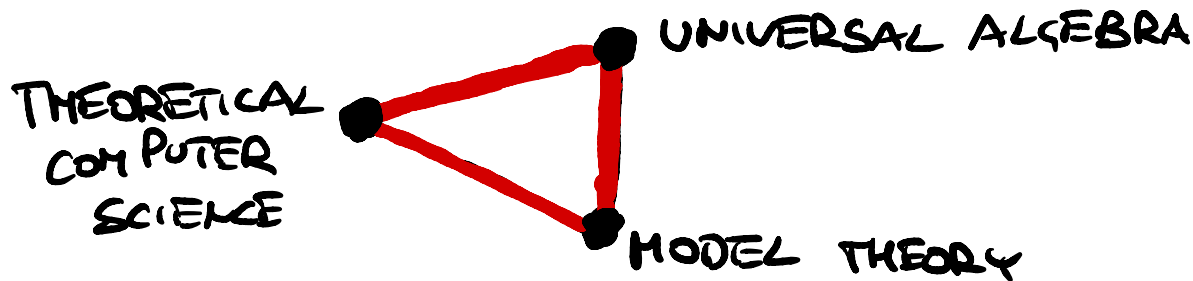


THE POWER OF POLYMORPHISMS



MICHAEL PINSKER
TU WIEN

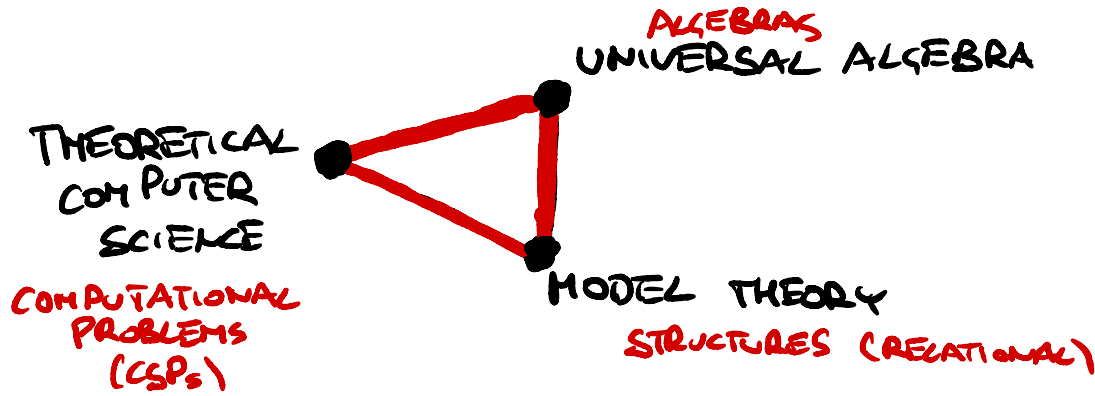
მზღორობი 9.9.2026

FWF Austrian
Science Fund

IS948



სყ რეკოპ
6A 101071674



PART I

SUDOKUS, CONSTRAINT SATISFACTION PROBLEMS, P vs. NP

PART II

CSPs, POLYMORPHISMS

PART I

SUDOKUS, CONSTRAINT SATISFACTION PROBLEMS, P vs. NP

	2							
			6					3
	7	4		8				
					3			2
	8			4			1	
6			5					
				1		7	8	
5					9			
							4	

1	2							
			6					3
	7	4		8				
					3			2
	8			4			1	
6			5					
				1		7	8	
5					9			
							4	

1	2	6						8
			6					3
	7	4		8				
					3			2
	8			4			1	
6			5					
				1		7	8	
5					9			
							4	

1	2	6						8
			6					3
	7	4		8				
					3			2
	8			4			1	
6			5					
				1		7	8	
5	4				9			
7							4	9

1	2	6						8
8	9	5	6					3
3	7	4		8				
4					3			2
9	8			4			1	
6			5					
2				1		7	8	
5	4				9			
7							4	9

1	2	6						8
8	9	5	6					3
3	7	4		8				
4	5	7	1	9	3			2
9	8	3	2	4	6		1	
6	1	2	5	7	8			
2				1		7	8	
5	4				9			
7							4	9

1	2	6						8
8	9	5	6					3
3	7	4		8				
4	5	7	1	9	3	8	6	2
9	8	3	2	4	6	5	1	7
6	1	2	5	7	8	3	9	4
2				1		7	8	5
5	4				9	2	3	1
7						6	4	9

1	2	6						8
8	9	5	6					3
3	7	4		8				
4	5	7	1	9	3	8	6	2
9	8	3	2	4	6	5	1	7
6	1	2	5	7	8	3	9	4
2	6	9	3	1	4	7	8	5
5	4	8	7	6	9	2	3	1
7	3	1	8	5	2	6	4	9

1	2	6	4	3	7	9	5	8
8	9	5	6	2	1	4	7	3
3	7	4	9	8	5	1	2	6
4	5	7	1	9	3	8	6	2
9	8	3	2	4	6	5	1	7
6	1	2	5	7	8	3	9	4
2	6	9	3	1	4	7	8	5
5	4	8	7	6	9	2	3	1
7	3	1	8	5	2	6	4	9

1	2	6	4	3	7	9	5	8
8	9	5	6	2	1	4	7	3
3	7	4	9	8	5	1	2	6
4	5	7	1	9	3	8	6	2
9	8	3	2	4	6	5	1	7
6	1	2	5	7	8	3	9	4
2	6	9	3	1	4	7	8	5
5	4	8	7	6	9	2	3	1
7	3	1	8	5	2	6	4	9

SANDIWAY FONG / UNIVERSITY OF ARIZONA
LINGUISTICS, AI

EASY - MEDIUM - DIFFICULT - VERY DIFFICULT

1	2	6	4	3	7	9	5	8
8	9	5	6	2	1	4	7	3
3	7	4	9	8	5	1	2	6
4	5	7	1	9	3	8	6	2
9	8	3	2	4	6	5	1	7
6	1	2	5	7	8	3	9	4
2	6	9	3	1	4	7	8	5
5	4	8	7	6	9	2	3	1
7	3	1	8	5	2	6	4	9

SANDIWAY FONG / UNIVERSITY OF ARIZONA
LINGUISTICS, AI

EASY - MEDIUM - DIFFICULT - VERY DIFFICULT - NOT FUN

"REAL" MATHEMATICS: SYSTEMS OF LINEAR EQUATIONS

"REAL" MATHEMATICS: SYSTEMS OF LINEAR EQUATIONS

DIFFICULT!

"REAL" MATHEMATICS: SYSTEMS OF LINEAR EQUATIONS

I

$$x_1 + x_2 + 4x_3 = -3$$

II

$$-3x_1 + 9x_2 - x_3 = -13$$

III

$$6x_1 - 2x_2 + 10x_3 = 10$$



DIFFICULT!

"REAL" MATHEMATICS: SYSTEMS OF LINEAR EQUATIONS



$$\text{I} \quad x_1 + x_2 + 4x_3 = -3$$

$$\text{II} \quad -3x_1 + 9x_2 - x_3 = -13$$

$$\text{III} \quad 6x_1 - 2x_2 + 10x_3 = 10$$

$$\text{I} \quad x_1 + x_2 + 4x_3 = -3$$

$$3\text{I} + \text{II} \quad 12x_2 + 11x_3 = -22$$

$$-6\text{I} + \text{III} \quad -8x_2 - 14x_3 = 28$$

"REAL" MATHEMATICS: SYSTEMS OF LINEAR EQUATIONS



$$\text{I} \quad x_1 + x_2 + 4x_3 = -3$$

$$\text{II} \quad -3x_1 + 9x_2 - x_3 = -13$$

$$\text{III} \quad 6x_1 - 2x_2 + 10x_3 = 10$$

$$\text{I} \quad x_1 + x_2 + 4x_3 = -3$$

$$3\text{I} + \text{II} \quad 12x_2 + 11x_3 = -22$$

$$-6\text{I} + \text{III} \quad -8x_2 - 14x_3 = 28$$

$$\text{I} \quad x_1 + x_2 + 4x_3 = -3$$

$$\text{II} \quad 12x_2 + 11x_3 = -22$$

$$2\text{II} + 3\text{III} \quad -20x_3 = 40$$

"REAL" MATHEMATICS: SYSTEMS OF LINEAR EQUATIONS



$$\begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \end{array} \quad \begin{array}{l} x_1 + x_2 + 4x_3 = -3 \\ -3x_1 + 9x_2 - x_3 = -13 \\ 6x_1 - 2x_2 + 10x_3 = 10 \end{array}$$

