Bachelor Thesis

# The perfect set property.

Analytic sets and the Axiom of Determinacy.

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Juni 2026

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# 1 Introduction

A very famous problem in mathematics is *Cantor's Continuum Problem*: It poses the question whether the following statement is true or false:

$$\exists A \subseteq \mathbb{R} : |\mathbb{N}| < |A| < |\mathbb{R}|, \tag{1}$$

where $|B| < |C|$ for sets $B, C$ means that there exists an injection from $B$ to $C$, but there exists no bijection from $B$ to $C$. This seems like a difficult question. And it is. Thanks to Kurt Gödel and Paul Cohen we now know that this is in fact independent of ZFC i.e., there exists a model of ZFC where such a set $A$ with the property from 1 exists, but there also exists a different model of ZFC where no such set $A$ exists.
However, if we cannot answer this difficult question, maybe we can look at a simpler variant of it: We do not ask if there exists any set with property 1, but we ask whether there exists a "nice" set that also fulfills this property.
To investigate this, we will use the *perfect set property* - that is a set is either countable or contains a perfect subset. The main theorem in the first part will show that *analytic sets* have the perfect set property. Many "nice" sets, e.g. closed sets, open sets, and Borel sets are in fact analytic. Additionally, by showing that perfect sets always have cardinality $|\mathbb{R}|$, we will have proved that these "nice" sets can never fulfill the property that is asked in the Continuum Problem.
In the second part, we will add an axiom: The Axiom of Determinacy, AD for short. This tells us if we have an infinite two player game then one player is guaranteed to have a winning strategy. Here we have to cut our losses, since AD contradicts the axiom of choice, which means we only work in ZF. Thus if we assume AD, then we can show that actually all subsets of reals have the perfect set property. So, if we assume the Axiom of Determinacy then the Continuum Problem has a negative answer, so no sets with property 1 exist.

Definitions, theorems and proofs from the first part are taken from [2], and in the second part from [1]. This Bachelor thesis aims to explain the topic in further detail and to fill in more steps in the proofs.

## 2 Analytic sets and the perfect set property

In this section our goal is to show that analytic sets satisfy the so called "perfect set property" and as a direct consequence that analytic sets cannot be witnesses to the Continuum Problem. In order to do so, we will first define what analytic sets are and what it means to have the "perfect set property". We will be working in a special space, namely the Baire space. The main source of this section will be [2].

### 2.1 The Baire Space

Before we define what the Baire space is, we want to look at more general cases and therefore, define a topology on the set of sequences ${}^{\omega}X := \{f : \omega \to X \mid \text{f is a function}\}$ for any arbitrary set $X$. We will also need the set of finite sequences, which we will denote by ${}^{<\omega}X := \{f : n \to X \mid n \in \omega, \text{ f is a function}\}$. We will call $s \in {}^{<\omega}X$ an *initial segment of* $x$ ($s \subseteq x$) for $x \in {}^{\omega}X$ or $x \in {}^{<\omega}X$, if the domain of $s$ (denoted $dom\, s$) is a subset of $dom\, x$ and $x \upharpoonright dom\, s = s$. Sometimes we want to extend a finite sequence $s \in {}^{n}X$ by one element $r \in X$. We will denote this extension by $s^\frown r$, or more formally: $s^\frown r : n+1 \to X, s^\frown r \upharpoonright n = s$ and $s^\frown r(n) := r$. Analogously, we define the shifted sequence that starts with $r$: $r^\frown s : n+1 \to X, r^\frown s(0) := r$ and $r^\frown s(k) = s(k-1)$ for $k > 0$, $k \leq n$.

**Definition 2.1.** Let $X \neq \emptyset$ be a set. For $s \in {}^{<\omega}X$ define

$$U_s := \{x \in {}^{\omega}X \mid s \subseteq x\}$$

and call it a *basic open set.*

$$\tau := \{A \in {}^{X}2 \mid A = \bigcup_{s \in I} U_s,\ I \subseteq {}^{<\omega}X\}$$

**Lemma 2.1.** *$\tau$ is a topology on ${}^{\omega}X$.*

*Proof.* In order to prove that $\tau$ is a topology on ${}^{\omega}X$ we need to show that the empty set and ${}^{\omega}X$ are in $\tau$ and that $\tau$ is closed under finite intersections and arbitrary unions: With $I = \emptyset$ we get $\emptyset \in \tau$ and taking $I = {}^{<\omega}X$ we get ${}^{\omega}X = \bigcup_{s \in {}^{<\omega}X} U_s \in \tau$.
Let $A, B \in \tau$. $A = \bigcup_{s \in I_A} U_s, B = \bigcup_{s \in I_B} U_s$.

$$A \cap B = \bigcup_{i \in I_A} (U_s \cap \bigcup_{t \in I_B} U_t) = \bigcup_{s \in I_A} \bigcup_{t \in I_B} (U_s \cap U_t) = \bigcup_{s \in I_A, t \in I_B} U^*_{s,t} \in \tau.$$

where $U^*_{s,t} := \begin{cases} s & t \subseteq s \\ t & \text{if } s \subseteq t \\ \emptyset & else. \end{cases}$
The union of sets in $\tau$ is obviously in $\tau$ as well as the union of unions of basic open sets. Therefore, $\tau$ is a topology on ${}^{\omega}X$. $\square$

**Definition 2.2.** For $X = \omega$ we call ${}^{\omega}X$ with the topology defined above the *Baire Space.* For $X = 2$ we call $({}^{\omega}X, \tau)$ the *Cantor Space.*

### 2.1.1 Perfect sets

**Definition 2.3.** Let $A \subseteq {}^{\omega}X$. $x \in {}^{\omega}X$ is called an *accumulation point of* $A$, if

$$\forall s \in {}^{<\omega}X : [x \in U_s \implies (A \setminus \{x\}) \cap U_s \neq \emptyset]$$

$A \neq \emptyset$ is called *perfect*, if

$$x \in A \iff x \text{ is an accumulation point of A.}$$

**Definition 2.4.** A subset of the Baire space has *the perfect set property*, if it is countable or if it contains a perfect subset.

An important reason why perfect sets are interesting in our quest of getting a better understanding of the Continuum Problem is the following lemma. It tells us that perfect sets are of size continuum. Therefore, if a set A fulfills the perfect set property, it is either at most countable or at least continuum big, so in particular it cannot be a witness to the Continuum Problem. But in order to prove this lemma about the size of perfect sets we need a practical theorem first, which tells us there exists a bijection between two sets, if there exist two injective functions from each set to the other respectively.

**Theorem 2.1** (Cantor-Schröder-Bernstein)**.** *Let $A, B$ be two arbitrary sets, $f : A \to B$, $g : B \to A$ injective functions, then there exists some function $h : A \to B$ bijective.*

*Proof.* We aim to define a function $h : A \to B$:
For $x \in A$ define the "$A$-Orbit of $x$" as the sequence

$$(g^{-1}(x), f^{-1}(g^{-1}(x)), g^{-1}(f^{-1}(g^{-1}(x))), ...)$$

as long as this exists. (This is well defined because $f, g$ are one-to-one.) Denote the length of this orbit by $k_A(x)$. If it is infinite, we set $k_A(x) := \infty$. Analogously, we define "$B$-Orbits of $y$" for $y \in B$ and its length by $k_B(y)$.
We partition $A$ and $B$ according to $k_A, k_B$ into disjoint sets:

$$A_0 := \{x \in A \mid k_A(x) = \infty\},\ B_0 := \{y \in B \mid k_B(y) = \infty\}$$

$$A_1 := \{x \in A \mid k_A(x) \text{ is even}\},\ B_1 := \{y \in B \mid k_B(y) \text{ is odd}\}$$

$$A_2 := \{x \in A \mid k_A(x) \text{ is odd}\},\ B_2 := \{y \in B \mid k_B(y) \text{ is even}\}$$

Now for $x \in A$ define:

$$h(x) := \begin{cases} f(x) & \text{if } x \in A_0 \cup A_1 \\ g^{-1}(x) & \text{if } x \in A_2. \end{cases}$$

This is well defined because if $x \in A_2$ then the orbit has at least length one so $g^{-1}(x)$ exists and $f(x)$ always exists.

