Universally Baire sets, determinacy and inner models

Sandra Müller

Jan 7, 2023

ASL Invited Address, JMM, Boston

Research supported by Austrian Science Fund (FWF) Elise Richter grant number V844, International Project I6087, and START Prize Y1498.

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- .... We study natural extensions of ZFC
  to find "the right axioms" for mathematics.

Games in set theory

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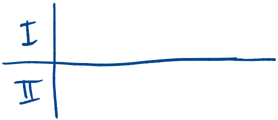

Games in set theory

Fix a set $A \subseteq {}^{\mathbb{N}}\mathbb{N}$.

I	
II	

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infinite sequences of natural numbers
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$$\begin{array}{c|l} \mathrm{I} & n_0 \\ \hline \mathrm{II} & \end{array}$$

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Player I wins iff $(n_0, n_1, \dots) \in A$.

O/w, Player II wins.

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winning strategy

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winning strategy for I in $G(A)$

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II	n_1	n_3

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A function $\sigma : \mathbb{N}^{<\mathbb{N}} \longrightarrow \mathbb{N}$ is a <u>winning strategy</u> for I in $G(A)$ iff

I	$\sigma(\emptyset)$		$\sigma(\sigma(\emptyset), n_1)$	
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<u>Def</u>: The set A is <u>determined</u> iff one of the players has a winning strategy.

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Def: The set A is determined iff one of the players has a winning strategy.

The Axiom of Determinacy says: Every set $A \subseteq {}^{\mathbb{N}}\mathbb{N}$ is determined.

What is determinacy good for?

Theorem (Mycielski, Swierczkowski, Mazur, Davis, 60's)

If all sets of reals are determined, then all sets of reals

- *are Lebesgue measurable,*
- *have the Baire property, and*
- *have the perfect set property.*

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If all sets of reals are determined and X is a zero-dimensional homogeneous space that is not locally compact, then X is strongly homogeneous.

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If all Borel *sets of reals are determined, then all* Borel *sets of reals*

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Theorem (Carroy-Medini-M, JML 2020) (van Engelen, 1986)

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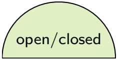
All of these results have *local versions.*

Which games are determined?



Gale-Stewart (1953), ZFC

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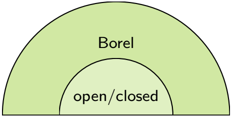

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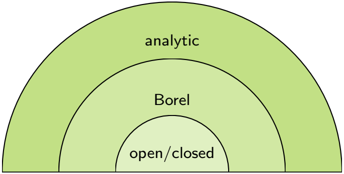

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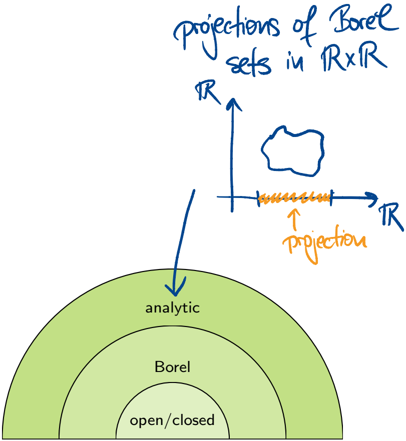

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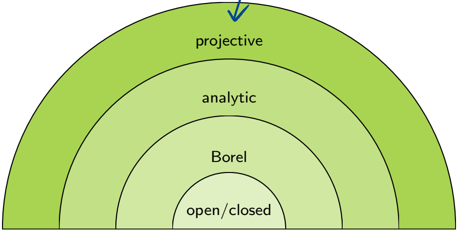
Which games are determined?