$$\begin{array}{l} \text{I} \\ 3\text{I} + \text{II} \\ -6\text{I} + \text{III} \end{array} \quad \begin{array}{l} x_1 + x_2 + 4x_3 = -3 \\ 12x_2 + 11x_3 = -22 \\ -8x_2 - 14x_3 = 28 \end{array}$$

$$\begin{array}{l} \text{I} \\ \text{II} \\ 2\text{II} + 3\text{III} \end{array} \quad \begin{array}{l} x_1 + x_2 + 4x_3 = -3 \\ 12x_2 + 11x_3 = -22 \\ -20x_3 = 40 \end{array} \quad \left. \vphantom{\begin{array}{l} \text{I} \\ \text{II} \\ 2\text{II} + 3\text{III} \end{array}} \right\} \Rightarrow \quad \begin{array}{l} x_3 = -2 \\ x_2 = 0 \\ x_1 = 5 \end{array}$$

"REAL" MATHEMATICS: SYSTEMS OF LINEAR EQUATIONS



FAIR???

	2						
			6				3
	7	4		8			
					3		2
	8			4		1	
6			5				
				1		7	8
5					9		
							4

$$a_{11}x_1 + \dots + a_{1q}x_q = b_1$$

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$$

$$a_{q1}x_1 + \dots + a_{qq}x_q = b_q$$

SAME METHOD WORKS !

[illegible]

	SUDOKU	ERU
IS "REAL" MATHEMATICS		

	SUDOKU	ERU
IS "REAL" MATHEMATICS		✓

	SUDOKU	ERU
IS "REAL" MATHEMATICS	✓	✓

	SUDOKU	ERU
IS "REAL" MATHEMATICS	✓	✓
MATHEMATICS HELPFUL		

	SUDOKU	ERU
IS "REAL" MATHEMATICS	✓	✓
MATHEMATICS HELPFUL		✓

	SUDOKU	ERU
IS "REAL" MATHEMATICS	✓	✓
MATHEMATICS HELPFUL	?	✓

	SUDOKU	LGS
IST ECHTE MATHEMATIK	✓	✓
MATHEMATIK HILFREICH	?	✓
FUN GAME		

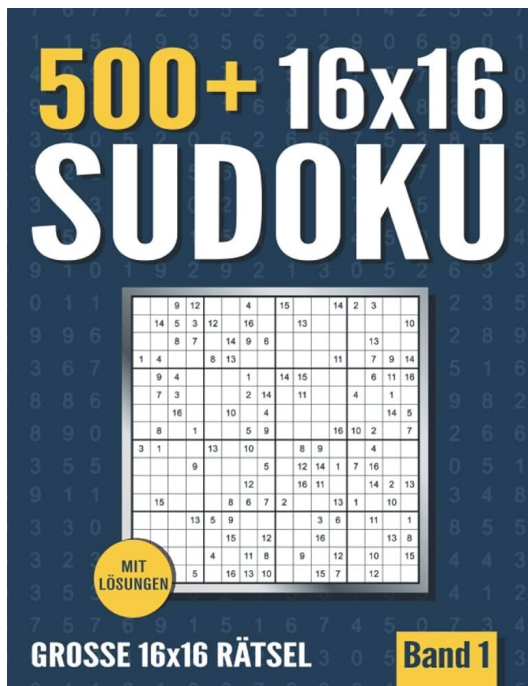
500+ 16x16 SUDOKU



MIT
LÖSUNGEN

GROSSE 16x16 RÄTSEL

Band 1



	SUDOKU	LGS
IST ECHTE MATHEMATIK	✓	✓
MATHEMATIK HILFREICH	?	✓
FUN GAME	✓	

500+ 16x16

SYSTEMS OF LINEAR EQUATIONS



WITH SOLUTIONS

BLK 16x16 SYSTEMS VOL. I

LINEAR 100% killers - no fillers!

ISSUE 38 £3.20

Killer

EQUATIONS

OVER 200!

PuzzleLife

Britain's best value killers!



Killer EQUATIONS

87 killer equations graded from easy to tricky

PuzzleLife

Barcode: 9 772622 597025

167 MARCH 2015

MATHEMATIK 5

Kompetent AUFSTEIGEN ...

- ✓ Kompetenzorientiert
- ✓ Bildungsstandards
- ✓ Extra Lösungsheft

5. Klasse AHS

Stefan Wurster

GRS

	SUDOKU	LGS
IST ECHTE MATHEMATIK	✓	✓
MATHEMATIK HILFREICH	?	✓
FUN GAME	✓	✗

	SUDOKU	LGS
IST ECHTE MATHEMATIK	✓	✓
MATHEMATIK HILFREICH	?	✓
FUN GAME	✓	✗
METHOD KNOWN		

	SUDOKU	LGS
IST ECHTE MATHEMATIK	✓	✓
MATHEMATIK HILFREICH	?	✓
FUN GAME	✓	✗
METHOD KNOWN		✓

	SUDOKU	LGS
IST ECHTE MATHEMATIK	✓	✓
MATHEMATIK HILFREICH	?	✓
FUN GAME	✓	✗
METHOD KNOWN	✓	✓

	SUDOKU	LGS
IST ECHTE MATHEMATIK	✓	✓
MATHEMATIK HILFREICH	?	✓
FUN GAME	✓	✗
METHOD KNOWN	✓	✓
EFFICIENT METHOD KNOWN		

	SUDOKU	LGS
IST ECHTE MATHEMATIK	✓	✓
MATHEMATIK HILFREICH	?	✓
FUN GAME	✓	✗
METHOD KNOWN	✓	✓
EFFICIENT METHOD KNOWN		✓

	SUDOKU	LGS
IST ECHTE MATHEMATIK	✓	✓
MATHEMATIK HILFREICH	?	✓
FUN GAME	✓	✗
METHOD KNOWN	✓	✓
EFFICIENT METHOD KNOWN	✗	✓

	SUDOKU	LGS
IST ECHTE MATHEMATIK	✓	✓
MATHEMATIK HILFREICH	?	✓
FUN GAME	✓	✗
METHOD KNOWN	✓	✓
EFFICIENT METHOD KNOWN	✗	✓

HOW MANY STEPS DEPENDING ON THE SIZE OF THE GAME?

	SUDOKU	LGS
IST ECHTE MATHEMATIK	✓	✓
MATHEMATIK HILFREICH	?	✓
FUN GAME	✓	✗
METHOD KNOWN	✓	✓
EFFICIENT METHOD KNOWN	✗	✓

HOW MANY STEPS DEPENDING ON THE SIZE OF THE GAME?

METHOD

- LOCAL CONSISTENCY
- LISTS OF POSSIBLE VALUES
- ELIMINATE, PROPAGATE
- **TRY**, BACKTRACK

$$\leq \underbrace{q \cdot \dots \cdot q}_{q^2}$$

$$= q^{q^2}$$

$$n^{n^2}$$

EXPONENTIAL

	SUDOKU	LGS
IST ECHTE MATHEMATIK	✓	✓
MATHEMATIK HILFREICH	?	✓
FUN GAME	✓	✗
METHOD KNOWN	✓	✓
EFFICIENT METHOD KNOWN	✗	✓

HOW MANY STEPS DEPENDING ON THE SIZE OF THE GAME ?

METHOD

- LOCAL CONSISTENCY
- LISTS OF POSSIBLE VALUES
- ELIMINATE, PROPAGATE
- TRY, BACKTRACK

$$\leq \underbrace{q \cdot \dots \cdot q}_{q^2}$$

$$= q^{q^2}$$

$$n^{n^2}$$

EXPONENTIAL

P

$$n^4$$

POLYNOMIAL

METHOD

GAUSS -
ALGORITHM

SUDOKU IN P ?