Claim: $h$ is injective:
Let $x_1, x_2 \in A$ with $h(x_1) = h(x_2)$.
If both $x_1, x_2 \in A_0 \cup A_1$, then $x_1 = x_2$ by the injectivity of $f$. If $x_1, x_2 \in A_2$, then $x_1 = x_2$ because $g^{-1}$ is injective, since $g$ is an injective function.
And if (without loss of generality) $x_1 \in A_0 \cup A_1, x_2 \in A_2$, then $f(x_1) = g^{-1}(x_2)$, so $x_1 = f^{-1}(g^{-1}(x_2))$. This implies that $k_A(x_2) = k_A(x_1) + 2$. However, that is impossible because $k_A(x_1)$ is odd or infinite and $k_A(x_2)$ is even. Therefore, this case never occurs and we get that $h$ is injective.

Claim: $h$ is surjective:
Take $y \in B = B_0 \cup B_1 \cup B_2$. This union is disjoint.
If $y \in B_0$ then $f^{-1}(y)$ exists in $A$. $k_A(f^{-1}(y)) = k_B(y) - 1 = \infty$ implies $f^{-1}(y) \in A_0$ and $h(f^{-1}(y)) = f(f^{-1}(y)) = y$.
If $y \in B_1$ then $f^{-1}(y)$ exists in $B$, since the $B$-Orbit of $y$ has an odd length. $k_A(f^{-1}(y)) = k_B(y) - 1$ is therefore even and $f^{-1}(y) \in A_1$ which yields: $h(f^{-1}(y)) = y$.
If $y \in B_2$, then $g(y) \in A$. The $A$-orbit of $g(y)$ looks like:

$$(y, f^{-1}(y), g^{-1}(f^{-1}(y)), ...)$$

and $k_A(g(y)) = k_B(y) + 1$. This gives $h(g(y)) = g^{-1}(g(y)) = y$.
This tells us that $h$ is bijective, which concludes the proof. □

**Lemma 2.2.** *For every perfect set $A \subseteq {}^\omega\omega$ there exists $f : A \to {}^\omega\omega$ bijective.*

*Proof.* Let $A \subseteq {}^\omega\omega$ perfect. We want to use Theorem 2.1 by finding injections from $A$ to ${}^\omega\omega$ and vice-versa. The injection from $A$ to ${}^\omega\omega$ is trivial, since $A \subseteq {}^\omega\omega$. So our goal is to find an injection $g : {}^\omega\omega \to A$:
Claim: There exists a bijection $\Psi : {}^\omega\omega \to {}^\omega 2$:
${}^\omega 2$ naturally embeds into ${}^\omega\omega$, so we just need an injection from ${}^\omega\omega$ to ${}^\omega 2$: We can use the injection $\varphi : {}^\omega\omega \to {}^\omega 2 \setminus E \subseteq {}^\omega 2$ that is defined later in the proof of Corollary 3.1. This is an injection from the Baire space into a subset of the Cantor space and hence defines an injection from ${}^\omega\omega$ to ${}^\omega 2$. Now using Theoem 2.1 we get the proposed bijection $\Psi$.
Therefore, it suffices to find an injective function $F : {}^\omega 2 \to A$: We start by recursively defining a function: We know $A \neq \emptyset$ which implies
$\exists a_\emptyset \in {}^{<\omega}\omega(A \cap U_{a_\emptyset} \neq \emptyset)$. Using this $a_\emptyset$ we can define:

$$\phi : {}^{<\omega}2 \to \{U_t \mid t \in {}^{<\omega}\omega\}$$

$$\phi(\emptyset) := U_{a_\emptyset}$$

If we have already defined $\phi(s) =: U_{a_s}$ for some $s \in {}^{<\omega}2$ such that $A \cap U_s \neq \emptyset$, we want to define $\phi(s^\frown 0)$ and $\phi(s^\frown 1)$. Because $A$ is perfect, there exist two distinct elements $b, c \in A \cap U_{a_s}$. So $\exists k \in \omega(b(k) \neq c(k))$. We define

$$a_{s^\frown 0} := b \restriction k + 1, a_{s^\frown 1} := c \restriction k + 1$$

and

$$\phi(s^\frown 0) := U_{a_{s^\frown 0}},\ \phi(s^\frown 1) := U_{a_{s^\frown 1}}.$$

Where $b \in A \cap U_{a_{s^\frown 0}} \neq \emptyset$, $c \in A \cap U_{a_{s^\frown 1}} \neq \emptyset$ and $a_s \subseteq a_{s^\frown i}$ for $i \in \{0,1\}$. Hence, $\phi : {}^{<\omega}2 \to \{U_t \mid t \in {}^{<\omega}\omega\}$ is a well defined, injective function. We want to "lift" $\phi$ to the Cantor space by defining

$$a_u := \bigcup_{n\in\omega} a_{u\restriction n}$$

for $u \in {}^{\omega}2$. This gives us a function $F : {}^{\omega}2 \to {}^{\omega}\omega, F(u) := a_u$.
Claim: $F$ is one-to-one
Let $u \neq v \in {}^{\omega}2$. This means that there exists $n \in \omega$ such that $u \restriction n \neq v \restriction n$. This implies $a_{u\restriction n} \neq a_{v\restriction n}$, and because $a_u, a_v$ are ascending unions it follows that $a_u \neq a_v$.
Claim: $a_u \in A$ for all $u \in {}^{\omega}2$:
We prove this by showing that $a_u$ is an accumulation point of $A$: Let $t \in {}^{<\omega}\omega$ such that $a_u \in U_t$, then for some $k$ large enough $a_{u\restriction k} \in U_t$. Because $a_u \notin U_{a_{u\restriction k^\frown(1-u(k))}}$, we have

$$(A \setminus \{a_u\}) \cap U_t \supseteq (A \setminus \{a_u\}) \cap U_{a_{u\restriction k^\frown(1-u(k))}} = A \cap U_{a_{u\restriction k^\frown(1-u(k))}} \neq \emptyset.$$

So $a_u$ is an accumulation point of $A$ and therefore an element of $A$.
We have found $g := F \circ \Psi : {}^{\omega}\omega \to A$, where $F$ and $\Psi$ are injective. This implies that $g$ is injective. Theorem 2.1 implies the existence of a bijection from $A \to {}^{\omega}\omega$. $\square$

### 2.1.2 Trees

A fruitful definition in sequence spaces ${}^{\omega}X$ is the concept of trees. We look at a set $T$ of finite sequences in $X$ that is closed under initial segments, and we say that $T$ *is a tree on* $X$. Elements of $T$ are called *nodes*. We call a node $t$ an *extension of* $s$ for any $t, s \in T$, if $s$ is an initial segment of $t$. We say that two such extensions $t, t'$ are *incompatible*, if neither is an initial segment of the other. We call a tree *perfect* if for every node there exist $|X|$-many incompatible extensions.
For such a tree $T$ it is interesting to look at all possible infinite branches, so all $\omega$-sequences in $X$, such that every finite initial segment is a node in $T$. We denote this by $[T]$, and one can think of this as some sort of closure of $T$.