Iteratively build projections & complements



Martin-Steel (1985), Woodin cardinals and a measurable cardinal

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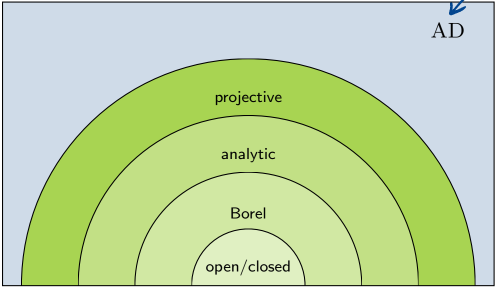
all games are determined



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Which games are determined?

all games on $\mathbb{R}$ are determined

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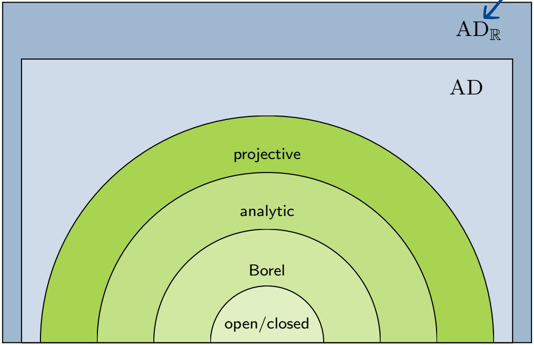

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Determinacy and large cardinals

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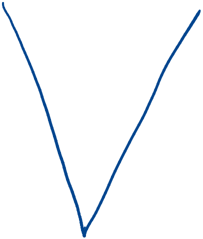

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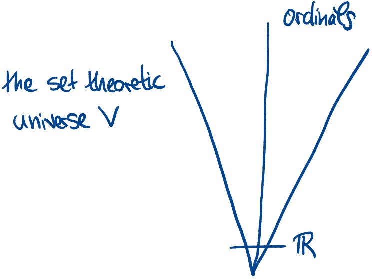

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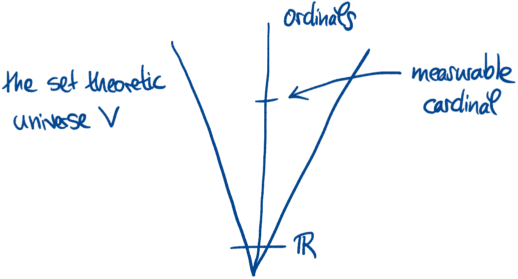

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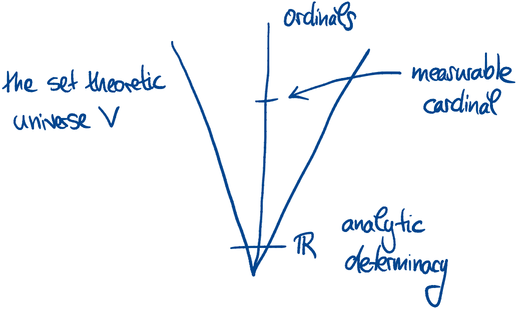

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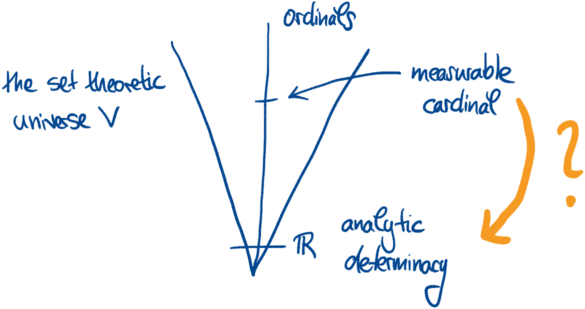

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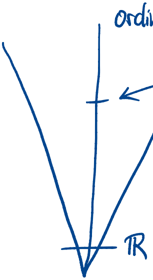

measurable cardinal

analytic determinacy

canonical model "mouse" for a measurable cardinal

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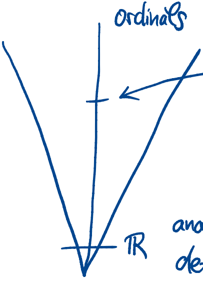

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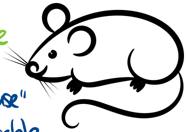

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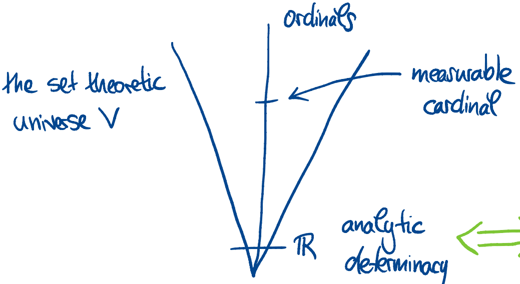
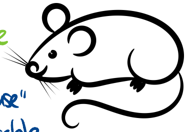


analytic determinacy $\Longleftrightarrow$ countable canonical model "mouse" for a measurable cardinal



Equivalences for analytic and projective determinacy

Theorem (Harrington, Martin)

The following are equivalent.