YES

THERE IS A POLYNOMIAL
SOLUTION METHOD

NO

THERE IS NO POLYNOMIAL
SOLUTION METHOD

SUDOKU IN P ?

YES

NO

THERE IS A POLYNOMIAL
SOLUTION METHOD

THERE IS NO POLYNOMIAL
SOLUTION METHOD

⇒

1000000 \$

!!!

SUDOKU IN P ?

YES

NO

THERE IS A POLYNOMIAL
SOLUTION METHOD

THERE IS NO POLYNOMIAL
SOLUTION METHOD

⇒

1000000 \$

MILLENNIUM-PROBLEMS OF CLAY MATHEMATICS INSTITUTE (2000)

- 1, CONJECTURE OF BIRCH & SWINNERTON-DYER
- 2, CONJECTURE OF HODGE
- 3, NAVIER-STOKES - EQUATIONS
- 4, SUDOKU IN P ?
- 5, POINCARÉ - CONJECTURE
- 6, RIEMANN - CONJECTURE
- 7, EQUATIONS OF YANG-MILLS

SUDOKU IN P ?

YES

NO

THERE IS A POLYNOMIAL
SOLUTION METHOD

THERE IS NO POLYNOMIAL
SOLUTION METHOD

⇒

1000000 \$

MILLENNIUM-PROBLEMS OF CLAY MATHEMATICS INSTITUTE (2000)

- 1, CONJECTURE OF BIRCH & SWINNERTON-DYER
- 2, CONJECTURE OF MURGE
- 3, NAVIER-STOKES - EQUATIONS
- 4, SUDOKU IN P ?
- ✓ 5, PONCARE - CONJECTURE
- 6, RIEMANN - CONJECTURE
- 7, EQUATIONS OF YANG-MILLS

CONSTRAINT SATISFACTION PROBLEMS (CSPs)

- GIVEN: VARIABLES $x_1, \dots, x_n$
& CONSTRAINTS
- QUESTION: FIND VALUES $\alpha_1, \dots, \alpha_n$
SATISFYING ALL CONSTRAINTS

CONSTRAINT SATISFACTION PROBLEMS (CSPs)

- GIVEN: VARIABLES $x_1, \dots, x_n$
& CONSTRAINTS
- QUESTION: FIND VALUES $\alpha_1, \dots, \alpha_n$
SATISFYING ALL CONSTRAINTS

SUDOKU

x_{11}	2	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}
x_{21}	x_{22}	x_{23}	6	x_{25}	x_{26}	x_{27}	x_{28}	3
x_{31}	7	4	x_{34}	8	x_{36}	x_{37}	x_{38}	x_{39}
x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	3	x_{47}	x_{48}	2
x_{51}	8	x_{53}	x_{54}	4	x_{56}	x_{57}	1	x_{59}
6	x_{62}	x_{63}	5	x_{65}	x_{66}	x_{67}	x_{68}	x_{69}
x_{71}	x_{72}	x_{73}	x_{74}	1	x_{76}	7	8	x_{79}
5	x_{82}	x_{83}	x_{84}	x_{85}	9	x_{87}	x_{88}	x_{89}
x_{91}	x_{92}	x_{93}	x_{94}	x_{95}	x_{96}	x_{97}	4	x_{99}

IN P ?

CONSTRAINT SATISFACTION PROBLEMS (CSPs)

- GIVEN: VARIABLES $x_1, \dots, x_n$
& CONSTRAINTS
- QUESTION: FIND VALUES $\alpha_1, \dots, \alpha_n$
SATISFYING ALL CONSTRAINTS

SUDOKU

x_{11}	2	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}
x_{21}	x_{22}	x_{23}	6	x_{25}	x_{26}	x_{27}	x_{28}	3
x_{31}	7	4	x_{34}	8	x_{36}	x_{37}	x_{38}	x_{39}
x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	3	x_{47}	x_{48}	2
x_{51}	8	x_{53}	x_{54}	4	x_{56}	x_{57}	1	x_{59}
6	x_{62}	x_{63}	5	x_{65}	x_{66}	x_{67}	x_{68}	x_{69}
x_{71}	x_{72}	x_{73}	x_{74}	1	x_{76}	7	8	x_{79}
5	x_{82}	x_{83}	x_{84}	x_{85}	9	x_{87}	x_{88}	x_{89}
x_{91}	x_{92}	x_{93}	x_{94}	x_{95}	x_{96}	x_{97}	4	x_{99}

IN P ?

LINEAR SYSTEMS OF EQUATIONS OVER $\mathbb{Q}$

$$\begin{aligned} x_1 + x_2 + 4x_3 &= -3 \\ -3x_1 + 9x_2 - x_3 &= -13 \\ 6x_1 - 2x_2 + 10x_3 &= 10 \end{aligned}$$

IN P

CONSTRAINT SATISFACTION PROBLEMS (CSPs)

- GIVEN: VARIABLES $x_1, \dots, x_n$
& CONSTRAINTS
- QUESTION: FIND VALUES $\alpha_1, \dots, \alpha_n$
SATISFYING ALL CONSTRAINTS

SUDOKU

x_{11}	2	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}
x_{21}	x_{22}	x_{23}	6	x_{25}	x_{26}	x_{27}	x_{28}	3
x_{31}	7	4	x_{34}	8	x_{36}	x_{37}	x_{38}	x_{39}
x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	3	x_{47}	x_{48}	2
x_{51}	8	x_{53}	x_{54}	4	x_{56}	x_{57}	1	x_{59}
6	x_{62}	x_{63}	5	x_{65}	x_{66}	x_{67}	x_{68}	x_{69}
x_{71}	x_{72}	x_{73}	x_{74}	1	x_{76}	7	8	x_{79}
5	x_{82}	x_{83}	x_{84}	x_{85}	9	x_{87}	x_{88}	x_{89}
x_{91}	x_{92}	x_{93}	x_{94}	x_{95}	x_{96}	x_{97}	4	x_{99}

IN P ?

LINEAR SYSTEMS OF EQUATIONS OVER $\mathbb{Q}$

$$\begin{aligned}x_1 + x_2 + 4x_3 &= -3 \\ -3x_1 + 9x_2 - x_3 &= -13 \\ 6x_1 - 2x_2 + 10x_3 &= 10\end{aligned}$$

IN P

LINEAR SYSTEMS OF EQUATIONS OVER $\mathbb{Z}$

$$\begin{aligned}x_1 + x_2 + 4x_3 &= -3 \\ -3x_1 + 9x_2 - x_3 &= -13 \\ 6x_1 - 2x_2 + 10x_3 &= 10\end{aligned}$$

IN P

CONSTRAINT SATISFACTION PROBLEMS (CSPs)

- GIVEN: VARIABLES $x_1, \dots, x_n$
& CONSTRAINTS
- QUESTION: FIND VALUES $\alpha_1, \dots, \alpha_n$
SATISFYING ALL CONSTRAINTS

DIOPHANTINE EQUATIONS OVER $\mathbb{Z}$

$$x_1^2 + x_2^2 = 1$$

$$x_1^3 + x_2^3 + x_3 x_1 = 7$$

NO METHOD

"UNDECIDABLE"

MATYASEVICH 1970

HILBERTS 10TH PROBLEM 1900

LINEAR SYSTEMS OF EQUATIONS OVER $\mathbb{Q}$

$$\begin{aligned}x_1 + x_2 + 4x_3 &= -3 \\ -3x_1 + 9x_2 - x_3 &= -13 \\ 6x_1 - 2x_2 + 10x_3 &= 10\end{aligned}$$

IN $\mathbb{P}$

LINEAR SYSTEMS OF EQUATIONS OVER $\mathbb{Z}$

$$\begin{aligned}x_1 + x_2 + 4x_3 &= -3 \\ -3x_1 + 9x_2 - x_3 &= -13 \\ 6x_1 - 2x_2 + 10x_3 &= 10\end{aligned}$$

IN $\mathbb{P}$

CONSTRAINT SATISFACTION PROBLEMS (CSPs)