**Definition 2.5.** For a set $X \neq \emptyset$.

- A subset $T \subseteq {}^{<\omega}X$ is called a *tree on* $X :\iff$
  $\forall s \in T\, \forall n \leq \mathit{dom}\, s : (s \restriction n \in T)$.
- For $T$ a tree on $X$ we define $[T] := \{x \in {}^{\omega}X : \forall n \in \omega(x \restriction n \in T)\}$.
- A tree $T \neq \emptyset$ on $X$ is *perfect* $:\iff$ $\forall s \in T\, \exists (t_i)_{i\in|X|} \subseteq T(\forall i \in |X| :$ $(s \subseteq t_i) \wedge (t_i \not\subseteq t_j) \wedge (t_j \not\subseteq t_i))$.

The definition of perfect trees is not really nice to grasp, but we will mostly use the following remark, which tells us an easier equivalent for trees on $\omega$ (or any countable set $X$).

*Remark* 2.1. A tree $T$ on $\omega$ is perfect if $\forall s \in T\, \exists t, t' \in T(s \subseteq t, s \subseteq t' \wedge t \nsubseteq t \wedge t' \nsubseteq t)$.

*Proof.* Suppose we only have a finite number of incompatible extensions for some $s \in T$. Let $k$ be minimal in $\omega$ such that there are not $k$-many pairwise incompatible extensions of $s$ in $T$. Let $\{t_n : 0 \leq n < k-1\}$ be $k-1$-many pairwise incompatible extensions. For $0 < n < k-1$ let $m_n \in dom\, t_0 \cap dom\, t_n$ be a witness to the incompatibility i.e., $t_0(m_n) \neq t_n(m_n)$. Since $t_n$ are extensions of $s$, $m_n \notin dom\, s$. We define $m := max_{n<k-1} m_n$. Now $t_0 \restriction (m+1) \in T$ has two incompatible extensions $t, t'$ by prerequisite and $s \subseteq t_0 \restriction (m+1) \subseteq t, s \subseteq t_0 \restriction (m+1) \subseteq t'$.
Claim: $\{t_n : 0 < n < k-1\} \cup \{t, t'\}$ are $k$-many pairwise incompatible extensions of $s$ in $T$:
Let $0 < n < k-1$, then $m_n < m+1 \implies t(m_n) = t_0 \restriction (m+1)(m_n) = t_0(m_n) \neq t_n(m_n)$ and therefore $t \nsubseteq t_n \wedge t_n \nsubseteq t$. Analogously for $t'$. Hence, we have $k$-many pairwise incompatible extensions of $s$, which contradicts the minimality of $k$. So we do in fact have $\omega$-many pairwise incompatible extensions of $s$ in $T$, which means that $T$ is perfect.

□

*Example* 1. $T := \{s \in {}^{<\omega}\omega \mid \forall m < k < dom\, s : s(m) < s(k)\}$ the set of all strictly monotone, finite sequences is a tree, because any initial segment of a strictly monotone sequence is a strictly monotone sequence. $T$ is perfect as well since any strictly monotone and finite sequence can be extended by $t(dom\, s) := s(dom\, s - 1) + 1$ and $t'(dom\, s) := s(som\, s - 1) + 2$. Those are two incompatible extensions that are still in $T$.

We now have talked about perfect sets and perfect trees, so it seems rather obvious that we somehow want to link those two concepts. The following lemma gives us this insight, at least in the case $A \subseteq {}^{\omega}\omega$ or $A \subseteq {}^{\omega}2$.

**Lemma 2.3.** *Let $X = \omega$ or $X = 2$ and $A \subseteq {}^{\omega}X$.*
*(i) A is closed $\iff \exists T$...tree on ${}^{\omega}X$ such that $A = [T]$.*
*(ii) A is perfect $\iff \exists T \neq \emptyset$...perfect tree on ${}^{\omega}X$ such that $A = [T]$.*

*Proof.* (i): "$\implies$": Let $A \subseteq {}^{\omega}X$ closed. $T := \{t \in {}^{<\omega}X \mid \exists x \in A(t \subseteq x)\}$.
Claim: $A = [T]$
"$\subseteq$": Let $x \in A, n \in \omega$. Then $x \restriction n \subseteq x \in A \implies x \restriction n \in T$, hence $x \in [T]$.
"$\supseteq$": Let $x \in [T]$, that means that $\forall n \in \omega \exists x_n \in A : x \restriction n \subseteq x_n$. This gives us $\lim_{n\to+\infty} x_n = x$ and because $A$ is closed we have that $x \in A$.

"$\impliedby$": Let $(I, \preceq)$ be a *directed set* i.e., $I$ is a set and $\preceq$ is a reflexive and transitive relation on $I$. Furthermore for every two elements $i, j \in I$ there exists some $k \in I$ such that $i \preceq k \wedge j \preceq k$. Now let $(x_i)_i \subseteq [T]$ be a *convergent net* with $\lim_{i\to+\infty} x_i = x$. That means

$$\forall n \in \omega \exists i_0 \in I \forall i \succeq i_0 : x_i \in U_{x\restriction n}.$$

Showing that the limit of every convergent net in $[T]$ is still in $[T]$ is equivalent to showing that $[T]$ is closed, so we want to prove that $x \in [T]$. Let $n \in \omega$: $\exists i_0 \forall i \succeq i_0 : x_i \in U_{x\restriction n}$, that means $x \restriction n \subseteq x_i$. Furthermore $x_i \in [T]$ yields $x \restriction n = x_i \restriction n \in T$, hence $x \in [T]$. So $[T]$ is closed.

(ii): " $\implies$ ": Let $A$ be perfect, define $T := \{t \in {}^{<\omega}X \mid \exists x \in A(t \subseteq x)\}$ as before. We already know from (i) (since $A$ perfect implies $A$ is closed) that $A = [T]$.
Since $A \neq \emptyset$ we can take $a \in A$, and $a \restriction n \in T$ for some $n \in \omega$, so $T \neq \emptyset$.

Claim: $T$ is perfect:
For $s \in T$ we want to find two incompatible extensions. By definition of $T$ we know that there exists $x \in A$ such that $s \subseteq x$. This implies $x \in U_s$.
$A$ being perfect implies $\exists y \in A \setminus \{x\} \cap U_s \neq \emptyset$. $x \neq y$ yields the existence of some $n < \omega$, such that $x(n) \neq y(n)$. But $s \subseteq x$ and $s \subseteq y \implies n \geq \mathit{dom}\, s$.
We now know that $x \restriction n+1$ and $y \restriction n+1$ are extensions of $s$ and
$x \restriction n+1 \nsubseteq y \restriction n+1 \wedge y \restriction n+1 \nsubseteq x \restriction n+1$, so they are incompatible extensions of $s$. Also $x \restriction n+1 \in T, y \restriction n+1 \in T$ because $x, y \in A$, which allows us to conclude that $T$ is perfect. (For $X = {}^{\omega}2$ this follows per definition and for $X = {}^{\omega}\omega$ we use Remark 2.1.)

" $\impliedby$ ": Let $x$ be an accumulation point of $[T]$ for a perfect tree $T$. We want to show that $x \in [T]$: So let $n \in \omega$, then $x \in U_{x\restriction n}$. $x$ being an accumulation point yields $\exists y \in ([T] \setminus \{x\}) \cap U_{x\restriction n}$. Therefore $x \restriction n \subseteq y \restriction n \in T$, and because $T$ is a tree $x \restriction n \in T$, this implies $x \in [T]$.
If, on the other hand, we have $x \in [T]$, we want to show that it is also an accumulation point of $[T]$: Take $s \in {}^{<\omega}\omega$ such that $x \in U_s$. It follows that $s \in T$ and therefore two incompatible extensions of $s$ in $T$ exist: We call them $t$ and $t'$. Without loss of generality, assume that $t \nsubseteq x$: Define $y \in {}^{\omega}\omega$ as follows:

$$y(n) := t(n), \text{ if } n \in \mathit{dom}\, t.$$

For $n \geq \mathit{dom}\, t$ we define $y(n)$ recursively: If $y(n)$ is already defined such that $y \restriction n+1 \in T$, we know that there exists an extension $\hat{t}$ of $y \restriction n+1$ in $T$ and we define $y(n+1) := \hat{t}(n+1)$ (We know there are two incompatible extensions and therefore there is an extension $\hat{t}$ with $\mathit{dom}\, \hat{t} \supsetneq \mathit{dom}\, y \restriction n+1$.) By construction $y \in [T]$, $x \neq y$ and $y \in U_s$ which proves that $x$ is an accumulation point of $T$.
In total we now know that $x \in [T]$ if and only if $x$ is an accumulation point of $[T]$ - in other words: $[T]$ is perfect. $\square$

As a final definition regarding trees, we will talk about trees on $\omega \times \alpha$ for some ordinal $\alpha$. These are sets of pairs of finite sequences with the same domain, that are closed under initial segments. As before, we can define $[T]$ naturally. We can also talk about the *projection of a tree* which contains all infinite sequences in $\omega$ such that there exists some corresponding infinite sequence in $\alpha$ such that every finite initial segment of the pair is a node in $T$.

