_\alpha), i_{\alpha,\gamma} \mid \alpha < \gamma < \lambda \rangle$ and $i_{\alpha,\lambda}$ are the direct limit embeddings.

We also write E^I_α, M^I_α and $i^I_{\alpha,\gamma}$ for E_α, M_α and $i_{\alpha,\gamma}$. We associate the "last model" M^I_∞ which is defined as the direct limit of $\langle M^I_\alpha, i^I_{\alpha,\gamma} \mid \alpha < \gamma < \beta \rangle$.

Remark 2.4.6.

- Lemma 2.4.4 yields that the construction is welldefined.
- We will sometimes suppress the $\mathcal{E}_\alpha$'s in favor of the readability.

- The models $\langle M_\alpha \mid \alpha < \beta \rangle$ and elementary embeddings $\langle i_{\alpha,\gamma} \mid \alpha < \gamma < \beta \rangle$ from Definition 2.4.5 are unique. Therefore we call them **the models and the elementary embeddings of I**.

- If $\beta \leq \omega_1$ then every M_α with $\alpha < \beta$ has the same cardinality as M.

Definition 2.4.7. Let θ be an ordinal. We call $(M, \mathcal{E})$ **$<\theta$-linearly iterable** iff M_∞^I is wellfounded for every linear iteration I of $(M, \mathcal{E})$ of length less than θ. If $(M, \mathcal{E})$ is $<\theta$-linearly iterable for every ordinal θ then we call $(M, \mathcal{E})$ **linearly iterable**.

We have the same theorem as Theorem 2.2.9, where we used M-nuf's instead of M-extenders.

Theorem 2.4.8. *If every $E \in \mathcal{E}$ is ω_1-complete then $(M, \mathcal{E})$ is linearly iterable.*

The proof works exactly the same as the proof in Chapter 2.2 using Lemma 2.4.9 and Lemma 2.4.10.

Lemma 2.4.9. *(Pull Back Linear Iterability) Let $(M, \mathcal{E})$ be $<\theta$-linearly iterable for some ordinal θ. Let $(N, \mathcal{F})$ be a good pair and let $\pi : (N, \mathcal{F}) \to (M, \mathcal{E})$ be an elementary map. Then $(N, \mathcal{F})$ is $<\theta$-linearly iterable, too.*

Lemma 2.4.10. *(Characterization of Linear Iterability) The following are equivalent*

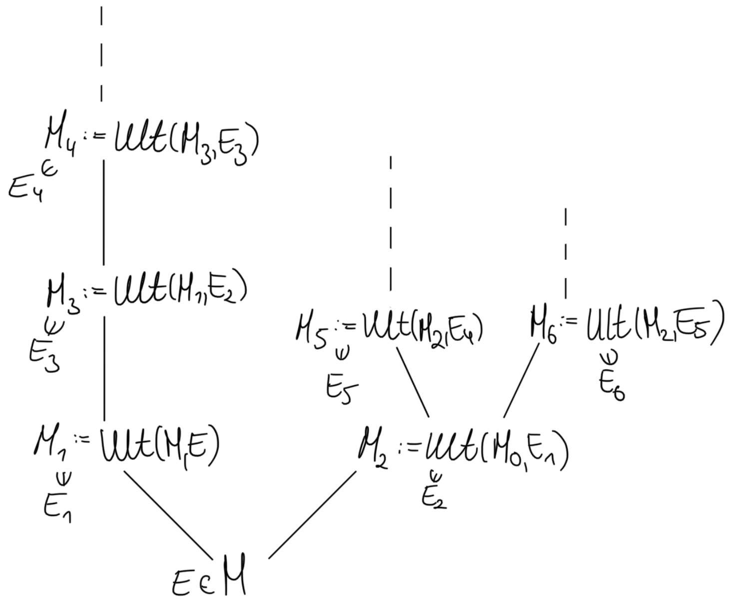
(i). $(M, \mathcal{E})$ is $<\omega_1$-linearly iterable,

(ii). If $(N, \mathcal{F})$ is a good pair, N is countable and transitive and $\pi : (N, \mathcal{F}) \to (M, \mathcal{E})$ is an elementary map. Then $(N, \mathcal{F})$ is $<\omega_1$-linearly iterable.

(iii). $(M, \mathcal{E})$ is linearly iterable.

2.5 Iteration Trees

In the previous chapter, we iterated ultrapowers by always using the last model for the next ultrapower. But we saw that a pre-extender E over one model is also a pre-extender over another model whenever the two models agree up to $crit(E) + 1$. Therefore we could also apply E to an earlier model in the iteration. This process creates an iteration tree.



Definitions and the Agreement Property

Definition 2.5.1. Let β be an ordinal. T is a **tree order on** β iff

(i). T an order on β, i.e. T is irreflexive, antisymmetric and transitive,

(ii). T is coarser than the usual ordering on the ordinals, i.e. $\mu T \xi \Rightarrow \mu < \xi$,

(iii). $\{\mu < \beta \mid \mu T \xi\}$ is linearly ordered by T for each $\xi < \beta$,

(iv). $\xi + 1$ is a successor in T for each $\xi < \beta$ with $\xi + 1 < \beta$. We denote the T-predecessor of $\xi + 1$ by $pred_T(\xi + 1)$ and

(v). $\{\mu < \beta \mid \mu T \gamma\}$ is cofinal in γ for $\gamma < \beta$ limit ordinal.

Definition 2.5.2. Let T be a tree order on β. For $\mu, \xi < \beta$, set

- $(\mu, \xi)_T := \{\alpha \mid \mu T \alpha T \xi\}$,
- $[\mu, \xi)_T := \{\mu\} \cup (\mu, \xi)_T$,
- $(\mu, \xi]_T := (\mu, \xi)_T \cup \{\xi\}$ and
- $[\mu, \xi]_T := \{\mu\} \cup (\mu, \xi)_T \cup \{\xi\}$.

Let $(M, \mathcal{E})$ be a good pair throughout this chapter.

Definition 2.5.3. Let β be an ordinal. $\mathcal{T} = (T, \langle E_\xi \mid \xi + 1 < \beta \rangle)$ is an **iteration tree of length** β **on** $(M, \mathcal{E})$ iff there are $\langle M_\xi, \mathcal{E}_\xi \mid \xi < \beta \rangle$ and elementary embeddings $\langle i_{\mu,\xi} : M_\mu \to M_\xi \mid \mu T \xi < \beta \rangle$ such that (i). T is a tree order on β and

(ii). $M_0 = M, \mathcal{E}_0 = \mathcal{E}$.

(iii). For each ξ with $\xi + 1 < \beta$:
M_ξ is a transitive model of ZFC^- and $E_\xi \in \mathcal{E}_\xi$.

(iv). For successors $\xi + 1 < \beta$, $\eta := pred_T(\xi + 1)$:
M_η and M_ξ agree up to $crit(E_\xi) + 1$,
$(M_{\xi+1}, \mathcal{E}_{\xi+1}) = Ult((M_\eta, \mathcal{E}_\eta), E_\xi)$,
$i_{\eta,\xi+1} = i^{M_\eta}_{E_\xi}$ and
$i_{\mu,\xi+1} = i_{\eta,\xi+1} \circ i_{\mu,\eta}$ for each $\mu T \eta$.

(v). For limit ordinals $\gamma < \beta$:
$(M_\gamma, \mathcal{E}_\gamma) := dirlim\langle (M_\xi, \mathcal{E}_\xi), i_{\mu,\xi} \mid \mu T \xi T \gamma \rangle$ and $i_{\xi,\gamma}$ is the direct limit embedding for each $\xi T \gamma$.

Remark 2.5.4.

- The condition that M_η and M_ξ agree up to $crit(E_\xi) + 1$ in (iv) implies that E_ξ is a M_η-pre-extender. Lemma 2.4.4 yields that (M_η, E_ξ) is a good pair. Therefore $Ult((M_\eta, \mathcal{E}_\eta), E_\xi)$ is welldefined.
- We will sometimes suppress the $\mathcal{E}_\xi' s$ in favor of the readability.
- The models and embeddings from Definition 2.5.3 are unique. Hence we call them **the models and embeddings of** $\mathcal{T}$ and denote them by $M_\xi^{\mathcal{T}}$ and $i^{\mathcal{T}}_{\mu,\xi}$. Whenever the iteration tree is clear from the context, we simply write M_ξ and $i_{\mu,\xi}$.

Definition 2.5.5. Let $\mathcal{T} = (T, \langle E_\xi \mid \xi + 1 < \beta \rangle)$ be an iteration tree on $(M, \mathcal{E})$. We say that $\mathcal{T}$ is

- **non-overlapping**[6] iff for each ξ with $\xi + 1 < \beta$, we have that $pred_T(\xi + 1)$ is the minimal η such that M_η and M_ξ agree up to $crit(E_\xi) + 1$.
- **length-increasing** iff $lh(E_\mu) < lh(E_\xi)$ for every $\mu < \xi + 1 < \beta$ (not only for $\mu T \xi$).
- **nice** iff E_ξ is nice in M_ξ, i.e. $str^{M_\xi}(E_\xi) = lh(E_\xi)$ is inaccessible in M_ξ, for every $\xi + 1 < \beta$ and $\mathcal{T}$ is non-overlapping and length-increasing.

Definition 2.5.6. A **branch** through an iteration tree $\mathcal{T}$ is a set which is linearly ordered by T and closed under $pred_T$. Assume that the length of $\mathcal{T}$ is a limit ordinal. For a cofinal branch b, i.e. $sup(b) = lh(\mathcal{T})$, we define the **direct limit along** b as

$$(M_b^{\mathcal{T}}, \mathcal{E}_b^{\mathcal{T}}) := dirlim\langle (M_\xi, \mathcal{E}_\xi), i_{\mu,\xi} \mid \mu T \xi \in b \rangle$$

---

[6] In Remark 2.5.13 we will see why this property is called non-overlapping.

and $i^{\mathcal{T}}_{\xi,b} : M_\xi \to M_b$ as the direct limit embeddings. Write $i^{\mathcal{T}}_b$ for $i^{\mathcal{T}}_{0,b}$. We call a cofinal branch b **wellfounded** iff $M^{\mathcal{T}}_b$ is wellfounded.

Definition 2.5.7. The **iteration game of length** β **on** $(M, \mathcal{E})$ is the following two-player-game of length β:
The beginning: $(M_0, \mathcal{E}_0) := (M, \mathcal{E})$.
$(\xi + 1)$-th stage: Player I plays an extender $E_\xi \in \mathcal{E}_\xi$ and chooses $\eta \leq \xi$ such that M_η and M_ξ agree up to $crit(E_\xi) + 1$. We set $(M_{\xi+1}, \mathcal{E}_{\xi+1}) := Ult((M_\eta, \mathcal{E}_\eta), E_\xi)$. If $M_{\xi+1}$ is illfounded, the game is over and Player I wins.
γ limit stage: Player II plays a cofinal branch $b_\gamma \subseteq \gamma$. We set $(M_\gamma, \mathcal{E}_\gamma) := (M_{b_\gamma}, \mathcal{E}_{b_\gamma})$. If M_γ is illfounded, the game is over and Player I wins.
The end: If they went through all the stages before β and Player I did not win at those stages then Player II wins.

Note that Player II only influences the iteration game at limit stages. So a winning strategy for him doesn't influence what's happening in the successor steps. It it not enough for Player II to make sure that M_λ is wellfounded. He also has to care about every possible $M_{\lambda+1}, M_{\lambda+2}, \dots$ This is a very hard job!

Definition 2.5.8. We say that $(M, \mathcal{E})$ is β**-iterable** iff Player II has a winning strategy in the iteration game of length β on $(M, \mathcal{E})$. We call such a winning strategy a β**-iteration strategy for** $(M, \mathcal{E})$. We say that $(M, \mathcal{E})$ is **(fully) iterable** if $(M, \mathcal{E})$ is β-iterable for every ordinal β.

Lemma 2.5.9. *(Pull Back Iterability) Let* $(M, \mathcal{E})$ *be* β*-iterable for some ordinal* β*. Let* $(N, \mathcal{F})$ *be a good pair and let* $\pi : (N, \mathcal{F}) \to (M, \mathcal{E})$ *be an elementary map. Then* $(N, \mathcal{F})$ *is* β*-iterable, too.*

Sketch of proof. The proof is analogous to the proof of pulling back linear iterability, see Lemma 2.2.11 and Lemma 2.4.9. For an iteration tree $\mathcal{S} = (S, \langle F_\xi \mid \xi + 1 < \beta \rangle)$ on $(N, \mathcal{F})$, we construct an iteration tree $\mathcal{T} = (T, \langle E_\xi \mid \xi + 1 < \beta \rangle)$ on $(M, \mathcal{E})$ and elementary embeddings $\pi_\xi : N_\xi \to M_\xi$ for every $\xi < \beta$. Since $(M, \mathcal{E})$ is β-iterable, we know that M_ξ is wellfounded for every $\xi < \beta$. Hence N_ξ is wellfounded for every $\xi < \beta$. □

We define a weakening of iterability which often suffices.

Definition 2.5.10. The **nice iteration game of length** β **on** $(M, \mathcal{E})$ is the iteration game of length β on $(M, \mathcal{E})$, where Player I can only play extenders which give rise to a nice iteration tree. She has to choose an extender E_ξ which is nice and has length above the previous extenders. She has only one option in choosing the model M_η because η is already determined by the non-overlapping property. We say that $(M, \mathcal{E})$ is β**-iterable for nice trees** iff Player II has a winning strategy in the nice iteration game of length β on $(M, \mathcal{E})$ and we say that $(M, \mathcal{E})$ is **(fully) iterable for nice trees** if $(M, \mathcal{E})$ is β-iterable for nice trees for every ordinal β.

Lemma 2.5.11. *Let $\mathcal{T}$ be an iteration tree of length β on $(M, \mathcal{E})$, which is nice except for the non-overlapping condition. Fix ξ be such that $\xi + 1 < \beta$. We assume that $\mathcal{T}$ is* ***non-overlapping below*** *ξ, i.e. for each ξ' with $\xi' + 1 < \xi$, we have that $pred_T(\xi' + 1)$ is the minimal μ such that M_μ and $M_{\xi'}$ agree up to $crit(E_{\xi'}) + 1$. Then every $\mu < \xi$ (not only $\mu T \xi$) satisfies:*

(i). M_μ and M_ξ agree up to $lh(E_\mu)$ and

(ii). M_μ and M_ξ do not agree up to $lh(E_\mu) + 1$.

Moreover, if ξ is a successor ordinal, say $\xi = \xi' + 1$, then

(iii). $pred_T(\xi'+1)$ is the minimal μ such that M_μ and $M_{\xi'}$ agree up to $crit(E_{\xi'})+1$ if and only if $pred_T(\xi'+1)$ is the minimal μ such that $crit(E_{\xi'}) < lh(E_\mu)$.

Proof. We show (i), (ii) and (iii) simultaneously by induction on ξ.
$\underline{\xi = 0}$: There is nothing to prove here.
$\underline{\text{successor } \xi + 1}$: We differentiate two cases for proving (i) and (ii).
Case 1 ($\mu = \xi$): $Ult(M_\xi, E_\xi)$ and $M_{\xi+1}$ agree up to $i^{M_\xi}_{E_\xi}(crit(E_\xi)) + 1$ by Proposition 2.3.17. Since E_ξ is nice in M_ξ, we know that M_ξ and $Ult(M_\xi, E_\xi)$ agree up to $lh(E_\xi)$. By Proposition 2.3.19, we know that M_ξ and $Ult(M_\xi, E_\xi)$ do not agree up to $lh(E_\xi) + 1$. Now $i^{M_\xi}_{E_\xi}(crit(E_\xi)) + 1 > lh(E_\xi)$. Hence M_ξ and $M_{\xi+1}$ agree up to $lh(E_\xi)$ and not up to $lh(E_\xi) + 1$.
Case 2 ($\mu < \xi$): The induction hypothesis together with Case 1 immediately solves this case.
In order to prove (iii), set $\eta := pred_T(\xi' + 1)$ and assume that η is the minimal μ such that M_μ and $M_{\xi'}$ agree up to $crit(E_{\xi'}) + 1$ and set $\eta := pred_T(\xi' + 1)$. We have that M_η and $M_{\xi'+1}$ do not agree up to $lh(E_\eta) + 1$ by (ii). Therefore $crit(E_{\xi'})+1 < lh(E_\eta)+1$ so $crit(E_{\xi'}) < lh(E_\eta)$. If $\alpha < \eta$ then by the assumption M_α and $M_{\xi'}$ do not agree up to $crit(E_{\xi'}) + 1$ but by (i), they agree up to $lh(E_\alpha)$ hence $crit(E_{\xi'}) + 1 > lh(E_\alpha)$ so $crit(E_{\xi'}) \not< lh(E_\alpha)$. On the other hand, assume that η is the minimal μ such that $crit(E_{\xi'}) < lh(E_\mu)$. Then M_η and $M_{\xi'}$ agree up to $crit(E_{\xi'}) + 1$ by (i) and for each $\alpha < \eta$, we have that M_α and M_η do not agree up to $crit(E_{\xi'}) + 1$ because $crit(E_{\xi'}) \geq lh(E_\alpha)$ and (ii).
$\underline{\xi \text{ limit ordinal}}$:
Take some $\alpha > \mu + 1$ such that $\alpha T \xi$. We can do this because $[0, \xi)_T$ is cofinal in ξ. We claim that every ε with $\varepsilon + 1 \in (\alpha, \xi)_T$ satisfies $crit(E_\varepsilon) \geq lh(E_{\mu+1})$. Suppose that there is some ε such that $\varepsilon + 1 \in (\alpha, \xi)_T$ and $crit(E_\varepsilon) < lh(E_{\mu+1})$. The induction hypothesis yields that $M_{\mu+1}$ and M_ε agree up to $lh(E_{\mu+1})$. So by

assumption, $M_{\mu+1}$ and M_ε agree up to $crit(E_\varepsilon)+1$. $\mathcal{T}$ is non-overlapping below ξ, hence $pred_T(\varepsilon+1) \leq \mu+1$. But on the other hand, $pred_T(\varepsilon+1) \geq \alpha > \mu+1$ which is a contradiction. The claim implies that $crit(i_{\alpha,\xi}) \geq lh(E_{\mu+1}) \geq lh(E_\mu) + 1$ and therefore M_α and M_ξ agree up to $lh(E_\mu) + 1$. By the induction hypothesis, M_μ and M_α agree up to $lh(E_\mu)$ but not up to $lh(E_\mu) + 1$. Hence M_μ and M_ξ agree up to $lh(E_\mu)$ but not up to $lh(E_\mu) + 1$. □

Corollary 2.5.12. *(Agreement property of nice trees) Let $\mathcal{T}$ be a nice iteration tree of length β on $(M, \mathcal{E})$. Then*

- *M_μ and M_ξ agree up to $lh(E_\mu)$ and*
- *M_μ and M_ξ do not agree up to $lh(E_\mu) + 1$*

for each $\mu < \xi < \beta$.