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Theorem (Neeman, Woodin)

The following are equivalent for all $n \geq 1$.

1. *All $\mathbf{\Sigma}^1_{n+1}$ sets are determined.*
2. *For every real x the ω_1-iterable countable model of set theory with n Woodin cardinals $M_n^{\#}(x)$ exists.*

For (1) $\Rightarrow$ (2) see (M-Schindler-Woodin) "Mice with Finitely many Woodin Cardinals from Optimal Determinacy Hypotheses", JML 2020.

For (2) $\Rightarrow$ (1) see (Neeman) "Optimal proofs of determinacy II", JML 2002.

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Games of countable length $\alpha > \omega$ and games on reals

Games	**Det $\Rightarrow$ Mice**	**Mice $\Rightarrow$ Det**
Analytic, length ω on $\mathbb{N}$	Martin, 1970	Harrington, 1978
Projective, length ω on $\mathbb{N}$, level by level	Woodin, appeared in M-Schindler-Woodin, JML 2020	Neeman, 2002, building on Martin-Steel, 1989
σ-projective, length ω on $\mathbb{N}$	Aguilera	Aguilera-M-Schlicht, APAL 2021, Aguilera
Projective, length ω^2 on $\mathbb{N}$	Aguilera-M, JSL 2020	Neeman, 2004
Analytic, length ω^α on $\mathbb{N}$	Trang, 2013, building on Woodin	Neeman, 2004
Projective, length ω^α on $\mathbb{N}$	M, 2020	Neeman, 2004
Projective, length ω on $\mathbb{R}$	Aguilera-M, NDJFL 2020	easy from Martin-Steel, 1989

Another approach to strengthen determinacy

Keep playing games of length ω and impose additional structural properties on the model.

$\mathrm{AD} +$ all sets of reals are Suslin

Being Suslin is a generalization of being analytic. More precisely, a set of reals is *Suslin* if it is the projection of a tree on $\omega \times \kappa$ for some ordinal κ.

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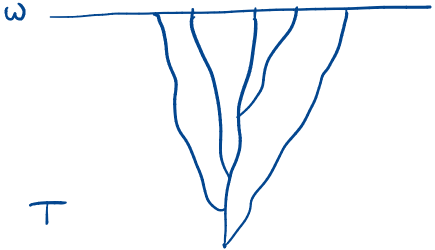

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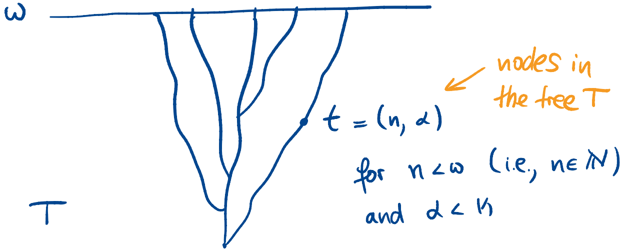

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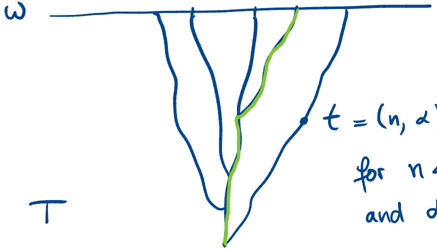

nodes in the tree T

$t = (n, \alpha)$

for $n < \omega$ (i.e., $n \in \mathbb{N}$)

and $\alpha < \kappa$

$[T]$ = the set of $\boxed{\text{branches}}$ through T

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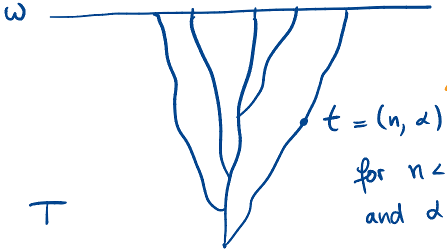