**Definition 2.6.** For $\alpha \geq \omega$ define:

- A *tree on* $\omega \times \alpha$ is a subset $T \subseteq \{(s,t) \in {}^{<\omega}\omega \times {}^{<\omega}\alpha \mid dom\, s = dom\, t\}$ and $\forall (s,t) \in T\, \forall n \leq dom\, s : (s \upharpoonright n, t \upharpoonright n) \in T$.
- $[T] := \{(x,y) \in {}^{\omega}\omega \times {}^{\omega}\alpha \mid \forall n \in \omega : (x \upharpoonright n, y \upharpoonright n) \in T\}$
- For a tree $T$ on $\omega \times \alpha$ we define the *projection of* $T$ $p[T] := \{x \in {}^{\omega}\omega \mid \exists y \in {}^{\omega}\alpha \forall n \in \omega : (x \upharpoonright n, y \upharpoonright n) \in T\}$

### 2.1.3 Analytic and $\alpha$-Souslin sets

We now have all the tools we need to talk about *analytic sets* which you can think of as the projections of Borel sets.

**Definition 2.7.** We call $A \subseteq {}^{\omega}\omega$ *analytic* $:\iff \exists T$ tree on $\omega \times \omega$ such that $A = p[T]$. We call it *coanalytic* $:\iff {}^{\omega}\omega \setminus A$ is analytic.

We want to show that analytic sets fulfill the perfect set property. In the following lemma, we will show that every Borel set is analytic; therefore, if we can prove that analytic sets satisfy the perfect set property, then so do Borel sets.

**Lemma 2.4.** *Every Borel set $A \subseteq {}^{\omega}\omega$ is analytic.*

*Proof.* Claim: If for $n < \omega : A_n$ is analytic, then $\bigcup_{n<\omega} A_n$ is analytic.
Since $A_n$ is analytic for every $n \in \omega$, there are trees $T_n$ on $\omega \times \omega$ such that $A_n = p[T_n]$. Define a new tree T on $\omega \times \omega$ by

$$(x,y) \in T :\iff \begin{cases} x = y = \emptyset & \text{or} \\ \exists n < \omega\, \exists y'\, (y = n^\frown y' \wedge (x \upharpoonright_{(dom\, x-1)}, y') \in T_n) \end{cases}$$

We want to show that $T$ is a tree i.e., that it is closed under initial segments: Let $(x,y) \in T$. (Without loss of generality $(x,y) \neq (\emptyset,\emptyset)$). This implies $t = n^\frown y'$ and $(x \upharpoonright_{(dom\,(x)-1)}, y') \in T_n$ for some $n \in \omega, y' \in \omega$. Because $T_n$ is a tree we know that $dom\ x \upharpoonright_{(dom\, x-1)} = dom\, y' < \omega$ which implies $dom\, x = dom\, y < \omega$. And for any $m \leq dom\, x : y \upharpoonright_m = n^\frown(y' \upharpoonright_{m-1})$ and $(x \upharpoonright_{m-1}, y' \upharpoonright_{m-1}) \in T_n$ yields $(x \upharpoonright_m, y \upharpoonright_m) \in T$. Therefore $T$ is a tree on $\omega \times \omega$.
We want to show: $p[T] = \bigcup_{n<\omega} A_n$:

"$\subseteq$": Let $x \in p[T]$, this means that $\exists y : \forall m < \omega (x \upharpoonright_m, y \upharpoonright_m) \in T$.
Define $n := y(0)$, and take $y'$ such that $y = n^\frown y'$. Then for every $m < \omega$: $(x \upharpoonright_m, y' \upharpoonright_m) \in T_n$, so $x \in A_n$.

"$\supseteq$": Take $x \in A_n = p[T_n]$, we get $\exists y' (\forall m : (x \upharpoonright_m, y' \upharpoonright_m) \in T_n)$.
Define $y := n^\frown y'$ and let $m < \omega$, then $y \upharpoonright_m = n^\frown(y' \upharpoonright_{m-1})$ and

$(x\restriction_{m-1}, y'\restriction_{m-1}) \in T_n$, therefore $(x\restriction_m, y\restriction_m) \in T$. And as a consequence $x \in p[T]$.

Claim: If for every $n < \omega$ $A_n$ is analytic, then so is $\bigcap_{n<\omega} A_n$:
Let $T_n$ be trees on $\omega \times \omega$ such that $A_n = p[T_n]$, and let $\varphi : \omega \times \omega \to \omega$ be a bijection that satisfies $\varphi(n,m) \leq \varphi(n,k)$ for $m \leq k$. For $t = (m_0, ..., m_j) \in {}^{<\omega}\omega$ and $n \in \omega$ we define $t^n := (m_{\varphi(n,0)}, ..., m_{\varphi(n,k-1)})$, where $k := min\{l \in \omega \mid \varphi(n,l) > j\}$.

$$T := \{(s,t) \in {}^{<\omega}\omega \times {}^{<\omega}\omega \mid dom\, s = dom\, t, \forall n < \omega (s \restriction dom\, t^n, t^n) \in T_n\}$$

We claim $T$ is a tree, so for $k < dom\, s = dom\, t$ we want to check whether $(s \restriction k, t \restriction k) \in T$. So, let $n < \omega$: We look at $(s \restriction dom\,(t \restriction k)^n, (t \restriction k)^n)$. $(t \restriction k)^n = (m_0, ..., m_{k-1})^n = (m_{\varphi(n,0)}, ..., m_{\varphi(n,l-1)})$, where $l$ is minimal such that $\varphi(n,l) > k-1$, $t^n = (m_{\varphi(n,0)}, .., m_{\varphi(n,l'-1)})$, where $l'$ is minimal such that $\varphi(n,l') > dom\, t - 1$. This implies $l \leq l'$ and therefore $\varphi(n,l) \leq \varphi(n,l')$ because $k-1 < dom\, t - 1$. This implies $(t \restriction k)^n \subseteq t^n$, hence $(s \restriction dom\,(t \restriction k)^n, (t \restriction k)^n) \in T_n$ as an initial segment of $(s \restriction dom\, t^n, t^n) \in T_n$. So $T$ is a tree and we claim $\bigcap_{n<\omega} A_n = p[T]$:
"$\subseteq$": Let $x \in A_n$ for every $n$, so there exist $y_n \in {}^{\omega}\omega$ such that $\forall k < \omega (x \restriction k, y_n \restriction k) \in T_n$. Define $y \in {}^{\omega}\omega, y(n) := y_{n_1}(n_2)$, where $\varphi^{-1}(n) = (n_1, n_2)$.
Now take $k < \omega$ and see that for every $n < \omega$:

$$((x \restriction k) \restriction dom\,(y \restriction k)^n, (y \restriction k)^n) = (x \restriction l, (y(\varphi(n,0)), ..., y(\varphi(n,l-1))))$$

$$= (x \restriction l, (y_n(0), ..., y_n(l-1))) = (x \restriction l, y_n \restriction l) \in T_n,$$

where $l$ minimal such that $\varphi(n,l) > k-1$. So $x \in p[T]$.