Remark 2.5.13. Lemma 2.5.11 (iii) says that we can exchange the non-overlapping condition in the definition of a nice iteration tree by the condition that $pred_T(\xi + 1)$ is the minimal μ such that $crit(E_\xi) < lh(E_\mu)$. This is the reason why this property is called non-overlapping. Consider $(\eta + 1)T(\xi + 1)$, i.e. the extenders E_η and E_ξ were used on the same branch. By the definition of $pred_T(\xi + 1)$, we have that $\mu := pred_T(\xi + 1) > \eta$. Therefore $crit(E_\xi) \geq lh(E_\eta)$, i.e. E_ξ is above E_η and they do not overlap.

Definition 2.5.14. (nSBH) The **nice Strategic Branches Hypothesis (nSBH)** asserts that every countable model which embeds into a rank initial segment of V is iterable for nice trees.

A proof of nSBH from ZFC, if it exists, would constitute a substantial breakthrough in the study of large cardinals, particularly in inner model theory. It is hard to find a proof of nSBH because the complexity of canonical inner models, i.e. the amount of large cardinals they can accommodate, directly influences the complexity of their iteration strategy. We are going to prove two special cases of nSBH. The first one is Theorem 2.5.15, where only nice iteration trees of length ω are considered. The second one is Corollary 2.5.25, where we replace iterability for nice trees by weak iterability.

$(\omega + 1)$-iterability of nice iteration trees

This subsection is based on the second chapter of Neeman's article *Determinacy in* $L(\mathbb{R})$ [Nee10]. The proof in [Ste15] is slightly different. The goal is to prove the following theorem.

Theorem 2.5.15. *Fix an ordinal θ. Let M be countable, let $\pi : M \to V_\theta$ be an elementary embedding and let $\mathcal{T}$ be a nice iteration tree of length ω on $(M, \mathcal{E})$. Then there is a cofinal branch b through $\mathcal{T}$ and an elementary embedding $\sigma : M_b \to V_\theta$ such that $\sigma \circ i_b = \pi$. In particular, b is a wellfounded cofinal branch through $\mathcal{T}$.*

$$\begin{tikzcd}
V_\theta & \\
M \arrow[u, "\pi"] \arrow[r, "i_b"'] & M_b \arrow[ul, "\sigma"']
\end{tikzcd}$$

For the proof of Theorem 2.5.15, we are going to use a special kind of nice iteration tree of length ω.

Definition 2.5.16. Let $\mathcal{T}$ be a nice iteration tree of length ω on $(M, \mathcal{E})$ with models $\langle M_n \mid n < \omega \rangle$ and embeddings $\langle i_{k,n} \mid kTn \rangle$. We say that $\mathcal{T}$ is **continuously illfounded** iff there is a sequence of ordinals $\langle \alpha_n \mid n < \omega \rangle$ with $\alpha_n \in M_n$ such that $M_n \models \alpha_n < i_{k,n}(\alpha_k)$ for each $kTn < \omega$.

Proposition 2.5.17. *If $\mathcal{T}$ is a continuously illfounded iteration tree then there is no wellfounded cofinal branch through $\mathcal{T}$.*

Proof. Fix a witness $\langle \alpha_n \mid n < \omega \rangle$ for $\mathcal{T}$ being continuously illfounded and a cofinal branch b through $\mathcal{T}$. For every $k, n \in b$ with kTn, we have that $M_n \models i_{n,n}(\alpha_n) = \alpha_n < i_{k,n}(\alpha_k)$. Since M_b is the direct limit of $\langle M_n \mid n \in b \rangle$, we also have that $M_b \models i_{n,b}(\alpha_n) < i_{k,b}(\alpha_k)$. So $\langle i_{n,b}(\alpha_n) \mid n < \omega \rangle$ is a strictly $\in$-decreasing. Therefore M_b is not wellfounded and hence b is not wellfounded. $\square$

Let M be a countable model which embeds into some V_θ. If there would be a continuously illfounded iteration tree on $(M, \mathcal{E})$ then this would contradict Theorem 2.5.15 by the previous proposition. We are going to show that every counterexample to Theorem 2.5.15 gives rise to a continuously illfounded iteration tree on V in Lemma 2.5.20. The next step is to show that there are no continuously illfounded iteration trees on V. This is done in Lemma 2.5.21.

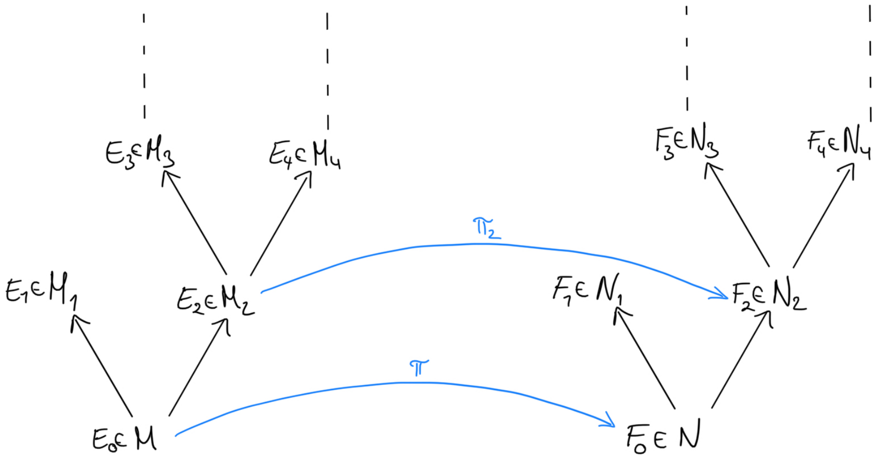
Lemma 2.5.18. *(Copy Construction) Let $\mathcal{T} = (T, \langle E_n \mid n < \omega \rangle)$ be a nice iteration tree of length ω on $(M, \mathcal{E})$ with models $\langle M_n \mid n < \omega \rangle$ and embeddings $\langle i_{m,n} \mid mTn \rangle$. Let $(N, \mathcal{F})$ be a good pair and let $\pi : (M, \mathcal{E}) \to (N, \mathcal{F})$ be an elementary embedding. Then there is a nice iteration tree $\mathcal{S} = (T, \langle F_n \mid n < \omega \rangle)$ of length ω on $(N, \mathcal{F})$ with models $\langle N_n \mid n < \omega \rangle$ and embeddings $\langle l_{k,n} \mid kTn \rangle$. Furthermore there are elementary embeddings $\langle \pi_n : (M_n, \mathcal{E}_n) \to (N_n, \mathcal{F}_n) \mid n < \omega \rangle$ such that*

(i). $\pi_0 = \pi$,

(ii). $F_n = \pi_n(E_n)$ *for each* $n < \omega$,

(iii). $l_{k,n} \circ \pi_k = \pi_n \circ i_{k,n}$ *for each* kTn,

(iv). π_k *and* π_n *agree up to* $lh(E_k)$ *for each* $k < n$.



Proof. We define $\langle N_n, F_n, \pi_n \mid n < \omega \rangle$ by recursion on $n < \omega$.
<u>n = 0:</u> Set $\pi_0 := \pi$.
<u>$n \to n+1$:</u> Set $m := pred_T(n+1)$ and $\kappa := crit(E_n)$. Let $F_n := \pi_n(E_n)$ be the shift of E_n to N_m via $\langle \pi_m, \pi_n \rangle$. Set $N_{n+1} := Ult(N_m, F_n)$ and $l_{m,n+1} := i^{N_m}_{F_n}$. $\mathcal{T}$ is a nice iteration tree, so M_m and M_n agree up to $lh(E_m)$ by Lemma 2.5.12. Since $m \leq n$, we constructed π_m and π_n such that they agree up to $lh(E_m)$. By the non-overlapping property, $\kappa < lh(E_m)$. We have that E_m is nice in M_m, hence $lh(E_m)$ is inaccessible. So in particular, π_m and π_n agree well beyond κ and we can use the Shift Lemma 2.3.20. Set π_{n+1} to be the shift map of $\langle \pi_m, \pi_n \rangle$ via E_n. Then $\pi_{m+1} : M_{n+1} \to N_{n+1}$ is an elementary embedding and

$$l_{m,n+1} \circ \pi_m = \pi_{n+1} \circ i_{m,n+1}.$$

This and the induction hypothesis imply that

$$\begin{aligned} l_{k,n+1} \circ \pi_k &= l_{m,n+1} \circ l_{k,m} \circ \pi_k = l_{m,n+1} \circ \pi_m \circ i_{k,m} \\ &= \pi_{n+1} \circ i_{m,n+1} \circ i_{k,m} = \pi_{n+1} \circ i_{k,n+1} \end{aligned}$$

for every kTm. Hence (iii) holds. The Shift Lemma also yields that π_n and π_{n+1} agree up to $lh(E_n)$. So π_k and π_{n+1} agree up to $lh(E_k)$ for every $k < n$ by the induction hypothesis and because $\mathcal{T}$ is length-increasing. This shows (iv). $\mathcal{S}$ is an iteration tree of length ω and what is left to show is that $\mathcal{S}$ is nice. $\mathcal{S}$ is length-increasing because every $k < n$ satisfy

$$lh(F_k) = \pi_k(lh(E_k)) = \pi_n(lh(E_k)) < \pi_n(lh(E_n)) = lh(F_n),$$

where we use that $\mathcal{T}$ is length-increasing for the inequality. In order to show that $\mathcal{S}$ is non-overlapping, set $m := pred_T(n+1)$. We will use the characterization of non-overlappingness from Remark 2.5.13 in the following arguments. We have that

$$crit(F_n) = \pi_n(crit(E_n)) = \pi_m(crit(E_n)) < \pi_m(lh(E_m)) = lh(F_m),$$

where we use that $crit(E_n) < lh(E_m)$ in the second and third step. For every $k < m$, we have that

$$crit(F_n) = \pi_n(crit(E_n)) \geq \pi_n(lh(E_k)) = \pi_k(lh(E_k)) = lh(F_k).$$

This shows that m is the least n' such that $crit(F_n) < lh(F_{n'})$. Finally,

$$N_n \models \text{“}F_n \text{ is nice”}$$

because π_n is elementary and $M_n \models$“E_n is nice”. $\square$

Definition 2.5.19. Let $(M, \mathcal{E})$, $(N, \mathcal{F})$, π and $\mathcal{T}$ be as in the previous lemma. We call $\mathcal{S}$ from above the **copy of $\mathcal{T}$ to N along π**, denoted by $\pi\mathcal{T}$, with associated **copy maps** $\langle \pi_n \mid n < \omega \rangle$.

Lemma 2.5.20. *Let M be countable and let $\pi : M \to V_\theta$ be an elementary embedding. Suppose that $\mathcal{T}$ is a counterexample to Theorem 2.5.15. Then $\pi\mathcal{T}$ is a continuously illfounded nice iteration tree of length ω on V.*

Proof. By Lemma 2.5.18, $\pi\mathcal{T}$ is a nice iteration tree of length ω on V. We need to show that $\pi\mathcal{T}$ is continuously illfounded. Denote $\pi\mathcal{T} = (T, \langle F_n \mid n < \omega \rangle)$ and let $\langle N_n \mid n < \omega \rangle$ be the associated models and let $\langle l_{m,n} \mid mTn \rangle$ be the associated embeddings. M is countable, hence M_n is countable for every $n < \omega$ and we enumerate M_n by $\bar{e}^n := \langle e^n_k \mid k < \omega \rangle$. Set $M_n \restriction l := \{e^n_k \mid k < l\}$ and if σ is a function with $dom(\sigma) = M_n$ then $\sigma \restriction l := \sigma \restriction (M_n \restriction l)$. We built the tree R of attempts to create a cofinal branch through T and to create a commuting system of embeddings realizing the models along this branch into V in the following way:
$\langle a, \langle \sigma_k \mid k \in a \rangle \rangle \in R \Leftrightarrow$

(i). a is a finite branch in T and $l := |a|$,

(ii). $\sigma_k : M_k \restriction l \to V$ for each $k \in a$, (iii). $\sigma_0 = \pi \restriction l$ and

(iv). if $k, m \in a$ with kTm and $x \in M_k \restriction l$ such that $i_{k,m}(x) \in M_m \restriction l$ then $\sigma_m(i_{k,m}(x)) = \sigma_k(x)$.

If R had a cofinal branch then its first coordinate b would be a cofinal branch of T. The direct limit of the embeddings in its second coordinate would be an elementary embedding $M_b \to V$. Therefore b would be a wellfounded cofinal branch of $\mathcal{T}$ which does not exist by the assumption. Hence R has no infinite branch. This implies that there is a rank function $\varphi : R \to Ord$, i.e. if $s, t \in R$ are such that s properly extends t then $\varphi(s) < \varphi(t)$. For $n < \omega$, set

$$s_n := ([0, n]_T, \bar{\sigma}^n)$$

where

- $[0, n]_T = \langle 0 = n_0, n_1, \ldots, n_{l-1} = n \rangle$,
- $\bar{\sigma}^n = \langle \sigma_i^n \mid i < l \rangle$ and
- $\sigma_i^n = \pi_n \circ i_{n_i,n} \restriction l$ for each $i < l$.