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nodes in the tree T

$[T]$ = the set of branches through T

$x \in p[T]$ iff
x is given by first coordinates of a branch through T

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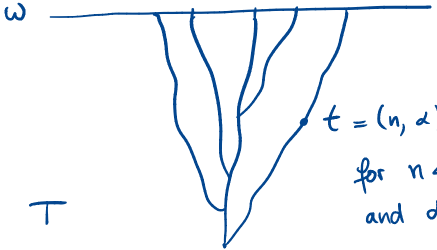

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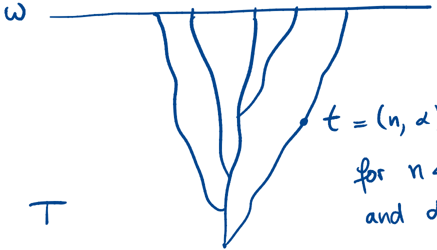
nodes in the tree T

$[T]$ = the set of branches through T

$x \in p[T]$ iff $x = (n_0, n_1, n_2, \dots)$ and there is a branch $b \in [T]$ with $(n_i, \alpha_i) \in b$ for some $\alpha_i < \kappa$ for all $i < \omega$

AD + all sets of reals are Suslin

Being Suslin is a generalization of being analytic. More precisely, a set of reals is *Suslin* if it is the projection of a tree on $\omega \times \kappa$ for some ordinal κ.



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Are the large cardinals necessary here?

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Theorem (Steel, 2008)

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A further strengthening: universally Baire sets

Definition (Schilling-Vaught, Feng-Magidor-Woodin)

A subset A of a topological space Y is *universally Baire* if $f^{-1}(A)$ has the property of Baire in any topological space X, where $f\colon X \to Y$ is continuous.

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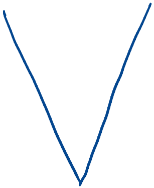

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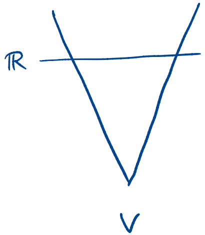

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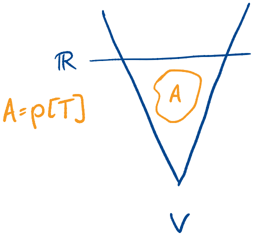

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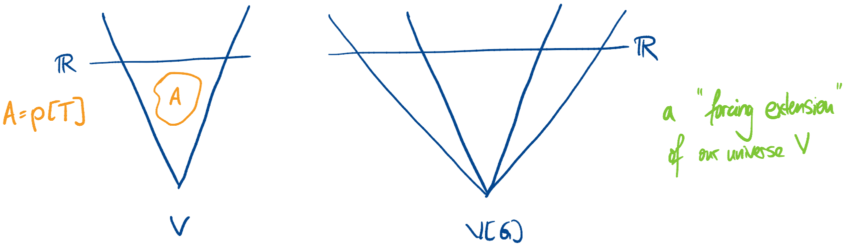

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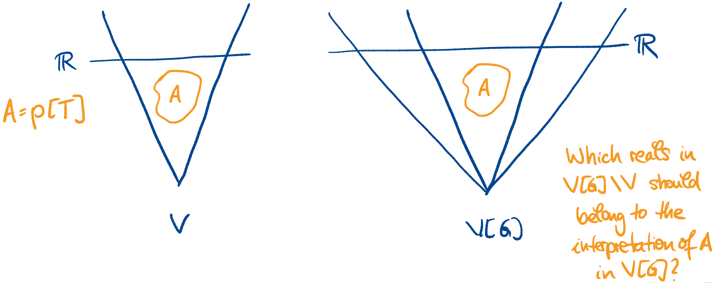

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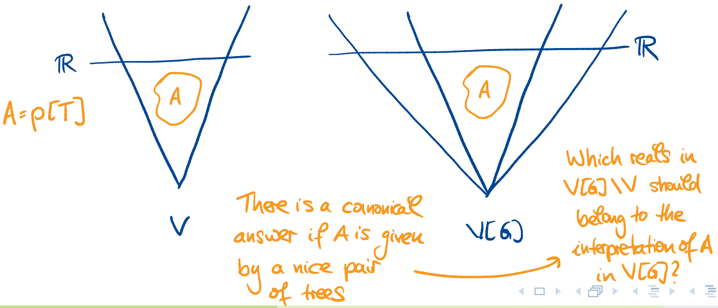