"$\supseteq$": Take $x \in p[T]$, so we have some $y \in {}^{\omega}\omega$ so that for every $n \in \omega$ $(x \restriction n, y \restriction n) \in T$. Define $y_n(m) := y(\varphi(n,m))$. We claim that $x \in p[T_n]$ is witnessed by $y_n$:
Let $k < \omega$. We regard $(x \restriction k, y_n \restriction k) = (x \restriction k, (y(\varphi(n,0)), ..., y(\varphi(n,k-1)))) = (x \restriction k, ((y \restriction \varphi(n,k))^n \restriction k)) \subseteq (x \restriction dom\,(y \restriction \varphi(n,k))^n, (y \restriction \varphi(n,k))^n) \in T_n$. And the latter is in fact in $T_n$ because $(x \restriction \varphi(n,k), y \restriction \varphi(n,k)) \in T$.

We know that basic open sets $U_s$ are closed sets ($(U_s)^c = \bigcup\{U_t \mid dom\, t \leq dom\, s \wedge t \not\subseteq s\}$ as its complement is a union of open sets). Lemma 2.3 tells us that closed sets are analytic and using the two results shown above, we get that Borel sets are analytic.

□

*Remark* 2.2. There is an analytic set that is not Borel.

*Proof.* See Theorem 7.5 and Lemma 7.11 in [2]. □

Our goal in the beginning was to prove something about analytic sets, but we will actually prove a more general result for a generalization of analytic sets, the so-called *α-Souslin sets*.

**Definition 2.8.** For $\omega \leq \alpha \in Ord$ we say that $A \subseteq {}^{\omega}\omega$ is $\alpha$-*Souslin* $:\iff \exists T$ tree on $\omega \times \alpha$ such that $A = p[T]$.

*Remark* 2.3. $\omega$-*Souslin*-sets are exactly the analytic sets.

**Theorem 2.2** (Souslin, Mansfield)**.** *Let $\alpha \in Ord$, $A \subseteq {}^{\omega}\omega$ be $\alpha$-Souslin. Then either $\exists B \subseteq A$ such that $B$ is perfect or $Card(A) \leq Card(\alpha)$.*

*Proof.* We start by giving two useful definitions regarding trees:
Let $U$ be a tree on $\omega \times \alpha$. Define for every $(s,t) \in U$:

$$U_{(s,t)} := \{(s_0, t_0) \in U \mid (s_0 \subseteq s \wedge t_0 \subseteq t) \vee (s_0 \supseteq s \wedge t_0 \supseteq t)\}$$

One can think of $U_{(s,t)}$ for a given node $(s,t) \in U$ as the tree consisting of all initial segments and extensions of $(s,t)$ in $U$. Since for $(s_0, t_0) \in U_{(s,t)}$ and for every $n \in \omega$ $(s_0 \upharpoonright n, t_0 \upharpoonright n) \in U_{(s,t)}$, $U_{(s,t)}$ is in fact a tree on $\omega \times \alpha$, therefore the following definition is justified.

$$U' := \{(s,t) \in U \mid Card(p[U_{(s,t)}]) > 1\}$$

Claim: $U'$ is a tree:
Let $(s,t) \in U'$, $n \in \omega$. $U_{(s \upharpoonright n, t \upharpoonright n)} \supseteq U_{(s,t)} \implies p[U_{(s \upharpoonright n, t \upharpoonright n)}] \supseteq p[U_{(s,t)}]$, so $(s \upharpoonright n, t \upharpoonright n) \in T'$ and therefore $U'$ a tree.
A node $(s,t) \in U$ is in $U'$, if there is more than one branch in $U$ with inital segment $(s,t)$, or in other words if the tree $U_{(s,t)}$ has more than one branch.

Now back to our analytic set $A$ and the corresponding tree $T$: Let $T$ be a tree on $\omega \times \alpha$ such that $A = p[T]$. Define

$$T_0 := T,\ T_{\beta+1} := (T_\beta)',\ T_\lambda := \bigcap_{\beta<\lambda} T_\beta$$

for $\beta$ an ordinal and $\lambda$ a limit ordinal. The intersection of trees is obviously a tree, so $T_\beta$ is a tree for every $\beta$. And $T_\beta \supseteq T_\gamma$ for every $\beta \leq \gamma$.

$$\implies Card(T_\beta) \leq Card(T_0) = Card(T) \leq Card(\omega \times \alpha) \leq Card(\alpha)$$

Claim: $\exists \delta < Card(\alpha)^+$ (where $\mu^+$ denotes the successor cardinal of $\mu$) such that $T_{\delta+1} = T_\delta =: T^\infty$:
Assume by contradiction that for every $\beta < |\alpha|^+$: $T_\beta \setminus T_{\beta+1} \neq \emptyset$. With the axiom of choice we then get a set $\{x_\beta \in T_\beta \setminus T_{\beta+1} \mid \beta < |\alpha|^+\} \subseteq T$, so $|T| \geq |\{x_\beta \in T_\beta \setminus T_{\beta+1} \mid \beta < |\alpha|^+\}| = |\alpha|^+ > |T|$, which contradicts regularity.

Case 1: $T^\infty = \emptyset$
For $x \in A = p[T]$ there exists a $y \in {}^{\omega}\alpha : \forall n \in \omega (x \upharpoonright n, y \upharpoonright n) \in T$.

$$\beta_0 := min\{\beta \mid \exists m : (x \upharpoonright m, y \upharpoonright m) \notin T_\beta\}$$

If $\beta_0$ were a limit ordinal, then $\exists m : (x \upharpoonright m, y \upharpoonright m) \notin T_{\beta_0} = \bigcap_{\gamma<\beta_0} T_\gamma$, which means $\exists \gamma < \beta_0 : (x \upharpoonright m, y \upharpoonright m) \notin T_\gamma$ this contradicts the minimality of $\beta_0$. So

there exists some $\beta$ with $\beta + 1 = \beta_0$.
For this $\beta$ define

$$n := min\{m \in \omega \mid (x \restriction m, y \restriction m) \notin T_{\beta+1}\}.$$

Claim: $p[(T_\beta)_{(x\restriction n, y\restriction n)}] = \{x\}$
"$\supseteq$": With $y \in {}^\omega\alpha$ and an arbitrary $m \in \omega$:

$$(x \restriction m, y \restriction m) \in T_\beta$$

due to the minimality of $\beta_0$. And either $(x \restriction m) \subseteq (x \restriction n) \wedge (y \restriction m) \subseteq (y \restriction n)$ or $(x \restriction n) \subseteq (x \restriction m) \wedge (y \restriction n) \subseteq (y \restriction m)$ and hence $(x \restriction m, y \restriction m) \in (T_\beta)_{(x\restriction n, y\restriction n)}$, which yields $x \in p[(T_\beta)_{(x\restriction n, y\restriction n)}]$.
"$\subseteq$": By our choice of $n$ we know: $(x \restriction n, y \restriction n) \notin T_{\beta+1} = (T_\beta)' \implies Card(p[(T_\beta)_{(x\restriction n, y\restriction n)}]) \leq 1$. Therefore we get $p[(T_\beta)_{(x\restriction n, y\restriction n)}] = \{x\}$.