- GIVEN: VARIABLES $x_1, \dots, x_n$
& CONSTRAINTS
- QUESTION: FIND VALUES $\alpha_1, \dots, \alpha_n$
SATISFYING ALL CONSTRAINTS

DIOPHANTINE EQUATIONS OVER $\mathbb{Z}$

$$x_1^2 + x_2^2 = 1$$

$$x_1^3 + x_2^3 + x_3 x_1 = 7$$

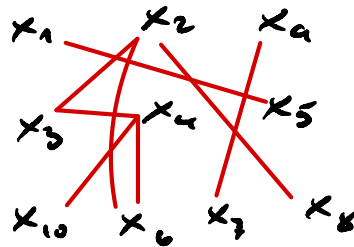
NO METHOD

"UNDECIDABLE"

MATYASEVICH 1970

HILBERTS 10TH PROBLEM 1900

2-COLOURING



LINEAR SYSTEMS OF EQUATIONS OVER $\mathbb{Z}$

$$x_1 + x_2 + 4x_3 = -3$$

$$-3x_1 + 9x_2 - x_3 = -13$$

$$6x_1 - 2x_2 + 10x_3 = 10$$

IN P

CONSTRAINT SATISFACTION PROBLEMS (CSPs)

- GIVEN: VARIABLES $x_1, \dots, x_n$
& CONSTRAINTS
- QUESTION: FIND VALUES $\alpha_1, \dots, \alpha_n$
SATISFYING ALL CONSTRAINTS

DIOPHANTINE EQUATIONS OVER $\mathbb{Z}$

$$x_1^2 + x_2^2 = 1$$

$$x_1^3 + x_2^3 + x_3 x_1 = 7$$

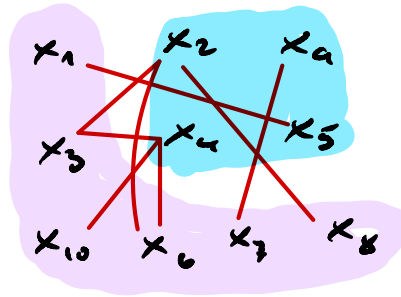
NO METHOD

"UNDECIDABLE"

MATYASEVICH 1970

HILBERTS 10TH PROBLEM 1900

2-COLOURING



IN P

LINEAR SYSTEMS OF EQUATIONS OVER $\mathbb{Z}$

$$x_1 + x_2 + 4x_3 = -3$$

$$-3x_1 + 9x_2 - x_3 = -13$$

$$6x_1 - 2x_2 + 10x_3 = 10$$

IN P

CONSTRAINT SATISFACTION PROBLEMS (CSPs)

- GIVEN: VARIABLES $x_1, \dots, x_n$
& CONSTRAINTS
- QUESTION: FIND VALUES $\alpha_1, \dots, \alpha_n$
SATISFYING ALL CONSTRAINTS

DIOPHANTINE EQUATIONS OVER $\mathbb{Z}$

$$x_1^2 + x_2^2 = 1$$

$$x_1^3 + x_2^3 + x_3 x_1 = 7$$

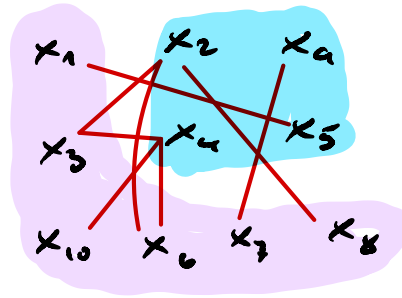
NO METHOD

"UNDECIDABLE"

MATYASEVICH 1970

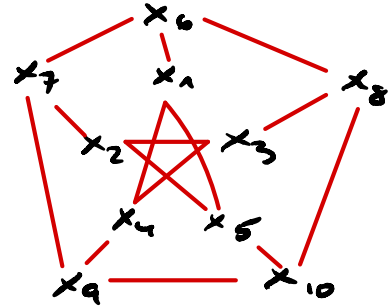
HILBERTS 10TH PROBLEM 1900

2-COLOURING



IN P

3-COLOURING



CONSTRAINT SATISFACTION PROBLEMS (CSPs)

- GIVEN: VARIABLES $x_1, \dots, x_n$
& CONSTRAINTS
- QUESTION: FIND VALUES $\alpha_1, \dots, \alpha_n$
SATISFYING ALL CONSTRAINTS

DIOPHANTINE EQUATIONS OVER $\mathbb{Z}$

$$x_1^2 + x_2^2 = 1$$

$$x_1^3 + x_2^3 + x_3 x_1 = 7$$

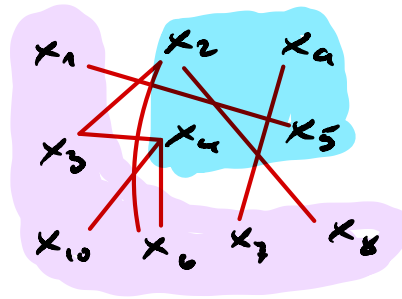
NO METHOD

"UNDECIDABLE"

MATYASEVICH 1970

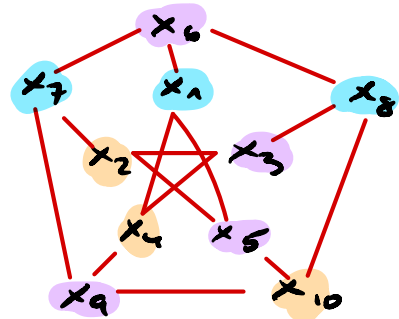
HILBERTS 10TH PROBLEM 1900

2-COLOURING



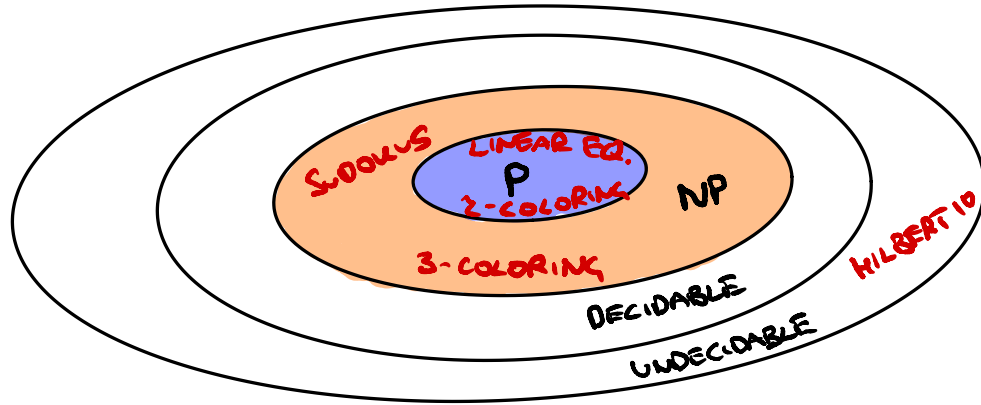
IN P

3-COLOURING

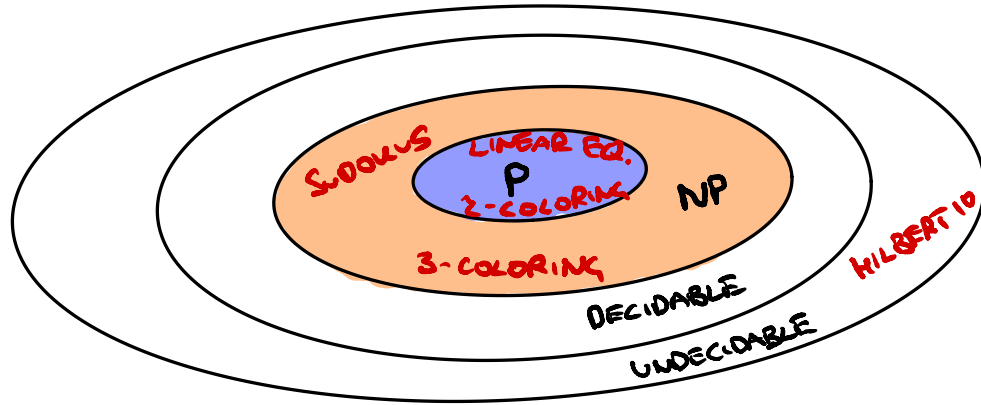


IN P ?