Claim 1. *$s_n \in l_{0,n}(R)$.*

$$\begin{tikzcd}
V = N_0 \arrow[r] \arrow[rrr, bend left, "l_{0,n}"] & N_{n_i} \arrow[r] & N_{n_{i'}} \arrow[r] & N_n \\
M = M_0 \arrow[u, "\pi"'] \arrow[r] \arrow[rrr, bend right, "i_{0,n}"'] & M_{n_i} \arrow[u] \arrow[r, "i_{n_i,n_{i'}}"] \arrow[rr, bend right, "i_{n_i,n}"'] & M_{n_{i'}} \arrow[u] \arrow[r, "i_{n_{i'},n}"] & M_n \arrow[u, "\pi_n"']
\end{tikzcd}$$

Proof of Claim 1. Since $T \subseteq \omega \times \omega$, $l_{0,n}(T) = T$, and therefore $[0, n]_T$ is a finite branch in $l_{0,n}(T)$. For every $i < l$, we have that

$$\sigma_i^n = \pi_n \circ i_{n_i,n} \restriction l : M_{n_i} \restriction l \to N_n.$$

Condition (iii) is also satisfied because

$$\sigma_0^n = \pi_n \circ i_{0,n} \restriction l = \pi \circ i_{0,n} \restriction l = (l_{0,n} \circ \pi) \restriction l.$$

For the last condition (iv), let $x \in M_{n_i} \restriction l$ such that $i_{n_i,n_{i'}}(x) \in M_{n_{i'}} \restriction l$. Then

$$\begin{aligned}\sigma^n_{i'}(i_{n_i,n_{i'}}(x)) &= (\pi_n \circ i_{n_{i'},n} \restriction l)(i_{n_i,n_{i'}}(x)) \\ &= \pi_n((i_{n_{i'},n} \circ i_{n_i,n_{i'}})(x)) \\ &= \pi_n(i_{n_i,n}(x)) = \sigma^n_i(x).\end{aligned}$$

■

Claim 2. *Let nTn'. Then $s_{n'}$ properly extends $l_{n,n'}(s_n)$.*

Proof of Claim 2. We have that $l_{n,n'}(s_n) = \langle [0,n]_T, \langle l_{n,n'} \circ \sigma^n_i \mid i < l\rangle\rangle$. Clearly $[0,n']_T$ properly extends $[0,n]_T$ and

$$\begin{aligned}l_{n,n'} \circ \sigma^n_i &= l_{n,n'} \circ \pi_n \circ i_{n_i,n} \restriction l \\ &= l_{n,n'} \circ l_{n_i,n} \circ \pi_{n_i} \restriction l \\ &= l_{n_i,n'} \circ \pi_{n_i} = \pi_{n'} \circ i_{n_i,n'} \restriction l = \sigma^{n'}_i.\end{aligned}$$

■

Set $\alpha_n := l_{0,n}(\varphi)(s_n)$. Note that α_n is welldefined since φ is a function in $V = N_0$ with domain R so $l_{0,n}(\varphi)$ is a function in N_n with domain $l_{0,n}(R)$ and $s_n \in l_{0,n}(R)$ by Claim 1. This also shows that $\alpha_n \in N_n$. φ is a rank function, so by Claim 2, we have that $\varphi(s_{n'}) < \varphi(l_{n,n'}(s_n))$. This implies that

$$\begin{aligned}\alpha_{n'} = l_{0,n'}(\varphi(s_{n'})) < l_{0,n'}(\varphi(l_{n,n'}(s_n))) &= (l_{n,n'} \circ l_{0,n})(\varphi(l_{n,n'}(s_n))) \\ &= (l_{n,n'} \circ l_{0,n})((l_{n,n'} \circ \varphi)(s_n)) \\ &= l_{n,n'}(l_{0,n}(\varphi(s_n))) \\ &= l_{n,n'}(\alpha_n).\end{aligned}$$

Hence $\langle \alpha_n \mid n < \omega\rangle$ witnesses that $\pi\mathcal{T}$ is continuously illfounded. □

Lemma 2.5.21. *Let $\mathcal{S} = (T, \langle F_n \mid n < \omega\rangle)$ be a nice iteration tree of length ω on V. Then $\mathcal{S}$ is not continuously illfounded.*

Proof. Let $\langle N_n \mid n < \omega\rangle$ be the models of $\mathcal{S}$ and $\langle l_{k,n} \mid kTn\rangle$ be the embeddings of $\mathcal{S}$. Suppose towards a contradiction that $\mathcal{S}$ is continuously illfounded with witness $\langle \beta^*_n \mid n < \omega\rangle$. Choose η large enough such that $\mathcal{S} \in V_\eta$. First of all, we manipulate the witness for continuously illfoundedness to get some extra properties. Let β_n be the β^*_n'th regular cardinal above $l_{0,n}(\eta)$ in N_n for each $n < \omega$. Then $\langle \beta_n \mid n < \omega\rangle$ witnesses that $\mathcal{S}$ is continuously illfounded. Take $k < n$. Then $l_{k,n}(\beta_k)$ is the $l_{k,n}(\beta^*_k)$'th regular cardinal above $l_{k,n}(l_{0,k}(\eta)) = l_{0,n}(\eta)$ in N_n by the elementarity of $l_{k,n}$. $\beta^*_n < l_{k,n}(\beta^*_k)$ by assumption, so $\beta_n < l_{k,n}(\beta_k)$. Now we have that $\beta_n > l_{0,n}(\eta)$ and that β_n is regular for every $n < \omega$. Choose θ large enough such that $\mathcal{S}, \langle \beta_n \mid n < \omega \rangle \in V_\theta$. Set $H := Hull_{V_\theta}(\{\mathcal{S}, \langle \beta_n \mid n < \omega \rangle\})$ and $M := mos(H)$. Note that M is countable. Let $\pi : M \to V_\theta$ be the anti-collapse embedding. Set

$$\mathcal{T} = (T, \langle E_n \mid n < \omega \rangle) := \pi^{-1}(\mathcal{S})$$

$$\text{and } \langle \alpha_n \mid n < \omega \rangle := \pi^{-1}(\langle \beta_n \mid n < \omega \rangle).$$

Then $\mathcal{T}$ is a nice iteration tree of length ω on M which is continuously illfounded and $\langle \alpha_n \mid n < \omega \rangle$ is a witness. We also have that α_n is regular in M_n, where $\langle M_n \mid n < \omega \rangle$ are the models of $\mathcal{T}$. $E_n \in M_n \cap V_{\alpha_n}$ because $\mathcal{S} \in V_\eta$ and therefore $F_n \in V_{i_{0,n}(\eta)} \subseteq V_{\beta_n}$. The plan is to produce models P_n and embeddings σ_n for each $n < \omega$ such that

(i). $\sigma_n : M_n \cap V_{\alpha_n} \to P_n$ is elementary,

(ii). $\sigma_n \in P_n$ and $P_n \models$"σ_n is countable",

(iii). σ_k and σ_n agree up to $lh(E_k)$ for $k < n$ and

(iv). $P_{n+1} \in P_n$.

If we find such models and embeddings then $\langle P_n \mid n < \omega \rangle$ is an infinite $\in$-decreasing sequence in V which is a contradiction. We construct the models and embeddings by recursion on $n < \omega$.

<u>$n = 0$:</u> Set $P_0 := V_{\beta_0}$ and $\sigma_0 := \pi \restriction (M \cap V_{\alpha_0})$. Note that $\sigma_0 \subseteq V_{\beta_0}$ and σ_0 is countable because M is countable. β_0 is regular, so in particular, $cof(\beta_0) > \omega$ and hence σ_0 is a bounded subset of V_{β_0}. Therefore σ_0 and all bijections between σ and ω are elements of $V_{\beta_0} = P_0$.

<u>$n \to n+1$:</u> Let $m := pred_T(n+1)$. Then $M_{n+1} = Ult(M_m, E_n)$ and we have the following situation.

$$\begin{tikzcd}
P_m & P_n & ? \\
M_m \cap V_{\alpha_m} \arrow[u, "\sigma_m"'] \arrow[rr, bend right, "i^{m,n+1}"'] & E_n \in M_n \cap V_{\alpha_n} \arrow[u, "\sigma_n"'] & M_{n+1} = Ult(M_m, E_n) \arrow[u, dashed]
\end{tikzcd}$$

We would like to shift E_n to P_m via $\langle \sigma_m, \sigma_n \rangle$. But this is not possible since σ_m and σ_n are only partial maps. What we can use is that

$$Ult(M_m \cap V_{\alpha_m}, E_n) = M_{n+1} \cap V_\gamma,$$

where $\gamma := i_{m,n+1}(\alpha_m)$ and $M_m \cap V_{\alpha_m}, M_n \cap V_{\alpha_n}$ agree up to $\lambda_n := lh(E_n)$. Let

- G_n be the shift of E_n (considered as an $(M_n \cap V_{\alpha_n})$-extender) to P_m via $\langle \sigma_m, \sigma_n \rangle$,
- $P_n^* := Ult(P_m, G_n)$ and
- $\sigma_n^* : Ult(M_{n+1}) \cap V_\gamma \to P_n^*$ be the shift map of $\langle \sigma_m, \sigma_n \rangle$ via E_n.

There are two major problems with P_n^* and σ_n^* namely $P_n^* \notin P_n$ and $\sigma_n^* \notin P_n^*$.

Claim. $\tau := \sigma_n^* \restriction (M_{n+1} \cap V_{\lambda_n}) \in P_n^*$.

Proof of Claim. First of all $\lambda_n \leq \gamma$, since $E_n \in V_{\alpha_n}$ implies

$$\lambda_n \leq \alpha_n < j_{m,n}(\alpha_m) < j_{n,n+1}(j_{m,n}(\alpha_m)) = j_{m,n+1}(\alpha_m) = \gamma.$$

By the Shift Lemma 2.3.20, σ_n and σ_n^* agree up to λ_n. Therefore

$$\tau = \sigma_n \restriction (M_{n+1} \cap V_{\lambda_n}).$$

By the induction hypothesis, $\sigma_n \in P_n$. $\lambda_n' := \sigma_n(\lambda_n)$ is inaccessible in P_n because E_n is nice and σ_n is elementary. Therefore $\tau \in P_n \cap V_{\lambda_n'} = P_n^* \cap V_{\lambda_n'}$, where we use that $str^{P_n}(G_n) = \lambda_n'$. ■

Claim. There is an elementary embedding $\sigma_n^{**} : M_{n+1} \cap V_{\alpha_{n+1}} \to P_n^{**}$, where $P_n^{**} := P_n^* \cap V_{\sigma_n^*(\alpha_{n+1})}$ such that

- $\sigma_n^{**} \restriction (M_{n+1} \cap V_{\lambda_n}) = \tau$,
- $\sigma_n^{**}(\lambda_n) = \lambda_n'$ and
- $\sigma_n^{**} \in P_n^*$ and $P_n^* \models$ "σ_n^{**} is countable".

Proof of Claim. First of all, $E_{n+1} \in V_{\alpha_{n+1}}$ implies that $\lambda_n < lh(E_{n+1}) \leq \alpha_{n+1}$, so the restriction of σ_n^{**} to $M_{n+1} \cap V_{\lambda_n}$ makes sense. Let R be the tree of attempts inside P_n^* to construct such a σ_n^{**}. We can do this because $\tau \in P_n^*$ according to the first claim. Then R has an infinite branch in V given by $\sigma_n^* \restriction (M_{n+1} \cap V_{\alpha_{n+1}})$ because $\alpha_{n+1} < i_{m,n+1}(\alpha_m) = \gamma$ and $\sigma_n^*(\lambda_n) = \sigma_n(\lambda_n) = \lambda_n'$ by the Shift Lemma 2.3.20. $\sigma_n^* \restriction (M_{n+1} \cap V_{\alpha_{n+1}})$ is countable since M_{n+1} is countable. Then by absoluteness, there is also an infinite branch inside P_n^*. ■

P_n^{**} and σ_n^{**} satisfy the first three conditions of the plan:

(i). σ_n^{**} is elementary by construction.

(ii). This is essentially the same arguing as in the case $n = 0$. We have that $\sigma_n^{**} \subseteq V_{\sigma_n^*(\alpha_{n+1})}$. M_{n+1} is countable so σ_n^{**} is countable. $\sigma_n^*(\alpha_{n+1})$ is regular in P_n^*. Therefore σ_n^{**} is a bounded subset of $V_{\sigma_n^*(\alpha_{n+1})}$ and all bijections between σ_n^{**} and ω are elements of $V_{\sigma_n^*(\alpha_{\alpha+1})}$. By the construction of σ_n^{**}, we know that $\sigma_n^{**} \in P_n^*$ and there is some bijection $\varphi : \sigma_n^{**} \to \omega$ in P_n^*. Hence $\sigma_n^{**} \in P_n^{**}$ and φ witnesses that $P_n^{**} \models$"σ_n^{**} is countable".

(iii). σ_n and σ_n^{**} agree up to $lh(E_n) = \lambda_n$ by construction. Let $k < n$. Then σ_k and σ_n are constructed such that σ_k and σ_n agree up to $lh(E_k)$. Since $\mathcal{T}$ is length-increasing we have that $lh(E_k) < \lambda_n$. Therefore σ_k and σ_n^{**} agree up to $lh(E_k)$.

We manipulate P_n^{**} and σ_n^{**} a little bit more such that they satisfy (iv), too. $\sigma_n^*(\alpha_{n+1}) \in P_n^*$ so $P_n^{**} = P_n^* \cap V_{\sigma_n^*(\alpha_{n+1})} \in P_n^*$ and P_n^{**} is a strict initial segment of P_n^*. Set

- $H := Hull_{P_n^{**}}((P_n^{**} \cap V_{\lambda'_n}) \cup \{\lambda'_n, \sigma_n^{**}\})$ and
- $P_{n+1} := mos(H)$.
- Let $\Phi : P_{n+1} \to H$ be the anti-collapse embedding and
- set $\sigma_{n+1} := \Phi^{-1}(\sigma_n^{**})$.

Then P_{n+1}, σ_{n+1} satisfy all the conditions that we wanted:

(i). Note that G_0 is a nice extender over V_{β_0} and $\lambda'_0 = str(G_0) > crit(G_0)$. $crit(G_0)$ is measurable so in particular uncountable hence λ'_0 is uncountable. $\mathcal{T}$ is length-increasing so

$$\lambda'_n = \sigma_n(lh(E_n)) > \sigma_n(lh(E_0)) = \sigma_0(lh(E_0)) = \lambda'_0.$$

Therefore λ'_n is uncountable. $\alpha_{n+1} \in M_{n+1}$ is countable because M_{n+1} is countable. Hence $\alpha_{n+1} < \lambda'_n \leq crit(\Phi)$ and

$$dom(\sigma_{n+1}) = \Phi^{-1}(dom(\sigma_n^{**})) = \Phi^{-1}(M_{n+1} \cap V_{\alpha_{n+1}}) = M_{n+1} \cap V_{\alpha_{n+1}}.$$

We also know that the codomain of σ_{n+1} is P_{n+1}. Therefore σ_{n+1} is an elementary embedding $M_{n+1} \cap V_{\alpha_{n+1}} \to P_{n+1}$.

(ii). $\sigma_n^{**} \in H$ and H is an elementary submodel of P_n^{**}. Therefore $H \models$"σ_n^{**} is countable" and $P_{n+1} \models$"σ_{n+1} is countable".

(iii). We have that $\sigma_n^{**}(\lambda_n) = \lambda'_n \in H$ and $crit(\Phi) \geq \lambda'_n$. This implies that σ_{n+1} and σ_n^{**} agree up to λ_n and we saw before that σ_n and σ_n^{**} agree up to λ_n. Hence σ_n and σ_{n+1} agree up to λ_n.

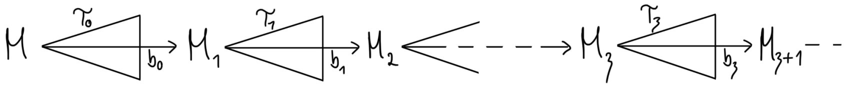
(iv). Since $P_n^{**}, \sigma_n^{**} \in P_n^*$, we have that $P_n^* \models |H| = \lambda_n'$. Therefore P_{n+1} can be coded by some $c \in \mathcal{P}(\lambda_n')^{P_n^*}$. We will show that $c \in P_n$. Recall that $P_n^* := Ult(P_m, G_n)$. M_m and M_n agree well beyond $crit(E_n)$ since $crit(E_n) < lh(E_m)$. Therefore P_m and P_n agree well beyond $crit(G_n)$ and by Proposition 2.3.17 P_n^* and $Ult(P_n, G_n)$ agree well beyond $i_{G_n}^{P_n}(crit(G_n))$. $\lambda_n' = lh(G_n) \leq i_{G_n}^{P_n}(crit(G_n))$ so $c \in \mathcal{P}(\lambda_n')^{P_n^*} = \mathcal{P}(\lambda_n')^{Ult(P_n,G_n)}$. $Ult(P_n, G_n)$ can be computed inside P_n since $G_n \in P_n$. Therefore $\mathcal{P}(\lambda_n')^{Ult(P_n,G_n)} \subseteq P_n$ and in particular $c \in P_n$.

☐

Proof of Theorem 2.5.15. The proof follows directly from Lemma 2.5.20 and Lemma 2.5.21 as discussed before. ☐

Weak Iterability

A weak iteration is a linear composition of iteration trees of length ω and looks like this:



The formal definition is as follows.

Definition 2.5.22. Let β be an ordinal. $\langle \mathcal{T}_\xi, b_\xi \mid \xi < \beta \rangle$ is a **weak iteration on** $(M, \mathcal{E})$ **of length** β iff there are $\langle M^\xi, \mathcal{E}^\xi \mid \xi < \beta \rangle$ and elementary embeddings $\langle i^{\mu,\xi} : M^\mu \to M^\xi \mid \mu < \xi < \beta \rangle$ such that

(i). $(M^0, \mathcal{E}^0) = (M, \mathcal{E})$.

(ii). For each $\xi < \beta$:
$\mathcal{T}_\xi$ is a nice iteration tree of length ω on $(M^\xi, \mathcal{E}^\xi)$,
b_ξ is a cofinal branch through $\mathcal{T}_\xi$,
$(M^{\xi+1}, \mathcal{E}^{\xi+1})$ is the direct limit along b_ξ and
$i^{\xi,\xi+1}$ is the direct limit embedding along b_ξ.

(iii). For each $\lambda < \beta$ limit:
$M^\gamma = dirlim\langle M^\xi, i^{\mu,\xi} \mid \mu < \xi < \gamma \rangle$ and
$i^{\xi,\gamma}$ is the direct limit embedding.

(iv). The remaining embeddings $i^{\mu,\xi}$ are obtained by composition.

Note the upper indices for the M's and i's so that one does not confuse M^n with $M_n^{\mathcal{T}_0}$.

As before, there is also a game corresponding to the construction of a weak iteration.