This gives us

$$p[T] \subseteq \bigcup\{p[(T_\beta)_{(s,t)}] \mid \beta < \delta, (s,t) \in T_\beta \setminus T_{\beta+1}, |p[(T_\beta)_{(s,t)}| = 1\} \subseteq p[T].$$

So we have

$$|A| = |\bigcup\{p[(T_\beta)_{(s,t)}] \mid \beta < \delta, (s,t) \in T_\beta \setminus T_{\beta+1}, |p[(T_\beta)_{(s,t)}| = 1\}|$$

$$\leq |\bigcup_{\beta<\delta} T_\beta \setminus T_{\beta+1} \times \{\beta\}| \leq |\bigcup_{\beta<\delta} T \times \{\beta\}| = |\delta| \leq |\alpha|.$$

So $Card(A) \leq Card(\alpha)$, which concludes the first part of the proof.
Case 2: $T^\infty \neq \emptyset$

$$T^\infty = (T^\infty)' = \{(s,t) \in T^\infty \mid Card(p[T^\infty_{(s,t)}]) > 1\}$$

implies that for all $(s,t) \in T^\infty$ the cardinality of $p[T^\infty_{(s,t)}]$ is greater than one. We want to find $(s_z, t_z) \in T^\infty$ for every $z \in {}^{<\omega}2$. We can achieve this via recursive construction:

$$(s_\emptyset, t_\emptyset) := (\emptyset, \emptyset) \in T^\infty$$

If $(s_z, t_z) \in T^\infty$ is already defined for $z \in {}^{<\omega}2$: we know that $p[T^\infty_{(s_z,t_z)}]$ has at least two distinct elements, call them $x_0, x_1$. For $x_0, x_1$ there are $y_0, y_1 \in {}^\omega\omega$ : $\forall n \in \omega : (x_i \restriction n, y_i \restriction n) \in T^\infty_{(s_z,t_z)}$ for $i \in \{0,1\}$. In particular we can choose $n > dom\, s_z$ such that $x_0 \restriction n \neq x_1 \restriction n$. Now we can define

$$(s_{z^\frown 0}, t_{z^\frown 0}) := (x_0 \restriction n, y_0 \restriction n) \in T^\infty, (s_{z^\frown 1}, t_{z^\frown 1}) := (x_1 \restriction n, y_1 \restriction n) \in T^\infty$$

with $dom\, s_{z^\frown 0} = dom\, s_{z^\frown 1} > dom\, s_z$. We note that $\forall i \in Ord : (\emptyset, \emptyset) \in T_i$, because $T_{i+1} \neq \emptyset \implies p[(T_i)_{(\emptyset,\emptyset)}] = p[T_i] \supseteq p[(T_i)_{(s,t)}]$ has more than one element for some $(s,t) \in T_{i+1}$, so our definition makes sense. Note as well that $s_a = s_b \implies a = b$ by construction.

Take $u \in {}^{\omega}2$ and define

$$x_u := \bigcup\{s_{u\restriction n} \mid n \in \omega\} \in {}^{\omega}\omega.$$

Then $x_u \in p[T] = A$, because

$$\forall n \in \omega : (x_u \restriction n, y_u \restriction n) = (s_u \restriction n, t_u \restriction n) \in T^{\infty} \subseteq T$$

with $y_u := \bigcup\{t_{u\restriction n} \mid n \in \omega\}$.
We get $B := \{x_u \mid u \in {}^{\omega}2\} \subseteq A$ and proving $B$ is perfect concludes the proof:

$$\tilde{T} := \{x_u \restriction n \mid n \in \omega, u \in {}^{\omega}2\} \subseteq {}^{<\omega}\omega$$

$\tilde{T}$ is obviously closed under initial segments and therefore a tree on $\omega$. Also $\tilde{T} \neq \emptyset$.

$$[\tilde{T}] = \{x \in {}^{\omega}\omega \mid \forall n \in \omega\, (x \restriction n \in \tilde{T})\} \supseteq B$$

Claim: $[\tilde{T}] \subseteq B$ and hence $B$ is a perfect subset of $A$:
Define for some fixed $v \in {}^{\omega}2$

$$\begin{aligned} k : \omega &\to \omega & k^* : \omega &\to \omega \\ n &\mapsto dom(s_{v\restriction n}) & n &\mapsto min\{m \in \omega \mid n \in k(m)\} \end{aligned}$$

$k$ is well defined and independent of $v$ because the domain of $s_x$ only depends on the domain of $x$ for $x \in {}^{<\omega}2$ and for any $v_1, v_2 \in {}^{\omega}2$:
$dom\, v_1 \restriction n = dom\, v_2 \restriction n = n$. $k$ is strictly monotone increasing and therefore unbounded as well by construction of $s_z$. This implies $k^*$ is well defined.
Let $x \in [\tilde{T}]$: For every $n \in \omega\, \exists u_n \in {}^{\omega}2$ s.t. $x \restriction n = x_{u_n} \restriction n$. This implies

$$s_{u_n\restriction k^*(n)} \restriction n = x_{u_n} \restriction n = x_{u_{n+1}} \restriction n = s_{u_{n+1}\restriction k^*(n)} \restriction n \subseteq s_{u_{n+1}\restriction k^*(n+1)} \restriction n+1$$

$$\implies s_{u_n\restriction k^*(n)-1} = s_{u_{n+1}\restriction k^*(n)-1}$$

$$\implies u_n \restriction k^*(n) - 1 = u_{n+1} \restriction k^*(n) - 1 \subseteq u_{n+1} \restriction k^*(n+1) - 1$$

So we can define

$$u := \bigcup_{n\in\omega} u_n \restriction k^*(n) - 1 \in {}^{\omega}2.$$

Now for every $n$:

$$x_u \restriction n = s_{u\restriction k^*(n)} \restriction n = x_{u_{n+1}} \restriction n = x_{u_n} \restriction n = x \restriction n,$$

therefore $x = x_u \in B$ and $B$ is perfect.

□

**Corollary 2.1.** *Let $A \subseteq {}^{\omega}\omega$ analytic. Then $A$ satisfies the perfect set property.*

**Corollary 2.2.** *For any $A \subseteq {}^{\omega}\omega$ that is open, closed, Borel or analytic:*

$$\neg(|\omega| < |A| < |{}^{\omega}\omega|)$$

*Proof.* $A$ is analytic (Lemma 2.4); therefore, $|A| \leq |\omega|$ or $\exists B \subseteq A$ is perfect (Theorem 2.2). And in the second case, Lemma 2.2 tells us that
$|{}^{\omega}\omega| = |B| \leq |A|$. □

# 3 Determinacy

In this next section, we will consider two players who take turns playing a natural number. If they take $\omega$-many turns, they generate a countable sequence. If this sequence is in a certain *payoff set* $A$, then Player I *wins*, if it is not in $A$, then Player II wins. We can use such *games* to find out more about the payoff sets of these games. If we require that for every game one of the players has to have a way of playing that ensures they will win (we say that the game is *determined*), then we will show that every subset of the Baire space has the perfect set property. The source of this section is [1].

## 3.1 Infinite Games

**Definition 3.1.** For a set $X \neq \emptyset$ and $A \subseteq {}^{\omega}X$ we define the game $G_X(A)$: This game consists of two players Player I and Player II, who take turns choosing some $x_i \in X$ resulting in an infinite sequence $x = (x_0, x_1, x_2, ...) \in {}^{\omega}X$. We call $x$ *a play*. We say *Player I wins* if $x \in A$, otherwise *Player II wins*.

| | | | | | |
|---|---|---|---|---|---|
| **Player I** | $x_0$ | | $x_2$ | | $\dots$ |
| **Player II** | | $x_1$ | | $x_3$ | $\dots$ |

A function $\sigma : \bigcup_{n\in\omega} {}^{2n}X \to X$ is called *a strategy for Player I.*
If $y = (y_0, y_1, ...) \in {}^{\omega}X$ are all the moves played by Player II, we denote by $\sigma * y := (\sigma(\emptyset), y_0, \sigma((\sigma(\emptyset), y_0), y_1, ...)$ the play we get if Player I responds to Player II playing $y$ according to strategy $\sigma$.
We call $\sigma$ a *winning strategy for Player I* $:\iff \{\sigma * y \mid y \in {}^{\omega}X\} \subseteq A$.