P ... POLYNOMIAL METHOD FOR SOLVING
NP ... POLYNOMIAL METHOD FOR VERIFYING



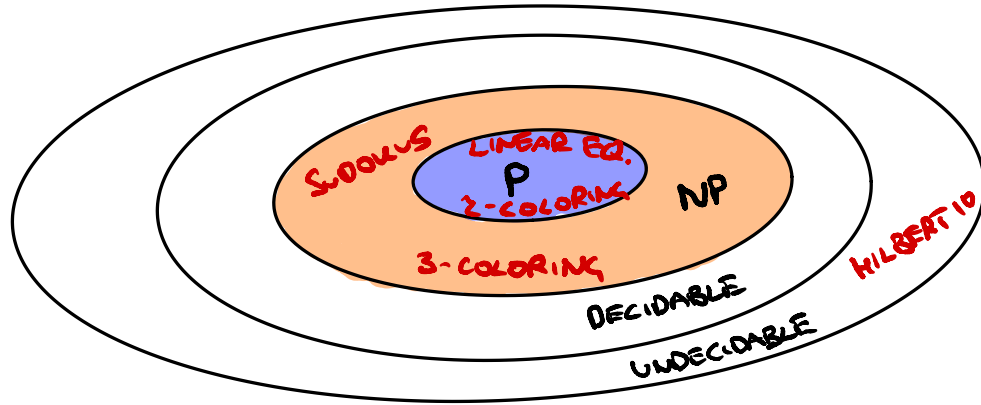
P ... POLYNOMIAL METHOD FOR SOLVING
NP ... POLYNOMIAL METHOD FOR VERIFYING



MILLENNIUM-PROBLEMS OF CLAY MATHEMATICS INSTITUTE (2000)

- 1, CONJECTURE OF BACH & SWINNERTON-DYER
- 2, CONJECTURE OF MOORE
- 3, NAVIER-STOKES-EQUATIONS
- 4, SUDOKU IN P?
- ✓ 5, Poincaré-conjecture
- 6, RIEMANN-CONJECTURE
- 7, EQUATIONS OF YANK-MILLS

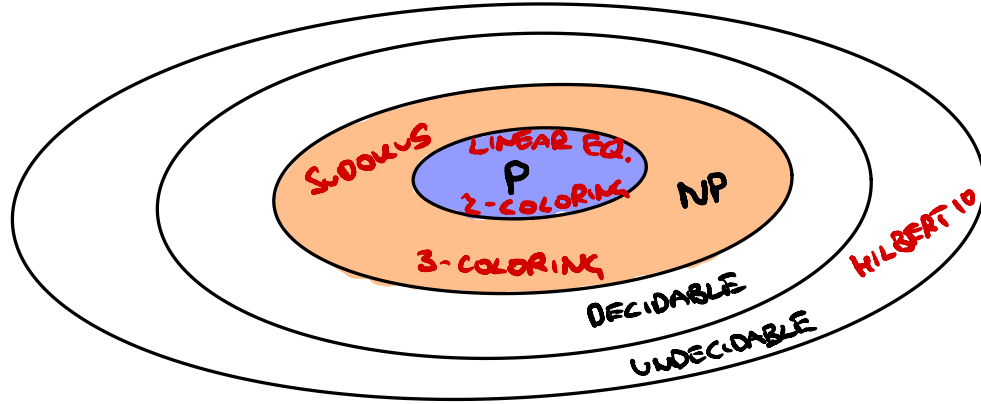
P ... POLYNOMIAL METHOD FOR SOLVING
NP ... POLYNOMIAL METHOD FOR VERIFYING



MILLENNIUM-PROBLEMS OF CLAY MATHEMATICS INSTITUTE (2000)

- 1, CONJECTURE OF BACH & SWINNERTON-DYER
- 2, CONJECTURE OF MOORE
- 3, NAVIER-STOKES-EQUATIONS
- 4, ~~SUDDOKU IN P?~~ $P = NP$
- ✓ 5, PONCAREÉ-CONJECTURE
- 6, RIEMANN-CONJECTURE
- 7, EQUATIONS OF YALK-MILLS

P ... POLYNOMIAL METHOD FOR SOLVING
NP ... POLYNOMIAL METHOD FOR VERIFYING

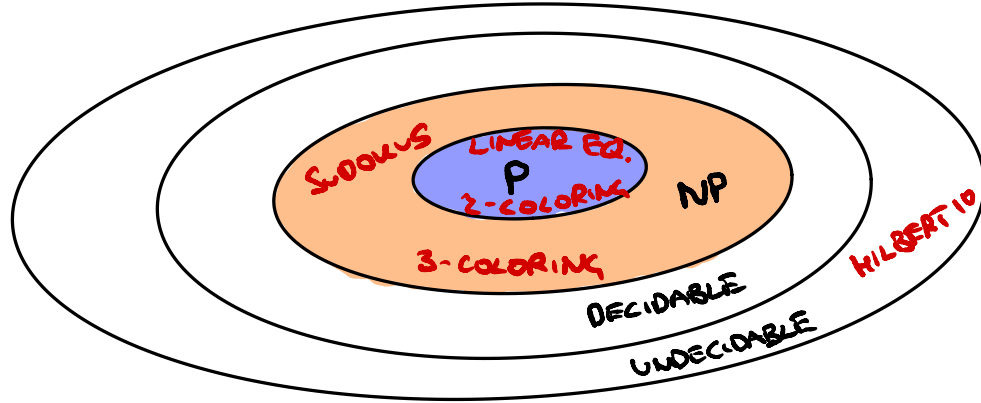


P = NP ?

- 1971 STEPHEN COOK



P ... POLYNOMIAL METHOD FOR SOLVING
NP ... POLYNOMIAL METHOD FOR VERIFYING

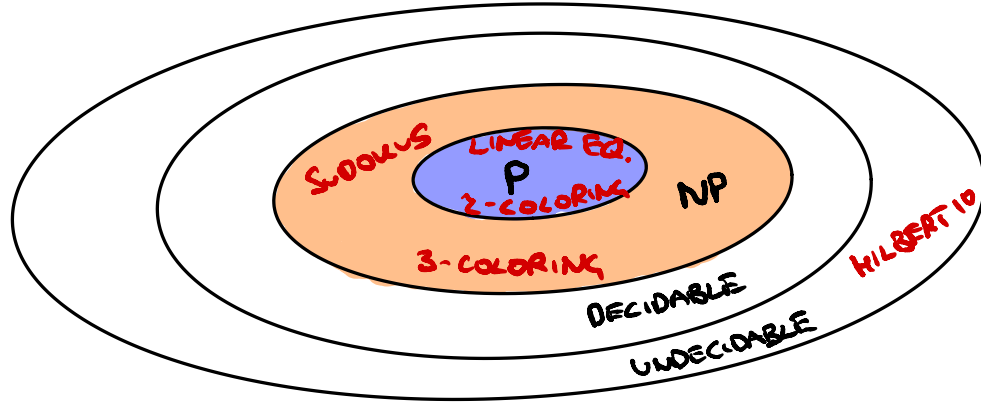


P = NP ?

- 1971 STEPHEN COOK
- NO MORE SUDOKUS



P ... POLYNOMIAL METHOD FOR SOLVING
NP ... POLYNOMIAL METHOD FOR VERIFYING

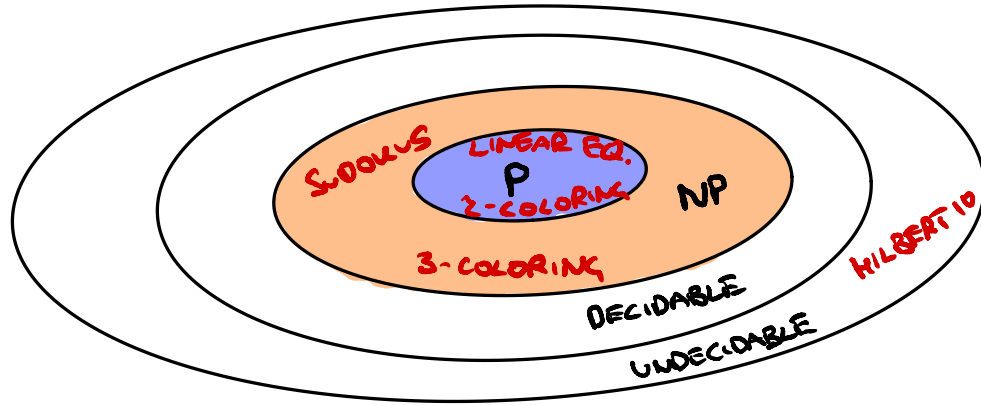


P = NP ?