Definition 2.5.23. The **weak iteration game on** $(M, \mathcal{E})$ is the following two-player-game of length ω_1:
The beginning: $(M^0, \mathcal{E}^0) := (M, \mathcal{E})$.
ξ-th stage: Player I yields a nice iteration tree $\mathcal{T}_\xi$ of length ω on $(M^\xi, \mathcal{E}^\xi)$. Player II chooses a cofinal branch b_ξ through $\mathcal{T}_\xi$. We set $(M^{\xi+1}, \mathcal{E}^{\xi+1})$ to be the direct limit along b_ξ. If $M^{\xi+1}$ is illfounded then the game is over and Player I wins.
γ limit: Set $M^\gamma := dirlim\langle M^\xi, i^{\mu,\xi} \mid \mu < \xi < \gamma\rangle$. If M^γ is illfounded, the game is over and Player I wins.
The end: If they went through all the stages before ω_1 and Player I did not win at those stages then Player II wins.

Definition 2.5.24. We say that $(M, \mathcal{E})$ is **weakly iterable** iff Player II has a winning strategy in the weak iteration game on $(M, \mathcal{E})$.

Lemma 2.5.25 is another weakening of nSBH, where iterability is replaced by weak iterability.

Lemma 2.5.25. *Let* M *be countable and* $\pi : M \to V_\theta$ *be an elementary embedding. Then* $(M, \mathcal{E})$ *is weakly iterable.*

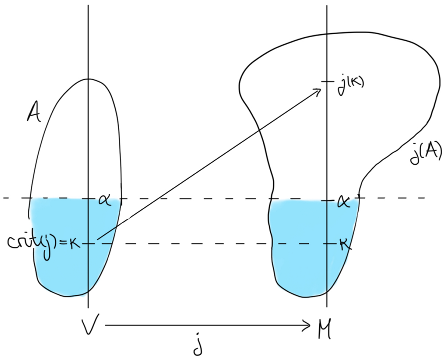
Proof. We need to find a winning strategy for Player II in the weak iteration game. Construct wellfounded branches $\langle b_\xi \mid \xi < \omega_1\rangle$ and elementary embeddings $\pi_\xi : M^\xi \to V_\theta$ by recursion on $\xi < \omega_1$:
$\xi = 0$: $M^0 := M, \pi_0 := \pi$.
$\xi \to \xi + 1$: Player I chooses some nice iteration tree $\mathcal{T}_\xi$ of length ω on M^ξ. π_ξ is elementary and M^ξ is wellfounded and countable. We can use Theorem 2.5.15 which yields a wellfounded cofinal branch b_ξ trough $\mathcal{T}_\xi$ and an elementary embedding

$$\pi_{\xi+1} : M^{\xi+1} \to V_\theta,$$

where $M^{\xi+1} := (M^\xi)_{b_\xi}$. So $M^{\xi+1}$ is wellfounded and countable.
γ limit: M^γ is set to be the direct limit of the system $\langle M^\xi, i^{\mu,\xi} \mid \mu < \xi < \gamma\rangle$ which is countable. We set $\pi_\gamma := dirlim\langle \pi_\xi \mid \xi < \gamma\rangle$. This is an elementary embedding $M^\gamma \to V_\theta$ hence M^γ is wellfounded. □

2.6 Woodin Cardinals

Definition 2.6.1. Let $j : V \to N$ be an elementary embedding. Let A be a set and let α be an ordinal. We say that j is **α-strong for** A iff $\alpha > crit(j)$, $V_\alpha \subseteq N$ and $j(A) \cap V_\alpha = A \cap V_\alpha$.



A V-extender E is **α-strong for** A iff i_E^V is α-strong for A.

The next lemma says that if j is α-strong for A then there is a V-extender which is α-strong for A.

Lemma 2.6.2. *Let $j : V \to N$ with $crit(j) = \kappa$ and let $\alpha > \kappa$ be such that $V_\alpha \subseteq N$ and $j(A) \cap V_\alpha = A \cap V_\alpha$. Let $\lambda := |V_\alpha|^+$ and set $E := E(j, \lambda)$. Then E is α-strong for A.*

Proof. Write i for i_E^V. We first show that $V_\alpha \subseteq Ult(V, E)$.
Use the factor map

$$k : V \to Ult(V, E), [a, f] \mapsto j(f)(a)$$

from Lemma 2.3.8. Set $\beta := |V_\alpha|$, fix a bijection $\Psi : \beta \to V_\alpha$ and define the relation

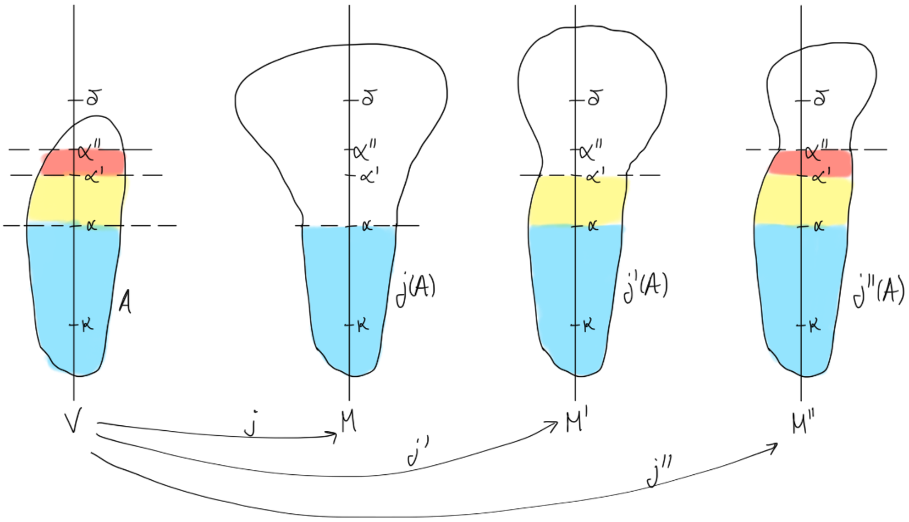
$$R := \{(x, y) \in \beta \times \beta \mid \Phi(x) \in \Phi(y)\}.$$

Since $\alpha, \beta < \lambda$ and $crit(k) \geq \lambda$ by Lemma 2.3.8, we know that $\alpha, \beta \in rng(k)$. Now $R \subseteq \beta^2$ so $R = k^{-1}(R) \in Ult(V, E)$ and therefore $(\beta, R) \subseteq Ult(V, E)$. Hence $V_\alpha \subseteq Ult(V, E)$. In order to show that $i(A) \cap V_\alpha = A \cap V_\alpha$, we use that $crit(k) \geq \lambda$ again:

$$i(A) \cap V_\alpha = k(i(A) \cap V_\alpha) = k(i(A)) \cap V_{k(\alpha)}^N = j(A) \cap V_\alpha = A \cap V_\alpha.$$

The last two equalities holds because j is α-strong for A. $\square$

Definition 2.6.3. Let $\kappa < \delta$ for some ordinals κ and δ and let $A \subseteq V_\delta$. We say that κ **reflects** A **in** δ iff for every $\alpha < \delta$ above κ, there is an α-strong embedding for A with critical point κ.



Definition 2.6.4. Let δ be inaccessible. δ is called **Woodin** iff for each $A \subseteq V_\delta$ there is some $\kappa < \delta$ which reflects A in δ.

Remark 2.6.5. Assume that δ is Woodin. By the previous Lemma 2.6.2, all the witnesses for δ being Woodin can be chosen as extenders. They can even be chosen in V_δ since δ is inaccessible. This is very useful because it yields a formalization of Woodinness as a first order property. So if δ is Woodin in M then there is a set $\mathcal{E} \in M$ witnessing that δ is Woodin in M and we have that $(M, \mathcal{E})$ is a good pair.

3 Using Iteration Trees

So far, we established the basic theory of iteration trees. The first part of this chapter is about developing the method of genericity iteration which heavily uses iteration trees. We will use this method in the second part to show that $L(\mathbb{R})$ is a model of the Axiom of Determinacy. We will work with ZFC instead of ZFC^- in this chapter just to be extra safe. So M will always be a model of ZFC which is transitive and hence wellfounded.

3.1 Genericity Iterations

The results in this chapter are due to W. Hugh Woodin and their presentation here is based on Farah's paper *The extender algebra and* Σ^2_1*-absoluteness*, see [Far16]. δ is an ordinal from now on. We identify $\mathbb{R}$, ω^ω and $\mathcal{P}(\omega)$ as usual in set theory.

Definition 3.1.1. Let $\mathbb{W}$ be a forcing notion and let a be a set. We say that a is **$\mathbb{W}$-generic over** M iff there is some g which is $\mathbb{W}$-generic over M and $a \in M[g]$.

Woodin's Genericity Iteration is a construction which makes a previously fixed real generic over a sufficiently iterable countable structure which has a Woodin cardinal. It uses a specific forcing notion called Extender Algebra which is based on an infinitary propositional logic.

Definition 3.1.2. Let δ be an infinite regular cardinal. $\mathcal{L}_\delta$ is the infinitary propositional logic with countably many variables $\{v_n \mid n < \omega\}$ and standard connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$. The additional connectives are conjunctions and disjunctions of length less than δ denoted by $\bigvee_{\xi<\beta}$ and $\bigwedge_{\xi<\beta}$ for each $\beta < \delta$. Besides the usual axioms and rules for classical propositional logic, we have

- $\vdash \bigvee_{\xi<\beta} \neg\Phi_\xi \leftrightarrow \neg \bigwedge_{\xi<\beta} \Phi_\xi$,
- $\vdash \bigwedge_{\xi<\beta} \Phi_\xi$ then $\vdash \Phi_\xi$ for all $\xi < \beta$ and
- if $\vdash \Phi_\xi$ for every $\xi < \beta$ then $\vdash \bigwedge_{\xi<\beta} \Phi_\xi$.

For a set $a \subseteq \omega$, and $n < \omega$ we set $a \models v_n$ iff $n \in a$. $a \models \Phi$ for arbitrary $\Phi \in \mathcal{L}_\delta$ is defined by recursion as expected. We code an $\mathcal{L}_\delta$-formula Φ as a set in V_δ by $A_\Phi := \{a \in \mathbb{R} \mid a \models \Phi\}$. This language is usually denoted by $\mathcal{L}_{\delta,\omega}$, where the second index indicates the number of variables. We will always have countably many variables and therefore omit the second index.

We define a theory in this language which depends on a Woodin cardinal and its witnesses. This theory yields a Boolean algebra which can be used as a forcing poset.

Definition 3.1.3. Let $\mathcal{E}$ be a set of extenders witnessing that δ is a Woodin cardinal. We define the $\mathcal{L}_\delta$-theory $\mathbb{T}_\delta(\mathcal{E})$:
If $E \in \mathcal{E}$, $crit(E) = \kappa$, $\vec{\Phi} = \langle \Phi_\xi \mid \xi < \delta \rangle \subseteq \mathcal{L}_\delta$ such that $\vec{\Phi} \restriction \kappa \subseteq V_\kappa$ and E is λ-strong for $\vec{\Phi}$. Then

$$\bigvee_{\xi<\kappa} \Phi_\xi \leftrightarrow \bigvee_{\xi<\lambda} \Phi_\xi \in \mathbb{T}_\delta(\mathcal{E}).$$

$\mathbb{T}_\delta(\mathcal{E})$ is the deductive closure of those sentences in $\mathcal{L}_\delta$. Note that

$$\bigvee_{\xi<\lambda} \Phi_\xi = i_E(\bigvee_{\xi<\kappa} \Phi_\xi) \restriction \lambda$$

since E is λ-strong for $\vec{\Phi}$.
The Lindenbaum algebra of $\mathbb{T}_\delta(\mathcal{E})$ is defined in the following way:
Let $\sim$ be the equivalence relation on $\mathcal{L}_\delta$, where

$$\Phi \sim \Phi' \text{ iff } \mathbb{T}_\delta(\mathcal{E}) \vdash \Phi \leftrightarrow \Phi'.$$

Denote the equivalence classes by $[\Phi]$. Set

$$0 := [\bot], 1 := [\top], [\Phi] \wedge [\Phi'] := [\Phi \wedge \Phi'], [\Phi] \vee [\Phi'] := [\Phi \vee \Phi'], \text{ and } \neg[\Phi] = [\neg\Phi].$$

This yields a Boolean algebra

$$\mathbb{W}_\delta(\mathcal{E}) := \big(\{[\Phi] \mid \Phi \in \mathcal{L}_\delta\}, 0, 1, \wedge, \vee, \neg\big)$$

which we call the **extender algebra**. Whenever the set of witnesses $\mathcal{E}$ is understood from the context, we will write $\mathbb{W}_\delta$ instead $\mathbb{W}_\delta(\mathcal{E})$. We often interpret $\mathbb{W}_\delta(\mathcal{E})$ as a forcing notion using the following partial order on the non-zero elements:

$$[\Phi] \leq [\Phi'] \text{ iff } \mathbb{T}_\delta(\mathcal{E}) \cup \{\Phi\} \vdash \Phi'.$$

Lemma 3.1.4. *$\mathbb{W}_\delta$ has the δ-c.c.*

Proof. Let $\vec{\Phi} = \langle \Phi_\xi \mid \xi < \delta \rangle \subseteq \mathcal{L}_\delta$ such that $[\Phi_\xi] \neq 0$ for each $\xi < \delta$. We want to show that $\{[\Phi_\xi] \mid \xi < \delta\}$ is not an antichain. Set

$$C := \{\beta < \delta \mid \vec{\Phi} \restriction \beta \subseteq V_\beta\}.$$

We claim that C is club in δ. That C is closed is immediate from its definition. In order to see that C is unbounded, we fix $\alpha_0 < \delta$ and we will find some $\beta \geq \alpha_0$ such that $\beta \in C$. Note that $\vec{\Phi} \subseteq V_\delta$ and $\vec{\Phi} \restriction \alpha_0$ is a bounded subset of V_δ since $\alpha_0 < \delta$. δ is regular, so we can find some $\alpha_1 < \delta$ such that $\vec{\Phi} \restriction \alpha_0 \subseteq V_{\alpha_1}$. We cannot use this α_1 for β because there might be some $\xi < \alpha_1$ such that $\Phi_\xi \notin V_{\alpha_1}$.

But we can "catch our tail" by repeating this. Let $1 \leq n < \omega$ and assume that $\alpha_n < \delta$ is already defined such that $\vec{\Phi} \restriction \alpha_{n-1} \subseteq V_{\alpha_n}$. Since $\alpha_n < \delta$ and δ is regular, there is some α_{n+1} such that $\vec{\Phi} \restriction \alpha_n \subseteq V_{\alpha_{n+1}}$. Set $\beta := sup_{n<\omega}\alpha_n$. If $\xi < \beta$ then there is some $n < \omega$ such that $\xi < \alpha_n$ and by the construction $\Phi_\xi \in V_{\alpha_{n+1}} \subseteq V_\beta$. This shows that $\beta \in C$ and hence C is club in δ.

$\mathcal{E}$ witnesses that δ is a Woodin cardinal, so we can pick an extender $E \in \mathcal{E}$ such that E is λ-strong for $\langle C, \vec{\Phi}\rangle$ and $[\kappa, \lambda) \cap C \neq \emptyset$, where $\kappa := crit(E) < \delta$. We use E to show that $\kappa \in C$. It is enough to show that for every $\alpha < \kappa$, there is some $\beta \in C$ such that $\alpha < \beta < \kappa$ because C is closed under limits. Set $i := i^V_E$. We have that

$$Ult(V, E) \models \exists \beta \big(\alpha < \beta < i(\kappa) \wedge \beta \in C \cap \lambda \big)$$

because every element of $[\kappa, \lambda) \cap C$ is a witness. E is λ-strong for C, i.e. $i(C) \cap V_\lambda = C \cap V_\lambda$. In particular, $i(C) \cap \lambda = C \cap \lambda$. Therefore

$$Ult(V, E) \models \exists \beta \big(\alpha < \beta < i(\kappa) \wedge \beta \in i(C) \big).$$

We use the elementarity of i and $\alpha < \kappa = crit(i)$ to see that

$$V \models \exists \beta \big(\alpha < \beta < \kappa \wedge \beta \in C \big).$$

So we have that $\Phi \restriction \kappa \subseteq V_\kappa$. By the definition of the theory,

$$(\bigvee_{\xi<\kappa} \Phi_\xi \leftrightarrow \bigvee_{\xi<\lambda} \Phi_\xi), (\Phi_\kappa \rightarrow \bigvee_{\xi<\lambda} \Phi_\xi) \in \mathbb{T}_\delta(\mathcal{E}).$$

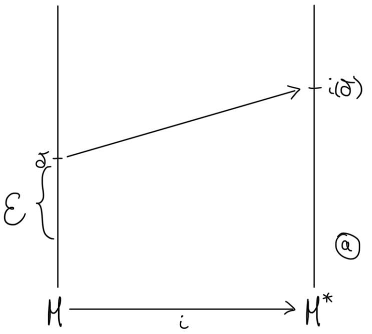
Therefore $(\Phi_\kappa \rightarrow \bigvee_{\xi<\kappa} \Phi_\xi) \in \mathbb{T}_\delta(\mathcal{E})$ and $[\Phi_\kappa] \leq [\bigvee_{\xi<\kappa} \Phi_\xi]$. We claim that there is some $\xi < \kappa$ such that $[\Phi_\xi]$ and $[\Phi_\kappa]$ are compatible. Suppose toward a contradiction that $[\Phi_\xi]$ and $[\Phi_\kappa]$ are not compatible for every $\xi < \kappa$. Then $T \cup \{\Phi_\xi \wedge \Phi_\kappa\} \vdash \bot$ (otherwise we would have $0 \neq [\Phi_\xi \wedge \Phi_\kappa] \leq [\Phi_\xi], [\Phi_\kappa]$). Compute that

$$\begin{aligned} & T \cup \{\Phi_\xi \wedge \Phi_\kappa\} \vdash \bot \text{ for each } \xi < \kappa, \\ & \quad \Rightarrow T \cup \{\Phi_\kappa\} \vdash \neg\Phi_\xi \text{ for each } \xi < \kappa, \\ & \quad \Rightarrow T \cup \{\Phi_\kappa\} \vdash \bigwedge_{\xi<\kappa} \neg\Phi_\xi \text{ and} \\ & \quad \Rightarrow T \cup \{\Phi_\kappa\} \vdash \neg \bigvee_{\xi<\kappa} \Phi_\xi. \end{aligned}$$

But we showed that $[\Phi_\kappa] \leq [\bigvee_{\xi<\kappa} \Phi_\xi]$, i.e. $T \cup \{\Phi_\kappa\} \vdash \bigvee_{\xi<\kappa} \Phi_\xi$, which is a contradiction. ☐

We have all the ingredients to formulate Woodin's Genericity Iteration.