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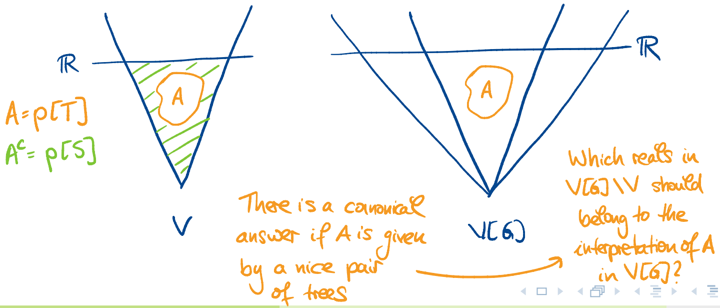

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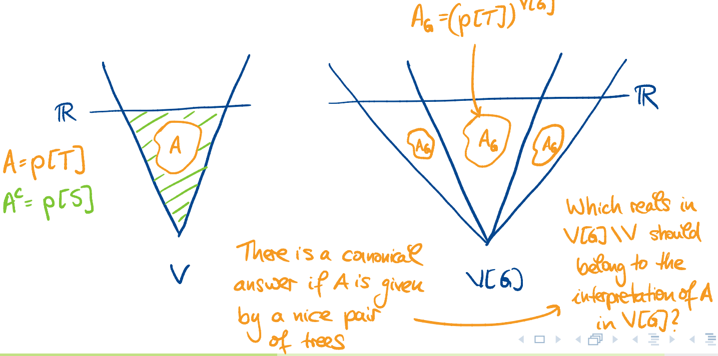

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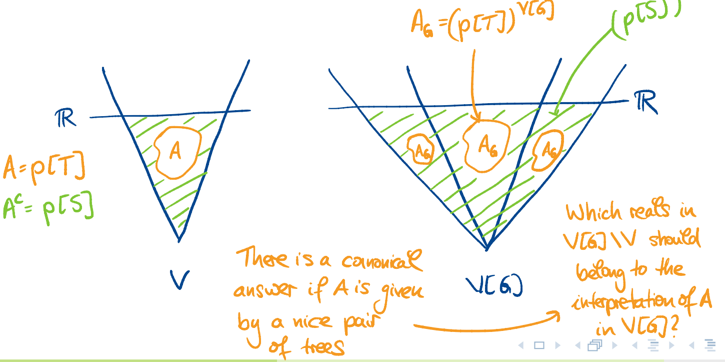

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Definition (Feng-Magidor-Woodin)

A set of reals A is *universally Baire (uB)* if for every Z, there are Z-absolutely complementing trees (S, T) with $p[S] = A$.

Can all sets be universally Baire?

Is there a model of determinacy in which all sets are universally Baire?

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Suppose there is a cardinal λ that is

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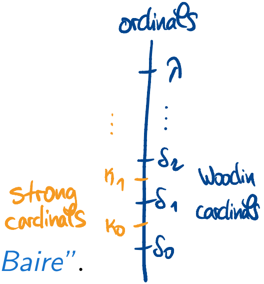

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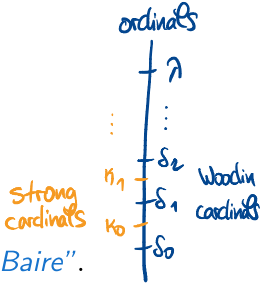

Conjecture (Sargsyan)

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Sargsyan's conjecture holds

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Suppose there is a proper class model of

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The proof is based on a new translation procedure to translate iteration strategies in hybrid mice into large cardinals. This extends work of Steel, Zhu, and Sargsyan.

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Not all questions in mathematics can be answered in ZFC

Abstractly: Gödel's incompleteness theorems.

→ There are statements that are independent from ZFC

standard axioms for set theory

Nowadays there are numerous concrete examples:

Is there a set A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$?

- Continuum Problem (set theory),
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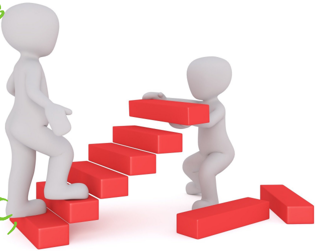

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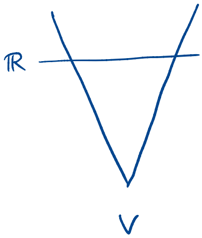
↳ We need new ideas and hard work to break this barrier!