- 1971 STEPHEN COOK
- NO MORE SUDOKUS
- EFFICIENT MATHEMATICAL PROVING
KURT GÖDEL 1956



P ... POLYNOMIAL METHOD FOR SOLVING
NP ... POLYNOMIAL METHOD FOR VERIFYING



P = NP ?

- 1971 STEPHEN COOK
- NO MORE SUDOKUS
- EFFICIENT MATHEMATICAL PROVING
KURT GÖDEL 1956
- EFFICIENT FACTORING OF PRIME NUMBERS

ALICE  BOB
 $n = p \cdot q$

SUDOKU IN $P \Rightarrow P = NP : \text{WHY?}$

SUDOKU IN $P \Rightarrow P = NP$: WHY?

SUDOKU IS NP-COMPLETE : EVERY PROBLEM IN NP
REDUCES TO SUDOKU !

SUDOKU IN $P \Rightarrow P = NP : \text{WHY?}$

SUDOKU IS NP-COMPLETE : EVERY PROBLEM IN NP
REDUCES TO SUDOKU !

3-COLOURING

SUDOKU

SUDOKU IN P $\Rightarrow$ P = NP : WHY?

SUDOKU IS NP-COMPLETE : EVERY PROBLEM IN NP
REDUCES TO SUDOKU !

3-COLOURING

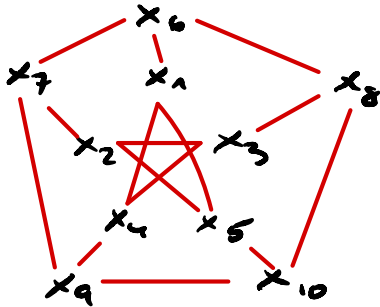
SUDOKU

TRANSFORMER

SUDOKU IN P $\Rightarrow$ P = NP : WHY?

SUDOKU IS NP-COMPLETE : EVERY PROBLEM IN NP
REDUCES TO SUDOKU !

3-COLOURING



SUDOKU

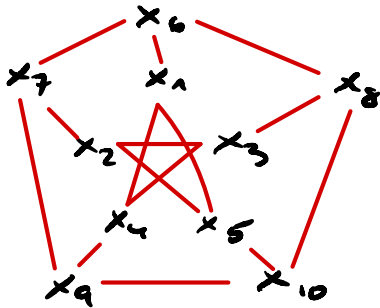
TRANSFORMER

SUDOKU IN P $\Rightarrow$ P = NP : WHY?

SUDOKU IS NP-COMPLETE : EVERY PROBLEM IN NP
REDUCES TO SUDOKU !

3-COLOURING

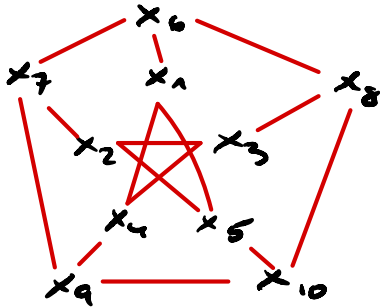
SUDOKU



SUDOKU IN P $\Rightarrow$ P = NP : WHY?

SUDOKU IS NP-COMPLETE : EVERY PROBLEM IN NP REDUCES TO SUDOKU !

3-COLOURING



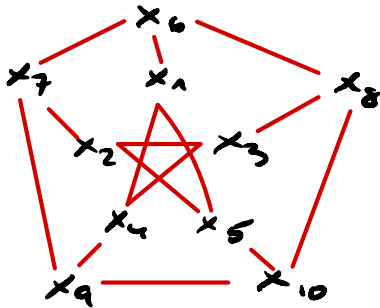
SUDOKU

$x_{1,1}$	2	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	$x_{1,6}$	$x_{1,7}$	$x_{1,8}$	$x_{1,9}$
$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	6	$x_{2,5}$	$x_{2,6}$	$x_{2,7}$	$x_{2,8}$	3
$x_{3,1}$	7	4	$x_{3,4}$	8	$x_{3,6}$	$x_{3,7}$	$x_{3,8}$	$x_{3,9}$
$x_{4,1}$	$x_{4,2}$	$x_{4,3}$	$x_{4,4}$	$x_{4,5}$	3	$x_{4,7}$	$x_{4,8}$	2
$x_{5,1}$	8	$x_{5,3}$	$x_{5,4}$	4	$x_{5,6}$	$x_{5,7}$	1	$x_{5,9}$
6	$x_{6,2}$	$x_{6,3}$	5	$x_{6,5}$	$x_{6,6}$	$x_{6,7}$	$x_{6,8}$	$x_{6,9}$
$x_{7,1}$	$x_{7,2}$	$x_{7,3}$	$x_{7,4}$	1	$x_{7,6}$	7	8	$x_{7,9}$
5	$x_{8,2}$	$x_{8,3}$	$x_{8,4}$	$x_{8,5}$	9	$x_{8,7}$	$x_{8,8}$	$x_{8,9}$
$x_{9,1}$	$x_{9,2}$	$x_{9,3}$	$x_{9,4}$	$x_{9,5}$	$x_{9,6}$	$x_{9,7}$	4	$x_{9,9}$

SUDOKU IN P $\Rightarrow$ P = NP : WHY?

SUDOKU IS NP-COMPLETE : EVERY PROBLEM IN NP REDUCES TO SUDOKU !

3-COLOURING



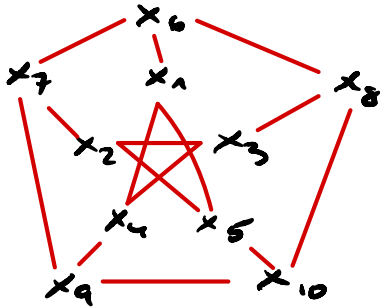
SUDOKU

1	2	6	4	3	7	9	5	8
8	9	5	6	2	1	4	7	3
3	7	4	9	8	5	1	2	6
4	5	7	1	9	3	8	6	2
9	8	3	2	4	6	5	1	7
6	1	2	5	7	8	3	9	4
2	6	9	3	1	4	7	8	5
5	4	8	7	6	9	2	3	1
7	3	1	8	5	2	6	4	9

SUDOKU IN P $\Rightarrow$ P = NP : WHY?

SUDOKU IS NP-COMPLETE : EVERY PROBLEM IN NP REDUCES TO SUDOKU !

3-COLOURING



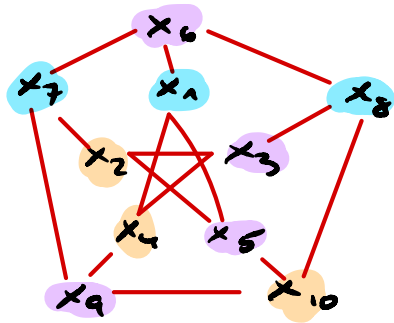
SUDOKU

1	2	6	4	3	7	9	5	8
8	9	5	6	2	1	4	7	3
3	7	4	9	8	5	1	2	6
4	5	7	1	9	3	8	6	2
9	8	3	2	4	6	5	1	7
6	1	2	5	7	8	3	9	4
2	6	9	3	1	4	7	8	5
5	4	8	7	6	9	2	3	1
7	3	1	8	5	2	6	4	9

SUDOKU IN P $\Rightarrow$ P = NP : WHY?

SUDOKU IS NP-COMPLETE : EVERY PROBLEM IN NP REDUCES TO SUDOKU !