Theorem 3.1.5. *(Woodin's Genericity Iteration) Let* $a \in \mathbb{R}$. *Suppose that* M *is countable and* $(M, \mathcal{E})$ *is a good pair such that* $\mathcal{E}$ *witnesses that* δ *is Woodin in* M. *Assume that* $(M, \mathcal{E})$ *is* $(\omega_1 + 1)$*-iterable and every extender in* $\mathcal{E}$ *is nice in* M. *Then there is a countable iteration* $i : M \to M^*$ *such that* a *is* $i(\mathbb{W}_\delta(\mathcal{E}))$*-generic over* M^*.



The next lemmas will be very useful for the proof of Theorem 3.1.5.

Lemma 3.1.6. *Let* a, $(M, \mathcal{E})$ *and* δ *be as in Theorem 3.1.5. If* $a \models \mathbb{T}_\delta(\mathcal{E})^M$ *then* a *is* $\mathbb{W}_\delta$*-generic over* M.

Proof. Set $h_a := \{[\Phi] \in \mathbb{W}_\delta \mid a \models \Phi\} \subseteq \mathbb{W}_\delta$. We claim that h_a is M-generic. If this is true then $a = \{n < \omega \mid [v_n] \in h_a\} \in M[h_a]$. It is easy to see that h_a is a filter, so we only need to show that h_a meets every maximal antichain in $\mathbb{W}_\delta$. We know that $\mathbb{W}_\delta$ has the δ-c.c. from Lemma 3.1.4, so it suffices to consider antichains of size less than δ. Let $A = \{[\Phi_\xi] \mid \xi < \beta\} \in M$ be an antichain in $\mathbb{W}_\delta$ for some $\beta < \delta$. Then $\mathbb{T}_\delta(\mathcal{E}) \vdash \bigvee_{\xi<\beta} \Phi_\xi$ since otherwise $A \cup \{[\bigwedge_{\xi<\beta} \neg\Phi_\xi]\}$ would be an antichain in $\mathbb{W}_\delta$. By the assumption, we have that $a \models \mathbb{T}_\delta(\mathcal{E})$. Therefore $a \models \bigvee_{\xi<\beta} \Phi_\xi$ and there is some $\xi < \beta$ such that $a \models \Phi_\xi$. Hence $[\Phi_\xi] \in h_a \cap A$. ☐

Lemma 3.1.7. *Let* $(M, \mathcal{E})$ *be a good pair and let* M *be countable. Suppose that* $\mathcal{T} = \langle T, M_\xi, E_\eta \mid \xi \leq \omega_1, \eta < \omega_1\rangle$ *is an iteration tree on* $(M, \mathcal{E})$ *with cofinal branch* b. *Assume that* H *is a countable elementary substructure of* V_θ, *where* θ *is large enough and* $\mathcal{T} \in H$. *Let* $\bar{H} := mos(H)$ *and* $\pi : \bar{H} \to V_\theta$ *be the anti-collapse embedding. Set* $\alpha := H \cap \omega_1$. *Then*

(i). $\alpha \in b$,

(ii). π *and* $i^{\mathcal{T}}_{\alpha,\omega_1}$ *agree on* $M_\alpha \cap \bar{H}$ *and*

(iii). $crit(i^{\mathcal{T}}_{\alpha,\omega_1}) = \alpha$.

Proof. Let $\bar{\mathcal{T}} \in \bar{H}$ be such that $\pi(\bar{\mathcal{T}}) = \mathcal{T}$. Note that π does not move countable ordinals, i.e. $\pi(\beta) = \beta$ for each $\beta < \alpha$. But $\pi(\alpha) = \omega_1$ because α is the set of countable ordinals in $\bar{H}$. Hence $crit(\pi) = \alpha$, $\alpha = \omega_1^{\bar{H}}$ and $\pi(\alpha) = \omega_1$.

(i). We know that $\mathcal{T} \upharpoonright \alpha = \bar{\mathcal{T}} \upharpoonright \alpha$ and $[0, \alpha)_{\bar{T}} = b \cap \alpha$. α is a limit ordinal, hence $[0, \alpha)_{\bar{T}}$ is cofinal in α by the definition of a tree order. Therefore $b \cap \alpha$ is cofinal in α. Since any branch of an iteration tree is closed below its supremum, we have that $\alpha \in b$.

(ii). Let $x \in M_\alpha \cap \bar{H}$. α is a limit ordinal, so $M_\alpha = dirlim\langle M_\xi, i^{\mathcal{T}}_{\mu,\xi} \mid \mu T \xi T \alpha\rangle$. Pick some $\xi < \alpha$ and $y \in M_\xi$ such that $x = i^{\mathcal{T}}_{\xi,\alpha}(y) = i^{\bar{\mathcal{T}}}_{\xi,\alpha}(y)$, where the last equality holds because $\mathcal{T} \upharpoonright (\alpha+1) = \bar{\mathcal{T}}$ by (i). M_ξ is countable because M is countable. Therefore $\pi(y) = y$. We compute

$$\pi(x) = \pi(i^{\bar{\mathcal{T}}}_{\xi,\alpha}(y)) = i^{\pi(\bar{\mathcal{T}})}_{\pi(\xi),\pi(\alpha)}(\pi(y)) = i^{\mathcal{T}}_{\xi,\omega_1}(y) = i^{\mathcal{T}}_{\alpha,\omega_1}(i^{\mathcal{T}}_{\xi,\alpha}(y)) = i^{\mathcal{T}}_{\alpha,\omega_1}(x).$$

(iii). $\alpha + 1 \subseteq M_\alpha \cap \bar{H}$ and $crit(\pi) = \alpha$. Hence (ii) implies that $crit(i^{\mathcal{T}}_{\alpha,\omega_1}) = \alpha$.

□

Proof of Theorem 3.1.5. The idea is to let Player I and Player II play a round of the nice iteration game of length $\omega_1 + 1$ on $(M, \mathcal{E})$ with resulting embedding $i : M \to M^*$. Player II uses his (ω_1+1)-iteration strategy, so we know that M^* is wellfounded. We let Player I use a special strategy which assures that $a \models i(\mathbb{T})^{M^*}$, where $\mathbb{T} := \mathbb{T}_\delta(\mathcal{E})^M$. We have that $i(\mathbb{T})^{M^*} = i(\mathbb{T}_\delta(\mathcal{E})^M) = \mathbb{T}_{i(\delta)}(i(\mathcal{E}))^M$ and therefore Lemma 3.1.6 implies that a is $i(\mathbb{W}_\delta)$-generic over M^*. In the end, we will prove that the construction actually stopped at a countable stage.

Let's start with the strategy for Player I. Let $M_0 := M$. Assume that $\alpha < \omega_1$ and

$$\langle T \upharpoonright \alpha, M_\xi, E_\mu \mid \xi \leq \alpha, \mu < \alpha\rangle$$

have already been constructed. Let

$$\langle i_{\mu,\xi} \mid \mu T \xi \leq \alpha\rangle$$

be the associated embeddings. If $a \models i_{0,\alpha}(\mathbb{T})^{M_\alpha}$ then I stops the construction. Otherwise there is a counterexample, i.e. there is

- an extender $E \in \mathcal{E}_\alpha$ with $\kappa := crit(E)$ and
- a δ^α-sequence of $\mathcal{L}_{\delta^\alpha}$-formulas $\vec{\Phi}$, where $\delta^\alpha := i_{0,\alpha}(\delta)$,

such that

- $\vec{\Phi} \upharpoonright \kappa \subseteq V_\kappa$,

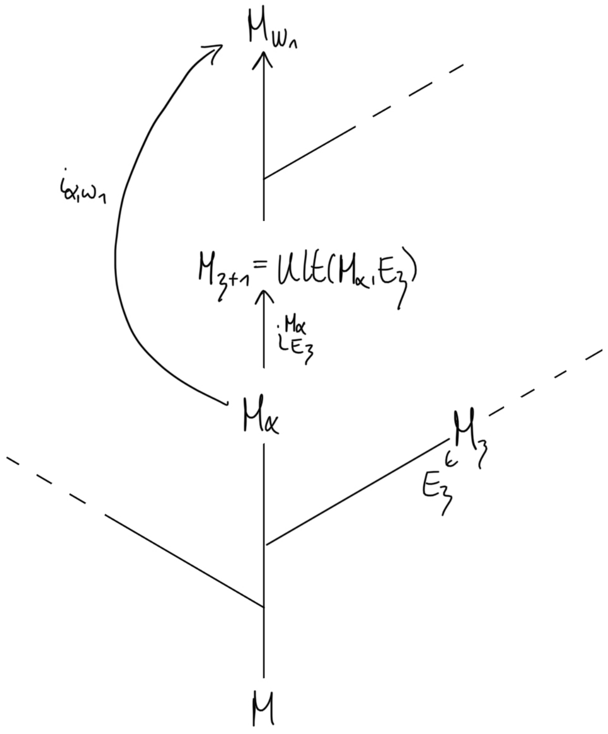
- E is λ-strong for $\vec{\Phi}$,
- $a \models i_E^{M_\alpha}(\bigvee_{\xi<\kappa} \Phi_\xi) \restriction \lambda$ and
- $a \not\models \bigvee_{\xi<\kappa} \Phi_\xi$.

Let λ be minimal such that there is a δ^α-sequence Φ and an extender E which is λ-strong for Φ and every condition of being a counterexample is satisfied. Let E_α be any such E.

Now let the two players play the nice iteration game of length $(\omega_1 + 1)$ on $(M, \mathcal{E})$. Player I uses the strategy described above and Player II uses his $(\omega_1 + 1)$-iteration strategy. Let

$$\mathcal{T} := \langle T, M_\xi, E_\mu \mid \xi \leq \beta, \mu < \beta \rangle$$

be the resulting tree. Note that $\beta \leq \omega_1$. Suppose towards a contradiction that $\beta = \omega_1$ and let b be a cofinal branch in $\mathcal{T}$. Fix a countable elementary substructure H of V_θ for a large enough θ such that H contains everything relevant. Let $\bar{H} := mos(H)$ and let $\pi : \bar{H} \to V_\theta$ be the anti-collapse map. Set $\alpha := H \cap \omega_1$. α is countable since H is countable. By Lemma 3.1.7, we know that $\alpha \in b$ and since $\alpha < \omega_1$, there is some $\xi \geq \alpha$ such that $M_{\xi+1} = Ult(M_\alpha, E_\xi)$.



Lemma 3.1.7 also shows that $crit(E_\xi) = crit(i^{M_\alpha}_{E_\xi}) = crit(i_{\alpha,\omega_1}) = \alpha$. Player I used her strategy to find E_ξ, so there is a δ^ξ-sequence of $\mathcal{L}_{\delta^\xi}$-formulas $\vec{\Phi} \in M_\xi$ with $\vec{\Phi} \upharpoonright \alpha \subseteq V_\alpha$ and E_ξ is λ-strong for $\vec{\Phi}$ such that

$$a \models i^{M_\alpha}_{E_\xi}(\bigvee_{\xi<\alpha} \Phi_\xi) \upharpoonright \lambda \text{ and} \tag{3.1}$$

$$a \not\models \bigvee_{\xi<\alpha} \Phi_\xi. \tag{3.2}$$

Since $\bigvee_{\xi<\alpha} \Phi_\xi \in M_\xi \cap V_{\alpha+1} = M_\alpha \cap V_{\alpha+1}$, we know that

$$i^{M_\alpha}_{E_\xi}(\bigvee_{\xi<\alpha} \Phi_\xi) = i_{\alpha,\omega_1}(\bigvee_{\xi<\alpha} \Phi_\xi).$$

Therefore

$$i^{M_\alpha}_{E_\xi}(\bigvee_{\xi<\alpha} \Phi_\xi) \upharpoonright \lambda = i_{\alpha,\omega_1}(\bigvee_{\xi<\alpha} \Phi_\xi) \upharpoonright \lambda = \pi(\bigvee_{\xi<\alpha} \Phi_\xi) \upharpoonright \lambda, \tag{3.3}$$

where we use Lemma 3.1.7 for the last equality. Now (3.1) and (3.3) imply that

$$a \models \pi(\bigvee_{\xi<\alpha} \Phi_\xi) \upharpoonright \lambda$$

and since $\pi(\bigvee_{\xi<\alpha} \Phi_\xi)$ is a disjunction, also

$$a \models \pi(\bigvee_{\xi<\alpha} \Phi_\xi).$$

a is a real, so $a = \pi(a)$ and therefore the elementarity of π implies that

$$a \models \bigvee_{\xi<\alpha} \Phi_\xi,$$

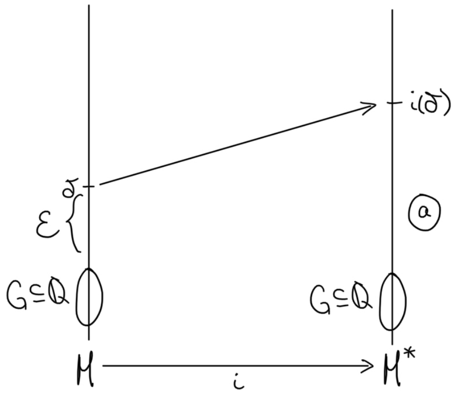
which is a contradiction to (3.2). This shows that the length β of the iteration tree $\mathcal{T}$ is strictly below ω_1. So Player I stopped the construction at a countable stage and she would only do that if $a \models i_{0,\beta}(\mathbb{T})^{M_\beta}$. Therefore, we can use $i := i_{0,\beta}$ and $M^* := M_\beta$ and we are done by the argument in the beginning of the proof. □

The result of Woodin's Genericity Iteration 3.1.5 can be slightly improved. We want to be able to use it while preserving another forcing extension which "happens below" the Genericity Iteration. Additionally, we can replace the forcing $\mathbb{W}_\delta$ by $Col(\omega, \delta)$.

Definition 3.1.8. Let $\mathcal{E}$ be a set of extenders. A forcing notion $\mathbb{Q}$ is called

small relative to $\mathcal{E}$ iff there is an ordinal ν such that $\mathbb{Q} \in V_\nu$ and $crit(E) > \nu$ for every $E \in \mathcal{E}$.