A *winning strategy for Player II* is a function $\tau : \bigcup_{n\in\omega} {}^{2n+1}X \to X$, and $z * \tau$ denotes the play resulting from Player II playing against $z \in {}^{\omega}X$ according to $\tau$.

**Definition 3.2.** We say a game $G_X(A)$ is *determined* if and only if there exists a winning strategy for Player I or Player II.
For $X = \omega$ we say $A \subseteq {}^{\omega}\omega$ *is determined* if $G_X(A)$ is determined.

### 3.1.1 The Davis Game and Davis Theorem

**Definition 3.3.** (Davis Game) For $X = 2$ and $A \subseteq {}^{\omega}2$ we consider the game $G_2^*(A)$ (*Davis Game*) consisting of two Players, Player I and Player II, where now Player I chooses $s_i \in {}^{<\omega}2$, Player II chooses a $k_i \in 2$ resulting in a play $x = (s_0, k_1, s_2, k_3, ...)$. We say *Player I wins*, if the generated sequence of this play $x_* := s_0^\frown k_1^\frown s_2^\frown ... \in A$, otherwise *Player II wins.*

The next theorem is the core part for proving that under the axiom of determinacy (AD) i.e. every set is determined, every set $A \subseteq {}^{\omega}\omega$ has the perfect set property. The statement is still slightly different: We prove that a subset of the Cantor space satisfies the perfect set property if and only if it is determined.

**Theorem 3.1.** *(Davis) For $A \subseteq {}^{\omega}2$:*

- *$A$ is countable $\iff$ Player II has a winning strategy in $G_2^*(A)$*
- *$A$ contains a perfect subset $B \iff$
Player I has a winnning strategy in $G_2^*(A)$.*

*Proof.* " $\implies$ ": We argue by some sort of diagonalization: As $A$ is countable, we want to write it as $A = \{a_n \mid n \in \omega\}$, now we define a winning strategy for player II:

$$\tau : \bigcup_{n\in\omega} {}^{n}({}^{<\omega}2 \times 2) \times {}^{<\omega}2 \to 2,$$

$${}^{n}({}^{<\omega}2 \times 2) \times {}^{<\omega}2 \ni s \mapsto \tau(s) := 1 - a_n(dom\, s).$$

That is to say Player II looks at the sequence that has been played so far after $2n+1$ turns, and then they play the value such that the emerging sequence does not match the $n$-th entry of $A$ at the respecting index $dom\, s$. Then $s^\frown\tau(s) \not\subseteq a_n$ (because $s^\frown\tau(s)(dom\, s) = \tau(s) \neq a_n(dom\, s)$). Therefore, $z * \tau \neq a_n$ for all $n \in \omega$ and Player II is sure to win.
" $\impliedby$ ": If conversely there is a winning strategy $\tau$ for Player II, we aim to find a countable superset of $A$. In order to do that we look at *partial plays* $p = (s_0, k_1, , s_2, ..., k_{2n+1})$, $s_i \in {}^{<\omega}2, k_j \in 2$ for $i, j < \omega$ according to $\tau$. We denote the corresponding sequence with $p_* = s_0^\frown(k_1)^\frown...^\frown(k_{2n+1}) \in {}^{<\omega}2$. We define sets

$$D_p := \{x \in {}^{\omega}2 \mid p_* \subseteq x \implies \exists t \in {}^{<\omega}2\, (p_*^\frown t^\frown\tau(p,t) \subseteq x)\}.$$

Claim: $A \subseteq \bigcup_p ({}^{\omega}2 \setminus D_p)$:
Let $x \in \bigcap_p D_p$ then for $p_* = \emptyset$ there exists a $t_0 \in {}^{<\omega}2$ such that $(t^\frown\tau(t)) \subseteq x$. Recursively we find $t_i : i \in \omega$ such that $x = t_0^\frown\tau(t_0)^\frown...t_i^\frown\tau(..., t_i)^\frown...$ a play according to $\tau$ - a winning strategy for player II - which tells us $x \notin A$, which proves the claim.
Now we take a closer look at ${}^{\omega}2 \setminus D_p$ for some $p$: We know that $x \in {}^{\omega}2 \setminus D_p \iff p_* \subseteq x \wedge \forall t \in {}^{<\omega}2\, (p_*^\frown t^\frown\tau(p,t) \not\subseteq x$.
Claim: $\exists y_p : \{y_p\} = {}^{\omega}2 \setminus D_p$:
It must hold $p_* \subseteq y_p$, then taking $t = \emptyset$ we gain that $y_p(dom\, p_*) = 1 - \tau(p, \emptyset)$. Recursively we get for all $i \in \omega$:

$$y_p(dom\, p_* + i) = 1 - \tau(p, (y_p(dom\, p_*), ..., y_p(dom\, p_* + i - 1))),$$

hence $y_p$ is uniquely defined.
There are only countably many partial plays $p$ according to $\tau$, so we have that $A$ is a subset of a countable union of sets that contain exactly one element, which yields that $A$ is countable. This concludes the first part of the theorem.

" $\implies$ ": For the second part we start with a perfect subset $B \subseteq A$. We define $T := \{x \upharpoonright n \mid n \in \omega, x \in B\}$. As in the proof of Lemma 2.3 we see that $T$ is a

perfect tree. A winning strategy $\sigma$ for Player I is defined recursively: The player starts by picking some finite sequence $s_0 \in {}^{<\omega}2$ such that both $s_0^\frown(0) \in T$ and $s_0^\frown(1) \in T$. Such $s_0$ exists, because $T \neq \emptyset$ and every element of T must have two incompatible extensions. For a given partial play $p = (s_0, (k_1), ..., s_{2n-1}, (k_{2n}))$, according to the strategy $p_* = s_0^\frown(k_1)^\frown ...^\frown s_{2n-1}^\frown(k_{2n}) \in T$ and again there exist two incompatible extensions, which means $\exists s_{2n+1} \in {}^{<\omega}2$ such that $p_*^\frown s_{2n+1}^\frown(0) \in T$ and $p_*^\frown s_{2n+1}^\frown(1) \in T$. From Lemma 2.3 we get $B = [T]$ and $\sigma * y \in [T] = B \subseteq A$ for all $y$, hence $\sigma$ is a winning strategy for Player I.
" $\Longleftarrow$ ": Assume we have a winning strategy $\sigma$ for Player I. Then we claim

$$B := \{\sigma * y \mid y \in {}^{\omega}2\} \subseteq A$$

is perfect:
We define

$$T := \{\sigma * y \upharpoonright n \mid y \in {}^{\omega}2, n \in \omega\}$$

a tree, $T \neq \emptyset$. $T$ is also perfect, because taking $\sigma * y \upharpoonright n \in T$, we know that this is an initial segment of a play, therefore we can extend it to a partial play $p = (s_0, (k_1), ..., s_{2n-1})$, where $p_* := s_0^\frown(k_1)^\frown ...^\frown s_{2n-1} \supseteq \sigma * y \upharpoonright n$. Therefore, both $p_*^\frown(0) \in T$ and $p_*^\frown(1) \in T$ are two incompatible extensions, so $T$ is a perfect tree and $[T] = B$. Using Lemma 2.3 we conclude $B$ is in fact a perfect subset of $A$.

□

## 3.2 AD and the perfect set property

Now we can show that, assuming that every subset of the Baire space is determined, then every set $A \subseteq {}^{\omega}\omega$ is either countable or else contains a perfect subset.