3-COLOURING



SUDOKU

1	2	6	4	3	7	9	5	8
8	9	5	6	2	1	4	7	3
3	7	4	9	8	5	1	2	6
4	5	7	1	9	3	8	6	2
9	8	3	2	4	6	5	1	7
6	1	2	5	7	8	3	9	4
2	6	9	3	1	4	7	8	5
5	4	8	7	6	9	2	3	1
7	3	1	8	5	2	6	4	9

HOW TO PROVE $P = NP$?

HOW TO PROVE $P = NP$?

1000000 \$



HOW TO PROVE $P = NP$?

1000000 \$

- FIND POLYNOMIAL METHOD FOR SUDOKUS

HOW TO PROVE $P = NP$?

1000000 \$



- FIND POLYNOMIAL METHOD FOR SUDOKUS
- CAN BE "INDIRECT"

HOW TO PROVE $P = NP$?

1000000 \$

- FIND POLYNOMIAL METHOD FOR SUDOKUS
- CAN BE "INDIRECT"

PRIME NUMBER PROBLEM

- GIVEN: NUMBER x
- QUESTION: DOES x HAVE A PROPER FACTOR?

IN NP:

GIVEN CANDIDATE y , DIVIDE!

HOW TO PROVE $P = NP$?

1000000 \$

- FIND POLYNOMIAL METHOD FOR SUDOKUS
- CAN BE "INDIRECT"

PRIME NUMBER PROBLEM

- GIVEN: NUMBER x
- QUESTION: DOES x HAVE A PROPER FACTOR?

IN NP:

GIVEN CANDIDATE y , DIVIDE!

THEOREM x PRIME $\Rightarrow 2^x \equiv 2 \text{ MODULO } x$

$$x=7: 2^7 = 128 \equiv 2 \text{ MODULO } 7$$

$$x=15: 2^{15} = 32768 \equiv 4 \text{ MODULO } 15 \Rightarrow 15 \text{ NOT PRIME}$$

HOW TO PROVE $P = NP$?

1000000 \$

- FIND POLYNOMIAL METHOD FOR SUDOKUS
- CAN BE "INDIRECT"

PRIME NUMBER PROBLEM

- GIVEN: NUMBER x
- QUESTION: DOES x HAVE A PROPER FACTOR?

IN NP:

GIVEN CANDIDATE y , DIVIDE!

IN P!

THEOREM x PRIME $\Rightarrow 2^x \equiv 2 \text{ MODULO } x$

$$x=7: 2^7 = 128 \equiv 2 \text{ MODULO } 7$$

$$x=15: 2^{15} = 32768 \equiv 4 \text{ MODULO } 15 \Rightarrow 15 \text{ NOT PRIME}$$

AGRAWAL -
KAZMAI -
SAXENA
2002

HOW TO PROVE $P \neq NP$?

1000000 \$

!!!

HOW TO PROVE $P \neq NP$?

1000000 \$

- SHOW THAT NO POLYNOMIAL METHOD SOLVES ALL SUDOKUS

HOW TO PROVE $P \neq NP$?

1000000 \$

- SHOW THAT NO POLYNOMIAL METHOD SOLVES ALL SUDOKUS
- FIND A PROPERTY WHICH :
 - ALL PROBLEMS IN P HAVE
 - SUDOKUS DO NOT HAVE

HOW TO PROVE $P \neq NP$?

1000000 \$

- SHOW THAT NO POLYNOMIAL METHOD SOLVES ALL SUDOKUS
- FIND A PROPERTY WHICH :
 - ALL PROBLEMS IN P HAVE
 - SUDOKUS DO NOT HAVE

LINEAR EQUATIONS : WHY IN P ?

$$\begin{aligned}x_1 + x_2 + 4x_3 &= -3 \\ -3x_1 + 9x_2 - x_3 &= -13 \\ 6x_1 - 2x_2 + 10x_3 &= 10\end{aligned}$$

HOW TO PROVE $P \neq NP$?

1000000 \$

- SHOW THAT NO POLYNOMIAL METHOD SOLVES ALL SUDOKUS
- FIND A PROPERTY WHICH :
 - ALL PROBLEMS IN P HAVE
 - SUDOKUS DO NOT HAVE

LINEAR EQUATIONS : WHY IN P ?

SOLUTIONS FORM AN AFFINE SPACE : POINT , LINE , PLANE

HOW TO PROVE $P \neq NP$?

1000000 \$

- SHOW THAT NO POLYNOMIAL METHOD SOLVES ALL SUDOKUS
- FIND A PROPERTY WHICH :
 - ALL PROBLEMS IN P HAVE
 - SUDOKUS DO NOT HAVE

LINEAR EQUATIONS : WHY IN P ?

SOLUTIONS FORM AN AFFINE SPACE : POINT , LINE , PLANE

INVARIANT UNDER $f(\bar{x}, \bar{y}, \bar{z}) = \bar{x} - \bar{y} + \bar{z}$

HOW TO PROVE $P \neq NP$?

1000000 \$

- SHOW THAT NO POLYNOMIAL METHOD SOLVES ALL SUDOKUS
- FIND A PROPERTY WHICH :
 - ALL PROBLEMS IN P HAVE
 - SUDOKUS DO NOT HAVE

LINEAR EQUATIONS : WHY IN P ?

SOLUTIONS FORM AN AFFINE SPACE : POINT , LINE , PLANE

INVARIANT UNDER $f(\bar{x}, \bar{y}, \bar{z}) = \bar{x} - \bar{y} + \bar{z}$

BLKATOV - DALMAU : FINITE SET OF VALUES
2006

SOLUTIONS INVARIANT UNDER : $f : f(a, a, b) = f(b, a, a) = b$

↪

IN P !

HOW TO PROVE $P \neq NP$?

1000000 \$

- SHOW THAT NO POLYNOMIAL METHOD SOLVES ALL SUDOKUS
- FIND A PROPERTY WHICH :
 - ALL PROBLEMS IN P HAVE
 - SUDOKUS DO NOT HAVE

LINEAR EQUATIONS : WHY IN P ?

SOLUTIONS FORM AN AFFINE SPACE : POINT, LINE, PLANE

INVARIANT UNDER $f(\bar{x}, \bar{y}, \bar{z}) = \bar{x} - \bar{y} + \bar{z}$

BULATOV - DALMAU : FINITE SET OF VALUES
2006

SOLUTIONS INVARIANT UNDER : $f : f(a, a, b) = f(b, a, a) = b$

BULATOV - ZHUK :
2017

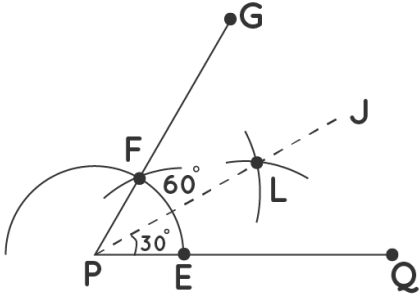
$$S(x, y, x, z, y, z) = S(y, x, z, x, z, y) \Rightarrow$$

IN P !

CLASSICAL PROBLEMS OF ANCIENT GREECE

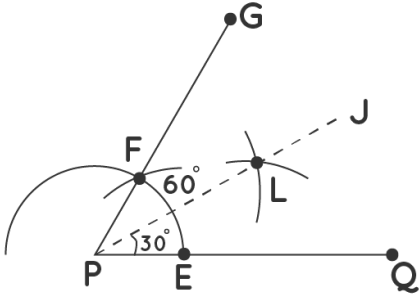
CLASSICAL PROBLEMS OF ANCIENT GREECE

ANGLE BISECTION (WITH COMPASS & RULER)



CLASSICAL PROBLEMS OF ANCIENT GREECE

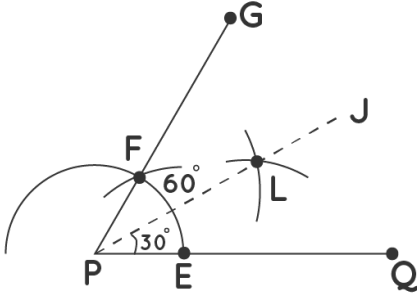
ANGLE BISECTION (WITH COMPASS & RULER)



METHOD !