Corollary 3.1.9. *(Improved Genericity Iteration 1) Let $a \in \mathbb{R}$. Suppose that M is countable and $(M, \mathcal{E})$ is a good pair such that $\mathcal{E}$ witnesses that δ is Woodin in M. Suppose that $(M, \mathcal{E})$ is $(\omega_1 + 1)$-iterable. Let $\mathbb{Q} \in M$ be a small forcing relative to $\mathcal{E}$ and let $G \subseteq \mathbb{Q}$ in V be M-generic. Then there is a countable iteration $i : M \to M^*$ such that a is $i(\mathbb{W}_\delta(\mathcal{E}))$-generic over $M^*[G]$.*



Proof. The critical point of i will be above the rank of $\mathbb{Q}$ because i will be built with the extenders in $\mathcal{E}$ and those have critical points above the rank of $\mathbb{Q}$. Therefore $G \subseteq \mathbb{Q}$ is also generic over M^*. The property that $\mathbb{Q}$ is small relative to $\mathcal{E}$ also yields that every extender $E \in \mathcal{E}$ defines an extender in $M[G]$ which has the same strength as E. We will abuse the notation and denote this extender by E, too. Then $\mathcal{E}$ witnesses that δ is Woodin in $M[G]$. These results are discussed in [HW00]. Then the proof is almost the same as the proof of Theorem 3.1.5. The only difference is that in the strategy for Player I she computes the extender E and the sequence $\vec{\Phi}$ in $M[G]$ instead of M. $G \in V$ so this is still a strategy for Player I. Note that this produces an iteration tree on $(M[G], \mathcal{E})$. Since $\mathbb{Q}$ is small relative to $\mathcal{E}$, this iteration tree is also an iteration tree on $(M, \mathcal{E})$. $\square$

Corollary 3.1.10. *(Improved Genericity Iteration 2) Let $a \in \mathbb{R}$. Suppose that M is countable and $(M, \mathcal{E})$ is a good pair such that $\mathcal{E}$ witnesses that δ is Woodin in M. Suppose that $(M, \mathcal{E})$ is $(\omega_1 + 1)$-iterable. Fix $q \in Col(\omega, \delta)$. Let $\mathbb{Q} \in M$ be a small forcing relative to $\mathcal{E}$ and let $G \subseteq \mathbb{Q}$ in V be M-generic. Then there is a countable iteration $i : M \to M^*$ and there is some $H \subseteq Col(\omega, i(\delta))$ which is generic over $M^*[G]$ such that $i(q) \in H$ and $a \in M^*[G][H]$.*

Proof. Let $i : M \to M^*$ be the countable iteration from Corollary 3.1.9, set $\delta^* := i(\delta)$, and let $h \subseteq \mathbb{W}_{\delta^*}$ be generic over $M^*[G]$ such that $a \in M^*[G][h]$. J. Cummings describes in Chapter 14 of [Cum10] how to absorb a forcing of size at most δ^* into $Col(\omega, \delta^*)$. We know that $\mathbb{W}_{\delta^*}$ has size exactly δ^*. Therefore $\mathbb{W}_{\delta^*}$ can be completely embedded into $Col(\omega, \delta^*)$. Set $q^* := i(q)$. We have that $Col(\omega, \delta^*)/q^*$ is isomorphic to $Col(\omega, \delta^*)$. Hence there is a complete embedding

$$k : \mathbb{W}_{\delta^*} \to Col(\omega, \delta^*)/q^*.$$

Choose $h' \subseteq Col(\omega, \delta^*)/q^*$ such that $M^*[G][h] \subseteq M^*[G][h']$. Every $M^*[G]$-generic in $Col(\omega, \delta^*)/q^*$ corresponds to some $M^*[G]$-generic in $Col(\omega, \delta^*)$ containing q^*, i.e. there is some $H \subseteq Col(\omega, \delta^*)$ such that $q^* \in H$ and $M^*[G][H] = M^*[G][h']$. H also satisfies that $a \in M^*[G][H]$ because $a \in M^*[G][h] \subseteq M^*[G][H]$. $\square$

3.2 AD in L($\mathbb{R}$)

In this chapter we will use genericity iterations to prove the consistency of the Axiom of Determinacy. The statement is that $L(\mathbb{R})$ is a model of the Axiom of Determinacy under large cardinal assumptions. The presented proof is a variation of the last section in [Nee10]. We identify $\mathbb{R}$ with ω^ω and $\mathcal{P}(\omega)$.

Definition 3.2.1. (Axiom of Determinacy) For a set $A \subseteq \mathbb{R}$ we define the game $G_\omega(A)$ in the following way. Player I chooses a natural number $x_0 \in \omega$. Then Player II chooses $y_0 \in \omega$. It's Player I's turn again and she chooses $x_1 \in \omega$ and so on. We end up with

Player I	x_0		x_1		...	
Player II		y_0		y_1		...

We say that **Player I wins** iff $(x_0, y_0, x_1, y_1, \dots) \in A$ and otherwise **Player II wins**. A **strategy for Player I** is a function $\sigma : \bigcup_{k\in\omega} \omega^{2k} \to \omega$. Set

$$\sigma \star y := (\sigma(\emptyset), y_0, \sigma(\sigma(\emptyset), y_0), y_1, \dots)$$

for $y = (y_0, y_1, \dots) \in \omega^\omega$. The strategy tells I what she should do in every step based on the previous choices of I and II. σ is a **winning strategy** iff $\sigma \star y \in A$ for every $y \in \omega^\omega$, i.e. whenever Player I plays according to σ, she will win. (Winning) strategies for Player II are defined analogously. We say that A is **determined** iff there is a winning strategy for one of the players. The **Axiom of Determinacy (**AD**)** says that every set of reals is determined.

Definition 3.2.2. Let Q be any model. We say that a set $X \subseteq Q$ is **definable over** Q iff there is an $\mathcal{L}_\in$-formula φ and elements $a_1, \dots, a_n \in Q$ for some $n < \omega$ such that

$$X = \{x \in Q \mid Q \models \varphi[x, a_1, \dots, a_n]\}.$$

Set

$$\begin{aligned} L_0(\mathbb{R}) &:= \mathbb{R}, \\ L_{\alpha+1}(\mathbb{R}) &:= \{X \subseteq L_\alpha \mid X \text{ is definable over } L_\alpha\} \text{ and} \\ L_\gamma(\mathbb{R}) &:= \bigcup_{\alpha<\gamma} L_\alpha(\mathbb{R}) \text{ for every limit ordinal } \gamma. \end{aligned}$$

$L(\mathbb{R}) := \bigcup_{\alpha \in On} L_\alpha(\mathbb{R})$ is the smallest model of ZF which contains $\mathbb{R} \cup On$.

This Axiom of Determinacy came up in the early 60's and it implies regularity properties for sets of reals, e.g. the Baire property, perfect set property or Lebesgue measurability. A lot of machinery was developed until Woodin showed in the mid 80s that $L(\mathbb{R})$ is a model of AD under large cardinal assumptions. I am going to prove a similar result using Woodin's Genericity Iterations.We need a slightly stronger iterability assumption than $(\omega_1 + 1)$-iterability.

Definition 3.2.3. Let $\mathcal{E} \in M$. We say that $(M, \mathcal{E})$ is $(2, \omega_1 + 1)$**-iterable** iff $(M, \mathcal{E})$ is $(\omega_1 + 1)$-iterable and every iteration $i : M \to M^*$ on $(M, \mathcal{E})$ of length $< \omega_1$ satisfies that $(M^*, i(\mathcal{E}))$ is also $(\omega_1 + 1)$-iterable.

Theorem 3.2.4. *Suppose that M is countable, $\vec{\delta} = \langle \delta_n \mid n < \omega \rangle$ is a strictly increasing sequence of ordinals of M and $\kappa > sup(\vec{\delta})$ is an ordinal in M. Suppose that $\langle \mathcal{E}_n \mid n < \omega \rangle \in M$, where $\mathcal{E}_n$ witnesses that δ_n is Woodin in M for each $n < \omega$ and $U \in M$ witnesses that κ is measurable in M. Assume that $(M, \bigcup_{n<\omega} \mathcal{E}_n \cup \{U\})$ is $(2, \omega_1 + 1)$-iterable. Then $L(\mathbb{R}) \models AD$.*

We will make use of the fact that the statement AD is of a very specific form.

Definition 3.2.5. Let $R \subseteq \mathbb{R}$. A $\Sigma_1(R)$**-statement** is a statement of the form

$$(\exists\, A \subseteq R)\, \psi[A, x_1, \ldots, x_n]$$

for some $x_1, \ldots, x_n \in R$ and some Δ_0-formula ψ, i.e. ψ only has $\forall x \in R$, $\exists x \in R$ as quantifiers.

Proposition 3.2.6. *$\neg AD$ is a $\Sigma_1(\mathbb{R})$-statement.*

Proof. $\neg AD$ says $(\exists\, A \subseteq \mathbb{R})$"$A$ is not determined". Note that a strategy for $G_\omega(A)$ can be coded as a real. "A is determined" can be expressed as the following Δ_0-statement:

$$\exists\, \sigma \in \mathbb{R}\, \forall\, y \in \mathbb{R}\, (\sigma \star y \in A).$$

Hence "A is not determined" is also a Δ_0-statement. $\square$

Definition 3.2.7. Let $R \subseteq \mathbb{R}$. We say $L_\alpha(R)$ is an **R-initial segment of** $L_\beta(R)$ iff $\alpha \leq \beta$ and $\mathbb{R}^{L_\alpha(R)} = \mathbb{R}^{L_\beta(R)} = R$.

It it easy to see the following.

Proposition 3.2.8. *("$\Sigma_1(R)$ goes up") If $L_\alpha(R)$ is an R-initial segment of $L_\beta(R)$ and φ is a $\Sigma_1(R)$-statement such that φ is true in $L_\alpha(R)$. Then φ is also true in $L_\beta(R)$.*

We will often use collapse forcing and introduce abbreviations for better readability.

Definition 3.2.9. For a finite sequence of ordinals $S = \langle s_0, \dots, s_k \rangle$ we let $\mathbb{Q}_S$ be the product $Col(\omega, s_0) \times \cdots \times Col(\omega, s_k)$. If $S = \langle s_n \mid n < \omega \rangle$ is a countable sequence of ordinals then $\mathbb{Q}_S$ denotes the finite support product $Col(\omega, s_0) \times Col(\omega, s_1) \times \dots$

Note that for an elementary embedding j we have that $j(\mathbb{Q}_S) = \mathbb{Q}_{j(S)}$ for finite and countable sequences S.

Definition 3.2.10. Let $\vec{\delta} = \langle \delta_n \mid n < \omega \rangle$ be a strictly increasing sequence of Woodin cardinals in M. Let $G = \langle g_n \mid n < \omega \rangle \subseteq \mathbb{Q}_{\vec{\delta}}$ be generic over M. Set $G \restriction d := g_0 \times \dots g_{d-1} \in \mathbb{Q}_{\langle \delta_0, \dots, \delta_{d-1} \rangle}$ for each $d < \omega$. Define $(\mathbb{R}^*)^{M[G]} := \bigcup_{n<\omega} \mathbb{R}^{M[G \restriction n]}$ called the **symmetric reals of M induced by G** and $Der(M,G) := L_{M \cap On}\big((\mathbb{R}^*)^{M[G]}\big)$ called the **derived model of M induced by G**.

Remark 3.2.11. Let $\vec{\delta}$ and G be as in the previous definition. Let $\vec{x}$ be a finite sequence of elements of $M[G \restriction d]$ for some $d < \omega$ and fix an $\mathcal{L}_\in$-formula φ. The forcing notion $\mathbb{Q}_{\langle \delta_d, \delta_{d+1}, \dots \rangle}$ is homogeneous. So if $\varphi[\vec{x}]$ holds in $M\big((\mathbb{R}^*)^{M[G]}\big)$, i.e. the minimal model containing M and $(\mathbb{R}^*)^{M[G]}$, then $\varphi[\vec{x}]$ is forced to hold in $M\big((\mathbb{R}^*)^{M[G]}\big)$ by the the empty condition in $\mathbb{Q}_{\langle \delta_d, \delta_{d+1}, \dots \rangle}$ over $M[G \restriction d]$. So if $A \subseteq M[G \restriction d]$ is definable in $M[G]$ from parameters in $M[G \restriction d]$, then $A \in M[G \restriction d]$.

Proposition 3.2.12. *Let $\vec{\delta}$ and G be as in Definition 3.2.10. Then $\mathbb{R}^{Der(M,G)} = (\mathbb{R}^*)^{M[G]}$.*

Proof. $(\mathbb{R}^*)^{M[G]} \subseteq \mathbb{R}^{Der(M,G)}$ follows from the definition of $Der(M,G)$. In order to show the other inclusion, let $A \in \mathbb{R}^{Der(M,G)}$. A is definable in $L_{M \cap On}\big((\mathbb{R}^*)^{M[G]}\big)$ from parameters in $(\mathbb{R}^*)^{M[G]} \cup (M \cap On)$. $L_{M \cap On}\big((\mathbb{R}^*)^{M[G]}\big)$ is definable in $M[G]$. So there is some $d < \omega$ such that A is definable in $M[G]$ from parameters in $M[G \restriction d]$. By the previous remark, $A \in M[G \restriction d]$. Therefore $A \in \mathbb{R}^{M[G \restriction d]} \subseteq (\mathbb{R}^*)^{M[G]}$. $\square$

Lemma 3.2.13. *Suppose that M is countable and $\vec{\delta} = \langle \delta_n \mid n < \omega \rangle$ is a strictly increasing sequence of ordinals of M. Suppose that $\langle \mathcal{E}_n \mid n < \omega \rangle \in M$, where $\mathcal{E}_n$ witnesses that δ_n is Woodin in M for each $n < \omega$. Assume that $(M, \bigcup_{n<\omega} \mathcal{E}_n)$ is $(\omega_1 + 1)$-iterable. Let $G = g_0 \times g_1 \times g_2 \times \cdots \subseteq \mathbb{Q}_{\vec{\delta}}$ be generic over M. Let $\varphi[\vec{x}]$ be a $\Sigma_1(\mathbb{R})$-statement, where $\vec{x}$ is a finite sequence of elements of $(\mathbb{R}^*)^{M[G]}$. If $Der(M, G) \models \varphi[\vec{x}]$ then $L(\mathbb{R}) \models \varphi[\vec{x}]$.*

Proof. Fix $d < \omega$ such that $\vec{x} \subseteq M[G \restriction d]$. We start with some preparation. Let θ be a large enough ordinal. Fix a countable substructure X of V_θ containing everything we need. Let P be the transitive collapse of X with anti-collapse embedding $\pi : P \to V_\theta$. Note that $\pi^{-1}(M) = M$ and $\pi^{-1}(\vec{x}) = \vec{x}$. We will show that $L(\mathbb{R})^P \models \varphi[\vec{x}]$. Then by the elementarity of π we have that $L(\mathbb{R})^{V_\theta} \models \varphi[\vec{x}]$. Since $L(\mathbb{R})^{V_\theta}$ is an $\mathbb{R}$-initial segment of $L(\mathbb{R})$ and $\varphi[\vec{x}]$ is an $\Sigma_1(\mathbb{R})$-statement, Lemma 3.2.8 implies that $L(\mathbb{R}) \models \varphi[\vec{x}]$. The plan is to find a wellfounded iterate M' of M and a set H such that

(1) M' has ω-many Woodin cardinals $\vec{\delta'} = \langle \delta'_n \mid < \omega \rangle$,

(2) H is $\mathbb{Q}_{\vec{\delta'}}$-generic over M',

(3) $(\mathbb{R}^*)^{M'[H]} = \mathbb{R}^P$,

(4) $M' \cap On \subseteq P \cap On$ and

(5) $Der(M', H) \models \varphi[\vec{x}]$.

We get that $Der(M', H)$ is an $\mathbb{R}^P$-initial segment of $L(\mathbb{R})^P$:
Both models are of the correct form since

$$L(\mathbb{R})^P = L_{P \cap On}(\mathbb{R}^P) \text{ and}$$
$$Der(M', H) = L_{M' \cap On}(\mathbb{R}^P) \text{ by (3).}$$

They have the same reals because

$$\mathbb{R}^{(L(\mathbb{R})^P)} = \mathbb{R}^P \text{ and}$$
$$\mathbb{R}^{Der(M',H)} = (\mathbb{R}^*)^{M'[H]} = \mathbb{R}^P \text{ by Proposition 3.2.12 and(3).}$$

So $Der(M', H)$ is an $\mathbb{R}^P$-initial segment of $L(\mathbb{R})^P$, since $M' \cap On \subseteq P \cap On$ by (4). By the assumption, $\varphi[\vec{x}]$ is a $\Sigma_1(\mathbb{R})$-statement and thus Proposition 3.2.8 and (5) imply that $L(\mathbb{R})^P \models \varphi[\vec{x}]$.

Let's start with the construction. First of all we make all the reals of P generic. Fix a $Col(\omega, \mathbb{R}^P)$-generic enumeration $\langle a_n \mid d \leq n < \omega \rangle$ of $\mathbb{R}^P$ in V. From now on, we work in $P[\langle a_n \mid d \leq n < \omega \rangle]$.