Sealing: Why is a Woodin limit of Woodins a barrier?

Shoenfield's Absoluteness Theorem implies that the truth of Σ^1_2-facts cannot be changed by forcing.



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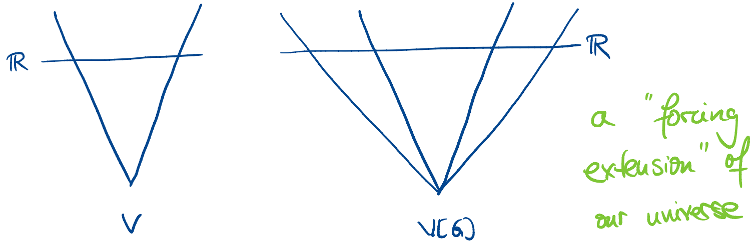

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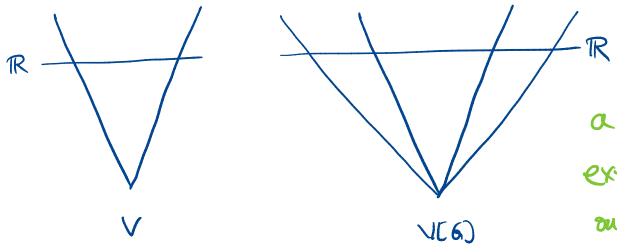

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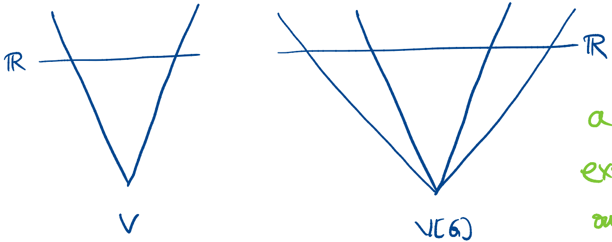

a "forcing extension" of our universe

$\psi(z)$ holds in V $\iff$ $\psi(z)$ holds in $V[G]$

for any $z \in V$

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Theorem (Woodin)

If A is a universally Baire set of reals and there is a class of Woodin cardinals, then the theory of $L(A, \mathbb{R})$ cannot be changed by forcing, i.e., for any set generic extensions $V[g] \subseteq V[g * h]$, there is an elementary embedding

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Sealing is a similar but stronger generic absoluteness statement: It asserts that the theory of all the universally Baire sets cannot be changed by forcing.

Sealing

Definition (Woodin)

Sealing is the conjunction of the following statements.

1. For every set generic g over V, $L(\Gamma^\infty_g, \mathbb{R}_g) \models \mathrm{AD}^+$ and $\mathcal{P}(\mathbb{R}_g) \cap L(\Gamma^\infty_g, \mathbb{R}_g) = \Gamma^\infty_g$.
2. For every set generic extensions $V[g] \subseteq V[g * h]$, there is an elementary embedding

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Sealing

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the universally Baire sets of reals in V[g]

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a technical strengthening of the Axiom of Determinacy

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2. For every set generic extensions $V[g] \subseteq V[g * h]$, there is an elementary embedding

$$j\colon L(\Gamma^\infty_g, \mathbb{R}_g) \to L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h})$$

such that for every $A \in \Gamma^\infty_g$, $j(A) = A_h$.

Sealing

the universally Baire sets of reals in $V[g]$

$\mathbb{R}^{V[g]}$

Definition (Woodin)

Sealing is the conjunction of the following statements.

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2. For every set generic extensions $V[g] \subseteq V[g * h]$, there is an elementary embedding

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→ Our current canonical inner models cannot satisfy Sealing

→ Our techniques, e.g., Core Model Induction break down (Sargsyan-Trang, 2019)

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- (M-Sargsyan, Woodin, 2022) Suppose there is a proper class of Woodin cardinals, κ is supercompact, and $\lambda > \kappa$ is an inaccessible limit of Woodin cardinals. Then Θ regular-Sealing holds after collapsing 2^{2^κ} to be countable.

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Is there a large cardinal that implies Sealing?

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