**Definition 3.4.** (Axiom of Determinacy (AD)) For all $A \subseteq {}^{\omega}\omega$, $A$ is determined.

**Corollary 3.1.** *(AD)* $\Longrightarrow$ $\forall A \subseteq {}^{\omega}\omega$*: A has the perfect set property.*

*Proof.* Let $A \subseteq {}^{\omega}\omega$. Because of the axiom of determinacy we know, that either Player I or Player II has a winning strategy for $G_\omega(A)$. Of course we want to use Theorem 3.1 from above. But to use this we need to say something about $G_2^*(A)$.
Therefore we want to define a homeomorphism $\varphi : {}^{\omega}\omega \to {}^{\omega}2 \setminus E$, where
$E := \{x \in {}^{\omega}2 \mid \exists n_0 \in \omega(\forall k \geq n_0 : x(k) = x(n_0))\}$. Take $n, k \in \omega$:

$$b_k^n : n+1 \to 2,\ b_k^n(i) := \begin{cases} k+1 \mod 2 & i < n \\ k \mod 2 & i = n \end{cases}$$

Using these $b_k^n$ we can define $\varphi$:

$$\varphi(x) := b_{x(0)}^0{}^\frown b_{x(1)}^1{}^\frown ... \in {}^{\omega}2$$

$\varphi$ injective and surjective onto ${}^{\omega}2 \setminus E$.

Claim: $\varphi$ is a homeomorphism:
Let $s \in {}^{<\omega}2$, then $U_s \setminus E$ is an open set in ${}^{\omega}2 \setminus E$. $s = (s_0, s_1, ..., s_n)$ and let $k < n$ be maximal such that $\forall l > k : s_l \neq s_k$. Then $(s_0, ..., s_k) = \varphi(t)$, where $t \in {}^{<\omega}\omega$ and $\varphi$ is defined analogously. Then

$$\varphi^{-1}[U_s \setminus E] = \bigcup_{l > n-k} U_{t^\frown l}$$

which is an open set, so $\varphi$ is continuous.
Take $t \in {}^{<\omega}\omega$, then $\varphi'' U_t = U_{\varphi(t)} \setminus E$, which is of course open. So $\varphi : {}^{\omega}\omega \to {}^{\omega}2 \setminus E$ is a homeomorphism.

We want to show that $G_2^*(\varphi'' A)$ is determined:
First we fix a bijection $\theta : \omega \to {}^{<\omega}2$. Then we define a new payoff set $B$ in the Baire space that lets player II loose instantly, if they play something different than $\theta^{-1}(0)$ or $\theta^{-1}(1)$. And that lets player I win, if the image of the generated sequence under $\theta$ is in $\varphi'' A$:
For the sake of readability, for a sequence $x = (x_1, x_2, ...) \in {}^{\omega}X$, and a function $f : X \to Y$ we denote $f(x) := (f(x_1), f(x_2), ...) \in {}^{\omega}Y$.

$$B := \{x \in {}^{\omega}\omega \mid \exists n \in \omega\, (\theta(x_{2n}) \notin \{0,1\})\} \cup \{x \in {}^{\omega}\omega \mid \theta(x_0)^\frown\theta(x_1)^\frown ... \in \varphi'' A\}$$

Because we assume $(AD)$, we know that either player I or player II has a winning strategy for $G_\omega(B)$.

Case 1: Player I has a winning strategy $\sigma$ for $G_\omega(B)$:
We need to define a winning strategy $\sigma^*$ for player I in $G_2^*(\varphi'' A)$: For a partial play $p^* := (s_0, (k_1), s_2, ..., (k_{2n-1}))$ we define:

$$\sigma^*(p^*) := \theta\left(\sigma\left(\theta^{-1}\left(s_0, k_1, s_2, ..., k_{2n-1}\right)\right)\right)$$

Is $\sigma^*$ a winning strategy for player I?
Take $k \in {}^{\omega}2$: $\sigma^* * k$ is the sequence obtained by the play $p = (s_0, k_0, s_1, k_1, ...)$ where Player I plays according to $\sigma^*$. If we look at the corresponding play $\theta^{-1}(p)$ in ${}^{\omega}\omega$, this is a play according to $\sigma$: Let $n$ be minimal in $\omega$ such that $\theta^{-1}(s_n)$ is not played according to $\sigma$. Then

$$s_n = \sigma^*(s_0, k_0, s_1, ..., k_{n-1}) = \theta\left(\sigma\left(\theta^{-1}\left(s_0, k_0, ..., k_{n-1}\right)\right)\right),$$

so

$$\theta(s_n) = \sigma\left(\theta^{-1}\left(s_0, k_0, ..., k_{n-1}\right)\right)$$

and $\theta^{-1}(s_0, k_0, ..., k_{n-1})$ is played according to $\sigma$ by the minimality of $n$, therefore $\theta^{-1}(p)$ is the play obtained by Player II playing $\theta^{-1}(k)$ and Player I playing according to $\sigma$. Which means $\sigma * \theta^{-1}(k) \in B$, which implies

$$\sigma * \theta^{-1}(k) \in \{x \in {}^{\omega}\omega \mid \theta(x_0)^\frown\theta(x_2)^\frown ... \in \varphi'' A\},$$

hence

$$\theta(\theta^{-1}(s_0))^\frown\theta(\theta^{-1}(k_0))^\frown ... = s_0^\frown k_0^\frown s_0^\frown ... \in \varphi'' A.$$

Which proves that $\sigma^*$ is a winning strategy for Player I in $G_2^*(\varphi''A)$.

Case 2: Player II has a winning strategy $\tau$ for $G_\omega(B)$:
We need to define a winning strategy for Player II in $G_2^*(\varphi''A)$: For a partial play $(s_0, k_1, ..., s_{2n})$ we define

$$\tau^*(p^*) := \theta\left(\tau\left(\theta^{-1}\left(s_0, k_1, ...\right)\right)\right)$$

This is well defined, because $\tau$ is a winning strategy for Player II, and if Player II were to play anything different that $\theta^{-1}(0)$ or $\theta^{-1}(1)$ then Player II would loose instantly, so $\tau^*(p^*) \in \{0,1\}$. Now as before for a play $p = (s_0, k_0, s_1, k_1, ...)$ according to $\tau^*$, $\theta^{-1}(p)$ is a play in ${}^\omega\omega$ according to $\tau$. Because $\tau$ is a winning strategy we then know that $\theta^{-1}(s) * \tau \notin B$ and therefore $\theta(\theta^{-1}(s_0))^\frown\theta(\theta^{-1}(k_0))^\frown ... = s_0^\frown k_0^\frown s_1^\frown ... \notin \varphi''A$ and $\tau^*$ is a winning strategy for Player II.

This yields $G_2^*(\varphi''A)$ is determined and Theorem 3.1 then tells us that $\varphi''A$ has the perfect set property. If $\varphi''A$ is countable then so is $A$ and if $\varphi''A$ contains a perfect subset $B$, then $B$ is also perfect in the subspace topology $({}^\omega 2 \setminus E, \tau\restriction_{E^C})$, therefore $\varphi^{-1}[B] \subseteq A$ is perfect as the homeomorph image of a perfect set.
This concludes that every set $A \subseteq {}^\omega\omega$ has the perfect set property under the Axiom of Determinancy.

$\square$

**Corollary 3.2.** *Under the Axiom of Determinacy the Continuum Problem has a negative answer, i.e.*

$$(AD) \implies \nexists A \subseteq {}^\omega\omega : |\omega| < |A| < |{}^\omega\omega|$$

*Proof.* Take $A \subseteq {}^\omega\omega$. Corollary 3.1 tells us $A$ has the perfect set property. So either $|A| \leq |\omega|$, or $\exists B \subseteq A$ perfect. Then Lemma 2.2 tells us
$|A| \geq |B| = |{}^\omega\omega|$. $\square$

## References


[1] Sandra Müller. *Determinancy. Lecture notes for Advanced Topics in Mathematical Logic.* 2020, **pages** 1–7.

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