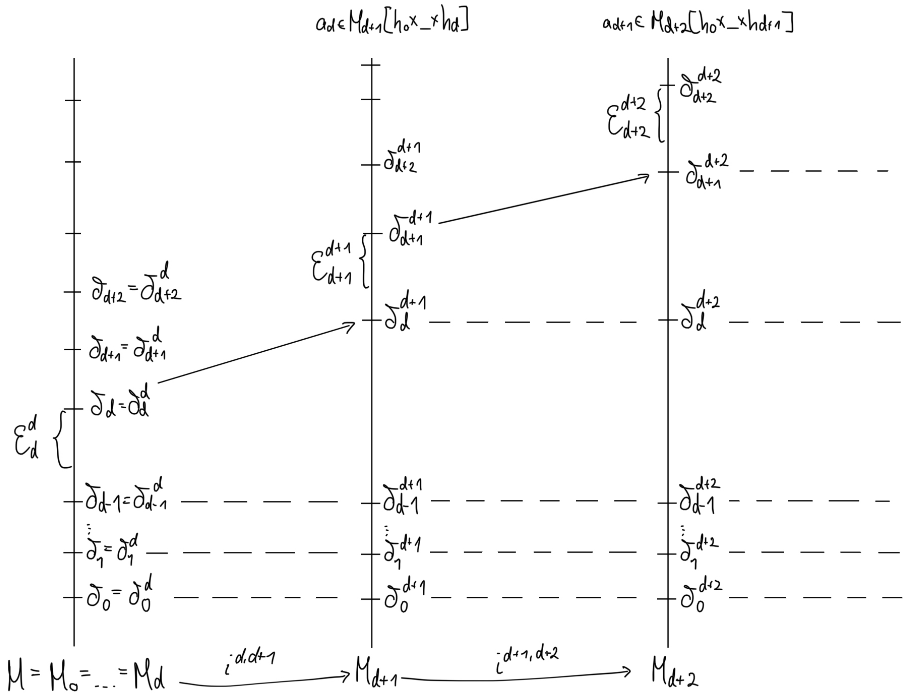
We will recursively construct models $\langle M^n \mid n < \omega\rangle$ and elementary embeddings $\langle i^{k,n} : M^k \to M^n \mid k < n < \omega\rangle$ such that $crit(i^{k,n}) > i^{0,k}(\delta_{k-1})$ for each $k < n < \omega$. We will denote

$$\vec{\delta}^k := i^{0,k}(\vec{\delta}) = \langle \delta^k_n \mid n < \omega\rangle$$

for each $k < \omega$. Furthermore, we will find generics $\langle h_n \subseteq Col(\omega, \delta^n_{n-1}) \mid n < \omega\rangle$ such that $a_n \in M^{n+1}[h_0 \times \cdots \times h_n]$ for each $n \geq d$.

We do nothing in the first d-many steps, i.e. for $m < k \leq d$ set $M^k := M$, $i^{m,k} := id : M^m \to M^k$ and $h_m := g_m$.

Fix $n \geq d$. Assume that $\langle M^k \mid k \leq n\rangle$, $\langle i^{k',k} : M^{k'} \to M^k \mid k' < k \leq n\rangle$, and $\langle h_k \mid k < n\rangle$ are already constructed. We want to use Woodin's Improved Genericity Iteration 2 from Corollary 3.1.10 for a_n, M^n, δ_n, and $\mathcal{E}^n_n := i^{0,n}(\mathcal{E}_n)$ with intermediate forcing extension by $h_0 \times \cdots \times h_{n-1} \subseteq \mathbb{Q}_{\langle \delta^1_0, \delta^2_1, \ldots, \delta^n_{n-1}\rangle}$. We will specify the choice of $q_n \in Col(\omega, \delta^n_n)$ later. The assumption that $\mathbb{Q}_{\langle \delta^1_0, \delta^2_1, \ldots, \delta^n_{n-1}\rangle}$ is small relative to $\mathcal{E}^n_n$ is not necessarily satisfied. We can arrange that by removing every extender $E \in \mathcal{E}^n_n$ with $crit(E) \leq \delta^n_{n-1}$ from $\mathcal{E}^n_n$. $\mathcal{E}^n_n$ still witnesses that δ^n_n is Woodin in M^n because $\delta^n_{n-1} < \delta^n_n$. The Genericity Iteration yields a countable iteration $i^{n,n+1} : M^n \to M^{n+1}$ and $h_n \subseteq Col(\omega, \delta^{n+1}_n)$ generic over M^{n+1} such that $a_n \in M^{n+1}[h_0 \times \cdots \times h_{n-1}][h_n]$ and $i^{n,n+1}(q_n) \in h_n$. We have that $crit(i^{n,n+1}) > \delta^n_{n-1}$ because only extenders with critical point above δ^n_{n-1} were used in the iteration.

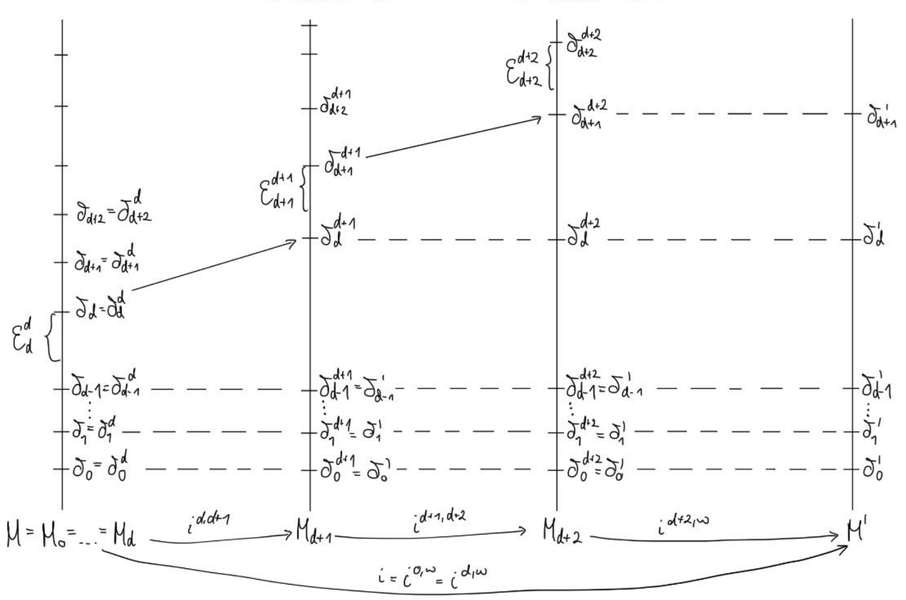

$a_d \in M_{d+1}[h_0 \times \_ \times h_d]$
$a_{d+1} \in M_{d+2}[h_0 \times \_ \times h_{d+1}]$
$\delta_{d+2} = \delta^d_{d+2}$
$\delta_{d+1} = \delta^d_{d+1}$
$\delta_d = \delta^d_d$
ε^d_d
$\delta_{d-1} = \delta^d_{d-1}$
$\delta_1 = \delta^d_1$
$\delta_0 = \delta^d_0$
δ^{d+1}_{d+2}
δ^{d+1}_{d+1}
ε^{d+1}_{d+1}
δ^{d+1}_d
δ^{d+1}_{d-1}
δ^{d+1}_1
δ^{d+1}_0
δ^{d+2}_{d+2}
ε^{d+2}_{d+2}
δ^{d+2}_{d+1}
δ^{d+2}_d
δ^{d+2}_{d-1}
δ^{d+2}_1
δ^{d+2}_0
$M = M_0 = \dots = M_d$
$i^{d,d+1}$
M_{d+1}
$i^{d+1,d+2}$
M_{d+2}

Define $i^{k,n} : M^k \to M^n$ for the remaining $k < n < \omega$ such that the maps commute. Set

$$M' := dirlim\langle M^n, i^{k,n} \mid k < n < \omega\rangle$$

and let $i^{n,\omega} : M^n \to M'$ be the direct limit embeddings. Set

$$\begin{aligned} i &:= i^{0,\omega}, \\ \vec{\delta}' &:= i(\vec{\delta}) = \langle \delta'_n \mid n < \omega\rangle, \\ \text{and } H &:= h_0 \times h_1 \times \ldots . \end{aligned}$$



We claim that M' is wellfounded. Fix $n \geq d$. We constructed $i^{n+1,n+2} : M^{n+1} \to M^{n+2}$ as an iteration on $(M^{n+1}, \mathcal{E}^{n+1}_{n+1})$, where $\mathcal{E}^{n+1}_{n+1} = i^{n,n+1}(\mathcal{E}^n_{n+1})$. We know that the iteration $i^{n,n+1} : M^n \to M^{n+1}$ happens below δ^{n+1}_n and we arranged that $\mathcal{E}^{n+1}_{n+1}$ is above δ^{n+1}_n. Therefore $i^{n,n+1}$ and $i^{n+1,n+2}$ can be "glued together" to an iteration $i^{n+1,n+2} \circ i^{n,n+1}$ on $(M^{n+1}, \mathcal{E}^n_n \cup \mathcal{E}^n_{n+1})$. We can repeat this ω-many times and get that i is an iteration on $(M, \bigcup_{n<\omega} \mathcal{E}_n)$. This iteration is of countable length since the individual steps are all of countable length. We assumed that $(M, \bigcup_{n<\omega} \mathcal{E}_n)$ is $(\omega_1 + 1)$-iterable, hence M' is wellfounded.

Note that $\delta^n_k = \delta'_k$ whenever $k < n$. We know that i is an elementary embedding from M to M'. Therefore $\vec{\delta}'$ is a strictly increasing sequence of Woodin cardinals in M' and condition (1) is satisfied.

But H is not necessarily $\mathbb{Q}_{\vec{\delta}'}$ -generic over M'. We specify the (q_n)'s to achieve that. We have to assure that H meets every dense open subset of $\mathbb{Q}_{\vec{\delta}'}$ in M'. Take such a dense open subset D. Since $M' = dirlim\langle M^n, i^{k,n} \mid k < n < \omega\rangle$ there is some $k < \omega$ and a dense open subset $D_k \in M^k$ of $\mathbb{Q}_{\vec{\delta}^k}$ such that $i^{k,\omega}(D_k) = D$. The idea is that M' has only countably many dense open subsets of $\mathbb{Q}_{\vec{\delta}'}$ and we can take care of one dense open subset in each step of the construction. Since we are constructing H in countably many steps, we can use some book-keeping to arrange that in the end we took care of each dense open subset. For simplicity and readability, we assume that $d = 0$ from now on. Otherwise we have to shift all the indices by $+d$. We fix a bijection

$$\varphi : \omega \times \omega \to \omega \text{ with } \varphi(k, l) \geq k.$$

We start with the 0-th step. Fix an enumeration

$$\psi_0 : \omega \to \{D \in M^0 \mid D \text{ is dense open in } \mathbb{Q}_{\vec{\delta}}\}.$$

There is such a $\psi_0 \in P$ because M^0 is countable. Let $\varphi^{-1}(0) = (0, l)$ and set $D := \psi_0(l)$. Pick any $p'_1 \in D \subseteq \mathbb{Q}_{\vec{\delta}}$ and set

$$\begin{aligned} &q_0 := (p'_1)_0 \in Col(\omega, \delta_0) \text{ and} \\ &p_1 := i^{0,1}(p'_1) \in \mathbb{Q}_{\vec{\delta}^1}. \end{aligned}$$

We used Corollary 3.1.10 to find $h_0 \subseteq Col(\omega, \delta_0^1)$ such that $h_0 \ni i^{0,1}(q_0) = p_1 \upharpoonright 1$. Fix an enumeration $\psi_1 : \omega \to \{D \in M^1 \mid D \text{ is dense open in } \mathbb{Q}_{\vec{\delta}^1}\}$ in P.

In the $(n+1)$-st step of the construction assume that we already constructed

(i). an enumeration $\psi_k : \omega \to \{D \in M^k \mid D \text{ inside } P \text{ is dense open in } \mathbb{Q}_{\vec{\delta}^k}\}$ for each $k \leq n$,

(ii). $p_n \in \mathbb{Q}_{\vec{\delta}^n}$ and

(iii). $p_n \upharpoonright n \in h_0 \times \cdots \times h_{n-1}$.

Set $\varphi^{-1}(n) := (k, l)$. Then $k \leq n$. We want to take care of $D := \psi_k(l)$ and need to find q_{n+1}, p_{n+1} and ψ_{n+1}. Choose $p^*_n \in \mathbb{Q}_{\vec{\delta}^n}$ such that

- $p^*_n \leq p_n$,
- $p^*_n \in i^{k,n}(D)$ and
- $p^*_n \upharpoonright n \in h_0 \times \cdots \times h_{n-1}$.

Why does such a p_n^* exist? Set

$$\begin{aligned}E_n := \{r \in \mathbb{Q}_{\langle \delta_0^n, \dots, \delta_{n-1}^n \rangle} \mid \\ r \le p_n \restriction n \text{ and } \exists p \in \mathbb{Q}_{\vec{\delta}^n} (p \le p_n, p \in i^{k,n}(D), p \restriction n = r)\}.\end{aligned}$$

E_n is dense in $\mathbb{Q}_{\langle \delta_0^n, \dots, \delta_{n-1}^n \rangle}$ below $p_n \restriction n$. Since $p_n \restriction n \in h_0 \times \cdots \times h_{n-1}$ by assumption, we can find some $r \in E_n \cap h_0 \times \cdots \times h_{n-1}$. Take p_n^* to be the witness for $r \in E_n$ and set $q_n := (p_n^*)_n$. Fix $p_{n+1} := i^{n,n+1}(p_n^*) \in \mathbb{Q}_{\vec{\delta}^{n+1}}$. We claim that $p_{n+1} \restriction (n+1) \in h_0 \times \cdots \times h_{n-1} \times h_n$. Compute that

$$\begin{aligned}p_{n+1} \restriction (n+1) &= i^{n,n+1}(p_n^*) \\ &= i^{n,n+1}(p_n^* \restriction n)^\frown \langle i^{n,n+1}\big((p_n^*)_n\big)\rangle \\ &= p_n^* \restriction n^\frown \langle i^{n,n+1}(q_n)\rangle \in (h_0 \times \dots h_{n-1}) \times h_n,\end{aligned}$$

where $i^{n,n+1}(p_n^* \restriction n) = p_n^* \restriction n$ because $crit(i^{n,n+1}) > \delta_{n-1}^n$ and $p_n^* \restriction n \in \mathbb{Q}_{\langle \delta_0^n, \dots, \delta_{n-1}^n \rangle} \subseteq V_{\delta_{n-1}^n}$. We have that $p_n^* \restriction n \in h_0 \times \dots h_{n-1}$ by assumption and $i^{n,n+1}(q_n) \in h_n$ since we used Corollary 3.1.10 to find h_n. Fix an enumeration

$$\psi_{n+1} : \omega \to \{D \in M^{n+1} \mid D \text{ is dense open in } \mathbb{Q}_{\vec{\delta}^{n+1}}\}.$$

Such an enumeration exists because M^{n+1} is countable and we can choose $\psi_{n+1} \in P$ since every finite initial segment of the process is happening inside P. This shows that H is $\mathbb{Q}_{\vec{\delta}'}$-generic over M'. Hence H satisfies (2).

We can also show that (3) holds for M' and H. Let $n \ge 1$. We made sure that the extenders in $\mathcal{E}_n^n$ have critical points above $\delta_{n-1}^n = \delta'_{n-1}$. Therefore $crit(i^{n,\omega}) = crit(i^{n,n+1}) > \delta'_{n-1}$. This implies that M^n and M' agree up to δ'_{n-1} and in particular $\mathbb{R}^{M'[H \restriction n]} = \mathbb{R}^{M^n[H \restriction n]}$. We have that M^n and $H \restriction n$ belong to P. Therefore $\mathbb{R}^{M'[H \restriction n]} = \mathbb{R}^{M^n[H \restriction n]} \subseteq P$ and since n is arbitrary, we have that $(\mathbb{R}^*)^{M'[H]} \subseteq \mathbb{R}^P$. On the other hand, $a_{n-1} \in \mathbb{R}^{M^n[H \restriction n]}$ implies that $a_{n-1} \in (\mathbb{R}^*)^{M'[H]}$. Again n is arbitrary, hence $\mathbb{R}^P = \{a_n \mid n < \omega\} \subseteq (\mathbb{R}^*)^{M'[H]}$.

Condition (4) is also satisfied. All the ordinals of M' belong to P because M' belongs to $P[\langle a_n \mid n < \omega \rangle]$.

In order to verify the last condition (5), we have to show that $Der(M', H) \models \varphi[\vec{x}]$. Set $R_G := (\mathbb{R}^*)^{M[G]}$ and $R'_H := (\mathbb{R}^*)^{M'[H]}$. Let $\psi(v_0, v_1)$ be the statement "$\varphi(v_0)$ holds in $L(v_1)$". By the assumption, $Der(M, G) \models \varphi[\vec{x}]$ and we have that

$$Der(M, G) = L(R_G)^M \in M(R_G).$$

Therefore $M(R_G) \models \psi[\vec{x}, R_G]$. Recall that $\vec{x} \subseteq M[G \restriction d]$. Remark 3.2.11 implies that $\psi[\vec{x}, R_G]$ is forced to hold in $M(R_G)$ by the empty condition in $\mathbb{Q}_{\langle \delta_d, \delta_{d+1}, \dots \rangle}$

over $M[G \upharpoonright d]$. By the construction, we have that $M[G \upharpoonright d] = M[H \upharpoonright d]$, $H \upharpoonright d \subseteq \mathbb{Q}_{\langle \delta_0, \ldots, \delta_{d-1} \rangle}$ and $crit(i) > \delta_{d-1}$. Therefore $i : M \to M'$ extends to an elementary embedding $i^* : M[H \upharpoonright d] \to M'[H \upharpoonright d]$. i^* does not move $\vec{x}$ because $\vec{x} \subseteq \mathbb{R}$. This implies that $\psi[\vec{x}, R'_H]$ is forced to hold in $M'(R'_H)$ over $M'[H \upharpoonright d]$. Therefore $Der(M', H) = L(R'_H)^{M'} \models \varphi[\vec{x}]$. □

Another important ingredient of the proof of AD in $L(\mathbb{R})$ is a simplified version of the Derived Model Theorem. The crucial step in the proof of the Simplified DMT is using universally Baire sets.

Definition 3.2.14. Let T be a tree on $\omega \times \omega$. We define the **projection of** T as $p[T] := \{x \in \mathbb{R} \mid \exists y \in \mathbb{R}$ such that (x, y) is a branch in $T\}$.

Definition 3.2.15. Let T_0 and T_1 be trees on $\omega \times \omega$ and let α be a regular cardinal. The pair $\langle T_0, T_1 \rangle$ is α**-absolutely complementing** iff whenever $g \subseteq Col(\omega, \alpha)$ is V-generic then $V[g] \models$ "$p[T_0] = \mathbb{R} \setminus p[T_1]$". A set $A \subseteq \mathbb{R}$ is called α**-universally Baire** iff there is a α-absolutely complementing pair of trees $\langle T_0, T_1 \rangle$ with $A = p[T_0]$.

The following lemma can be found in Chapter 6 of [Nee10].

Lemma 3.2.16. *(Neeman) Let δ be a Woodin cardinal. Every δ-universally Baire set of reals is determined.*

Theorem 3.2.17. *(Simplified DMT) Suppose that $\vec{\delta}$ is a strictly increasing sequence of cardinals in V. Let $\langle \mathcal{E}_n \mid n < \omega \rangle \in V$ be such that $\mathcal{E}_n$ witnesses that δ_n is Woodin in V. Assume that $(V, \bigcup_{n<\omega} \mathcal{E}_n)$ is $(\omega_1 + 1)$-iterable. Let H be $\mathbb{Q}_{\vec{\delta}}$-generic over V. Then $Der(V, H) \models AD$.*

Proof. Set $R := (\mathbb{R}^*)^{V[H]}$. Suppose that there is some $A \subseteq \mathbb{R}^{Der(V,H)} = R$ which is not determined in $Der(V, H) = L(R)$. By definition, there is a parameter $a \in R$, ordinals γ, ζ and a formula φ such that

$$x \in A \Leftrightarrow L_\gamma(R) \models \psi[x, a, \zeta].$$

We may assume that $a \in V = V[H \upharpoonright 0]$. Otherwise replace V by $V[H \upharpoonright n]$, where n is such that $a \in \mathbb{R}^{V[H \upharpoonright n]}$. We may also assume that $\langle \gamma, \zeta \rangle$ is $<_{lex}$-minimal such that

$$\{x \mid L_\gamma(R) \models \psi[x, a, \zeta]\}$$

is non-determined. Fix θ large enough. Working in V let $T_{in} \subseteq \omega \times V_\theta$ be the tree of attempts to construct a real x and a sequence $\langle \langle e_n, f_n \rangle \mid n < \omega \rangle \in (V_\theta)^\omega$ such that

- $\{e_n \mid n < \omega\}$ is an elementary substructure of V_θ.

Set $M_x := mos(\{e_n \mid n < \omega\})$ and let $\pi_x : M_x \to V_\theta$ be the anti-collapse embedding.

- $e_0 = a$, $e_1 = \vec{\delta}$, $e_2 = \mathbb{Q}_{\vec{\delta}}$, $e_3 = \dot{R}$ the canonical name for R, $e_4 = \gamma$, $e_5 = \zeta$ and e_6 is a name for a real in R and
- the empty condition in $\mathbb{Q}_{\vec{\delta}}$ forces that $L_{\check{\gamma}}(\dot{R}) \models \psi(e_6, \check{a}, \check{\zeta})$.

Set $\dot{x} := \pi_x^{-1}(e_6)$ and $\vec{\delta}_x := \pi_x^{-1}(\vec{\delta})$.

- $G_x := \{\pi_x^{-1}(e_{f_n}) \mid n < \omega\}$ forms a $\mathbb{Q}_{\vec{\delta}_x}$-generic filter over M_x and
- $\dot{x}[G_x] = x$.

$T_{out} \subseteq \omega \times V_\theta$ has the same definition except that the empty condition in $\mathbb{Q}_{\vec{\delta}}$ forces that $L_{\check{\gamma}}(\dot{R}) \not\models \psi(e_6, \check{a}, \check{\zeta})$.

For $x \in \mathbb{R}$ let $\varphi_{in}[a, x]$ be the statement "there is a non-determined set definable from a and ordinal parameters and x belongs to the least such set" and let $\varphi_{out}[a, x]$ be the statement "there is a non-determined set definable from a and ordinal parameters and x belongs to the complement of the least such set". Note that $\varphi_{in}[a, x]$ and $\varphi_{out}[a, x]$ are $\Sigma_1(\mathbb{R})$-statements. The definitions of T_{in} and T_{out} imply that

$$x \in p[T_{in}] \Rightarrow Der(M_x, G_x) \models \varphi_{in}[a, x]$$
$$x \in p[T_{out}] \Rightarrow Der(M_x, G_x) \models \varphi_{out}[a, x]$$

Let $x \in \mathbb{R}^V$ and assume that $x \in p[T_{in}]$. Set $\mathcal{F}_n := \pi_x^{-1}(\mathcal{E}_n)$ for each $n < \omega$. Then $\pi_x : (M_x, \bigcup_{n<\omega} \mathcal{F}_n) \to (V, \bigcup_{n<\omega} \mathcal{F}_n)$ is an elementary embedding and Lemma 2.5.9 implies that $(M_x, \bigcup_{n<\omega} \mathcal{F}_n)$ is $(\omega_1 + 1)$-iterable. Use Lemma 3.2.13 for M_x, $\vec{\delta}_x$, $\langle \mathcal{F}_n \mid n < \omega \rangle$, $G_x \subseteq \mathbb{Q}_{\vec{\delta}_x}$ and $\varphi_{in}[a, x]$. It yields that

$$Der(M_x, G_x) \models \varphi_{in}[a, x] \Rightarrow L(\mathbb{R}) \models \varphi_{in}[a, x].$$

Analogously for $x \in p[T_{out}]$:

$$Der(M_x, G_x) \models \varphi_{out}[a, x] \Rightarrow L(\mathbb{R}) \models \varphi_{out}[a, x].$$

Together with the above, we have

$$x \in p[T_{in}] \Rightarrow L(\mathbb{R}) \models \varphi_{in}[a, x]$$
$$x \in p[T_{out}] \Rightarrow L(\mathbb{R}) \models \varphi_{out}[a, x].$$

Claim. $\langle T_{in}, T_{out} \rangle$ is δ_0-absolutely complementing.

Proof of Claim. We show that $V \models$"$p[T_{in}] = \mathbb{R} \setminus p[T_{out}]$" first. Fix $x \in \mathbb{R}^V$ and assume that $x \in A$. In this case, we have that $L_\gamma((\mathbb{R}^*)^{V[H]}) \models \psi[x, a, \zeta]$ holds in V. Therefore we can find a countable elementary substructure M of V_θ containing $\vec{\delta}$ and H such that $L_\gamma((\mathbb{R}^*)^{M[H]}) \models \psi[x, a, \zeta]$ holds in M. Set $M_x := mos(M)$ with anti-collapse embedding $\pi_x : M_x \to V_\theta$. Find G_x such that $\pi_x(G_x) = H$. Then M_x and G_x witness that $x \in p[T_{in}]$. If $x \notin A$, then by the analogous argument, $x \in p[T_{out}]$. Hence $\mathbb{R} = p[T_{in}] \cup p[T_{out}]$ in V. Clearly $p[T_{in}] \cap p[T_{out}] = \emptyset$ because otherwise there would be a real x such that $L(\mathbb{R}) \models \varphi_{in}[a, x] \wedge \varphi_{out}[a, x]$. Now take some $g \subseteq Col(\omega, \delta_0)$ which is V-generic. Note that there is some $H^* \subseteq Col(\omega, \delta_0)$ which is $V[g]$-generic such that $V[H] = V[g][H^*]$ because $Col(\omega, \delta_0)$ is absorbed into $\mathbb{Q}_{\vec{\delta}}$. Therefore the exact same argument as for V works for $V[g]$. ■

On the one hand, this implies that $p[T_{in}]$ is δ_0-universally Baire. Hence $p[T_{in}]$ is determined by Lemma 3.2.16. On the other hand, we have that $x \in p[T_{in}]$ implies that x is in the least non-determined set definable from a and ordinal parameters in $L(\mathbb{R})$. Call this set LNS. If $x \notin p[T_{in}]$ then $x \in p[T_{out}]$ by the claim and hence $x \notin LNS$. Therefore $LNS = p[T_{in}]$. In particular, $p[T_{in}]$ is non-determined which is a contradiction. □

We are ready for the proof of Theorem 3.2.4.

Proof of Theorem 3.2.4. We start with some preparation. Let θ be a large enough ordinal. Fix a countable substructure X of V_θ containing everything we need and let P be the transitive collapse of X with anti-collapse embedding $\pi : P \to V_\theta$. We will show that $L(\mathbb{R})^P \models AD$. Then, by the elementarity of π, we have that $L(\mathbb{R})^{V_\theta} \models AD$. Since θ can be arbitrarily large, $L(\mathbb{R}) \models AD$ holds. The plan is to find a model N of ZFC and a set H such that

(1) N has a strictly increasing sequence $\vec{\delta}' = \langle \delta'_n \mid < \omega \rangle$ and a sequence $\langle \mathcal{F}_n \mid n < \omega \rangle$ such that $\mathcal{F}_n$ witnesses that δ'_n is Woodin in N for each $n < \omega$ and $(N, \bigcup_{n<\omega} \mathcal{F}_n)$ is $(\omega_1 + 1)$-iterable.

(2) H is $\mathbb{Q}_{\vec{\delta}'}$-generic over N,

(3) $(\mathbb{R}^*)^{N[H]} = \mathbb{R}^P$ and

(4) $P \cap On \subseteq N \cap On$.

If we find such N and H then (1) and (2) allow us to use the Simplified DMT 3.2.17 to conclude that $Der(N, H) \models AD$. We also get that $L(\mathbb{R})^P$ is an $\mathbb{R}^P$-initial segment of $Der(N, H)$:

Both models are of the correct form since

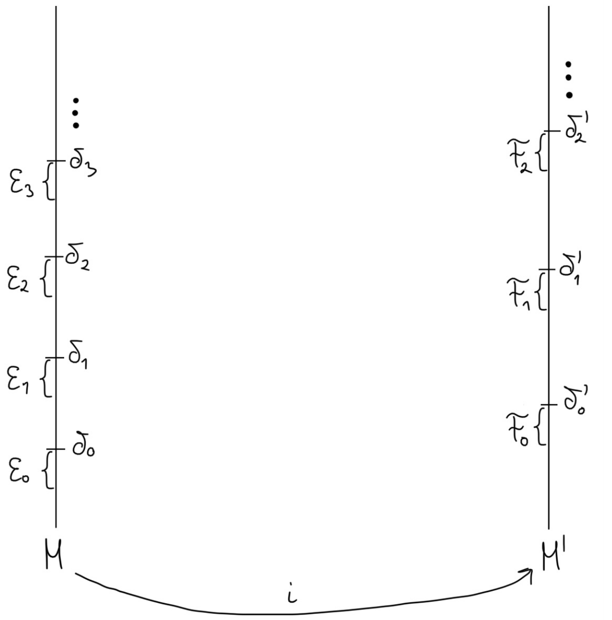
$$L(\mathbb{R})^P = L_{P\cap On}(\mathbb{R}^P) \text{ and}$$
$$Der(N,H) = L_{N\cap On}(\mathbb{R}^P) \text{ by (3).}$$

They have the same reals because

$$\mathbb{R}^{(L(\mathbb{R})^P)} = \mathbb{R}^P \text{ and}$$
$$\mathbb{R}^{Der(N,H)} = (\mathbb{R}^*)^{N[H]} = \mathbb{R}^P \text{ by Proposition 3.2.12 and (3).}$$

So $L(\mathbb{R})^P$ is an $\mathbb{R}^P$-initial segment of $Der(N,H)$ by (4). We saw that $\neg AD$ is a $\Sigma_1(\mathbb{R})$-statement in Proposition 3.2.6. We know that $\Sigma_1(\mathbb{R})$-statements go up from Proposition 3.2.8. Therefore $Der(N,H) \models AD$ implies that $L(\mathbb{R})^P \models AD$.

In the first step we make exactly the same construction as in the proof of Lemma 3.2.13 with $d := 0$. This yields a countable iteration $i : M \to M'$ where M' is wellfounded. Set $\vec{\delta}' := i(\vec{\delta})$ and $\langle \mathcal{F}_n \mid n < \omega \rangle := i(\langle \mathcal{E}_n \mid n < \omega \rangle)$.

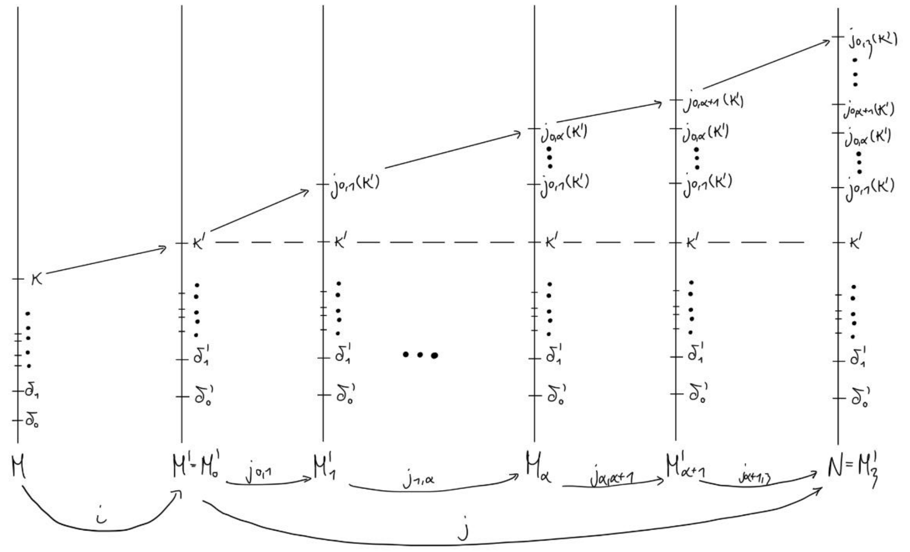


Since i is a countable iteration on $(M, \bigcup_{n<\omega} \mathcal{E}_n)$ and $(M, \bigcup_{n<\omega} \mathcal{E}_n)$ is $(2, \omega_1 + 1)$-iterable by assumption, we have that $(M', \bigcup_{n<\omega} \mathcal{F}_n)$ is $(\omega_1 + 1)$-iterable. The construction also produces $H \subseteq \mathbb{Q}_{\vec{\delta}'}$ generic over M' such that $(\mathbb{R}^*)^{M'[H]} = \mathbb{R}^P$. Therefore M' and H satisfy the conditions (1), (2), and (3).

But M' and H do not necessarily satisfy the last condition (4). In fact, we know that $M' \cap On \subseteq P \cap On$. We make another iteration to fix the ordinal height of M' by using the ultrafilter $U \in M$ witnessing that $\kappa > sup(\vec{\delta})$ is measurable in M. Set $U' := i(U)$, $\kappa' := i(\kappa)$ and $\xi := P \cap On \leq \omega_1$. Let

$$\langle M'_\alpha, j_{\alpha,\beta} : M'_\alpha \to M'_\beta \mid \alpha < \beta \leq \xi \rangle$$

be a linear iteration on (M', U'). Set $N := M'_\xi$, $j := j_{0,\xi} : M' \to N$ and $\langle \mathcal{F}_n \mid n < \omega \rangle := (j \circ i)\langle \mathcal{E}_n \mid n < \omega \rangle$.



j elongates the iteration i because i is a countable iteration and $\kappa' > \sup(\vec{\delta}')$. We have that $\xi \leq \omega_1$, so $j \circ i$ is an iteration of length $\leq \omega_1$ on $(M, \bigcup_{n<\omega} \mathcal{E}_n \cup \{U\})$. Again by the $(2, \omega_1 + 1)$-iterability and since $crit(j) > \sup(\vec{\delta}')$, we have that $(N, \bigcup_{n<\omega} \mathcal{F}_n)$ is $(\omega_1 + 1)$-iterable.

Note that

$$\kappa' < j_{0,1}(\kappa') < j_{0,2}(\kappa') < \cdots < j_{0,\xi}(\kappa')$$

are different ordinals in N. Therefore $\xi \leq N \cap On$ and N satisfies (4). κ' is the critical point of j and $\kappa' > \delta'_n$ for every $n < \omega$. Thus H is not touched by j and the conditions (2) and (3) still hold in N. $\Box$

Remark 3.2.18. Theorem 3.2.17 (Simplified DMT) and Theorem 3.2.4 are not optimal. In fact, Neeman proved in [Nee10] that if there are ω many Woodin cardinals and a measurable above those in V then AD holds in $L(\mathbb{R})$. This can be shown by replacing Woodin's Genericity Iteration by Neeman's Genericity Iteration. This Genericity Iteration doesn't need $(\omega_1 + 1)$-iterability. The construction is very different and more complicated. This is why I used Woodin's Genericity Iteration in this thesis.

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