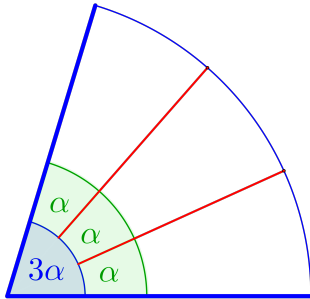
CLASSICAL PROBLEMS OF ANCIENT GREECE

ANGLE BISECTION (WITH COMPASS & RULER)



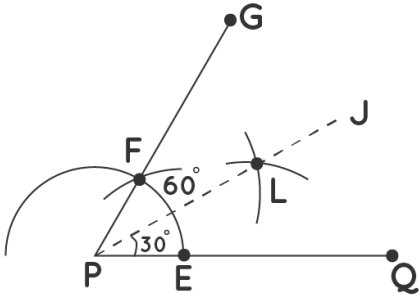
METHOD !

ANGLE TRISECTION



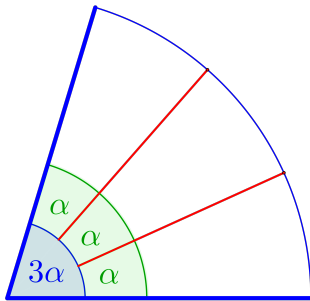
CLASSICAL PROBLEMS OF ANCIENT GREECE

ANGLE BISECTION (WITH COMPASS & RULER)



METHOD !

ANGLE TRISECTION



NO METHOD CAN EXIST ! (WANTZEL 1837)

"DIMENSION" OF COORDINATES
OF CONSTRUCTIBLE POINTS
IS POWER OF 2

TRISECTION $\Rightarrow$ DIMENSION 3

SUMMARY

SUMMARY

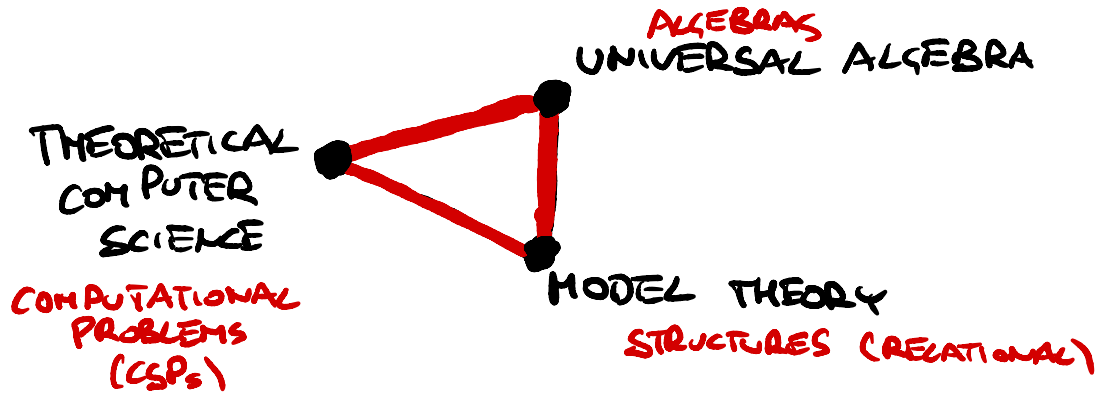
- NO EFFICIENT METHOD KNOWN FOR SUDOKUS (ONLY EXPONENTIAL)

SUMMARY

- NO EFFICIENT METHOD KNOWN FOR SUDOKUS (ONLY EXPONENTIAL)
- POLYNOMIAL METHOD $\Rightarrow$ $P=NP$ (MILLENNIUM PROBLEM)

SUMMARY

- NO EFFICIENT METHOD KNOWN FOR SUDOKUS (ONLY EXPONENTIAL)
- POLYNOMIAL METHOD $\Rightarrow$ $P=NP$ (MILLENNIUM PROBLEM)
- $P \neq NP$: SUDOKUS LACK STRUCTURE OF PROBLEMS IN P



PART I

SUDOKUS, CONSTRAINT SATISFACTION PROBLEMS, P vs. NP

PART II

CSPs, POLYMORPHISMS

PART II

CSPs, POLYMORPHISMS

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS ON THEM

QUESTION

- $\exists$ VALUES FOR $x_1 \dots x_n$
SATISFYING ALL CONSTRAINTS
?

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS ON THEM

QUESTION

- $\exists$ VALUES FOR $x_1 \dots x_n$
SATISFYING ALL CONSTRAINTS?

EXAMPLE

- SUDOKU
- SCHEDULING
- SOLVING EQUATIONS

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS ON THEM

QUESTION

- $\exists$ VALUES FOR $x_1, \dots, x_n$ SATISFYING ALL CONSTRAINTS?

EXAMPLE

- SUDOKU
- SCHEDULING
- SOLVING EQUATIONS

MODEL:

- SET V OF POSSIBLE VALUES (FIXED)
E.G. $\{0,1\}$, $\{0,1,2\}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, ...
- ALLOWED CONSTRAINTS (FIXED)
 $C_1, \dots, C_m$ RELATIONS ON V
 $C_i \in V^{d_i}$

SO $\mathcal{A} := (V, C_1, \dots, C_m)$ RELATIONAL STRUCTURE

CSP($\mathcal{A}$)

= "TEMPLATE"

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $C_{i_1}(\text{VARIABLES})$
 $C_{i_2}(\text{VARIABLES})$
 $\vdots$
 $C_{i_\ell}(\text{VARIABLES})$

QUESTION

- $\exists$ SOLUTION $x_i \mapsto a_i \in A$?

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS ON THEM

QUESTION

- $\exists$ VALUES FOR $x_1 \dots x_n$ SATISFYING ALL CONSTRAINTS?

EXAMPLE

- SUDOKU
- SCHEDULING
- SOLVING EQUATIONS

MODEL:

- SET V OF POSSIBLE VALUES (FIXED)
E.G. $\{0,1\}$, $\{0,1,2\}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, ...
- ALLOWED CONSTRAINTS (FIXED)
 $C_1, \dots, C_m$ RELATIONS ON V
 $C_i \in V^{d_i}$

SO $\mathcal{A} := (V, C_1, \dots, C_m)$ RELATIONAL STRUCTURE

CSP($\mathcal{A}$)

= "TEMPLATE"

GIVEN

- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $C_{i_1}(\text{VARIABLES})$
 $C_{i_2}(\text{VARIABLES})$
 $\vdots$

QUESTION

$\exists$ SOLUTION $x_i \mapsto a_i \in A$?

ALTERNATIVE: GIVEN PP-SENTENCE

$\varphi \equiv \exists x_1 \dots \exists x_n C_{i_1}(\text{VAR}) \wedge \dots \wedge C_{i_r}(\text{VAR})$

QUESTION: $\mathcal{A} \models \varphi$?

PP... PRIMITIVE
POSITIVE

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS ON THEM

QUESTION

- $\exists$ VALUES FOR $x_1 \dots x_n$ SATISFYING ALL CONSTRAINTS?

EXAMPLE

- SUDOKU
- SCHEDULING
- SOLVING EQUATIONS

MODEL:

- SET V OF POSSIBLE VALUES (FIXED)
E.G. $\{0,1\}$, $\{0,1,2\}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, ...
- ALLOWED CONSTRAINTS (FIXED)
 $C_1, \dots, C_m$ RELATIONS ON V
 $C_i \subseteq V^{\delta_i}$

SO $\langle A := (V, C_1, \dots, C_m) \rangle$ RELATIONAL STRUCTURE

CSP(A)

= "TEMPLATE"

GIVEN

- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $C_{i_1}(\text{VARIABLES})$
 $C_{i_2}(\text{VARIABLES})$

QUESTION

$\exists$ SOLUTION $x_i \mapsto a_i \in A$?

ALTERNATIVE: GIVEN PP-SENTENCE

$\varphi \equiv \exists x_1 \dots \exists x_n C_{i_1}(\text{VAR}) \wedge \dots \wedge C_{i_r}(\text{VAR})$

QUESTION: $\langle A \rangle \models \varphi$?

PP... PRIMITIVE
POSITIVE

META-QUESTION

FOR WHAT $\langle A \rangle$ IS CSP(A) EASY/HARD?

P:

$\exists$ ALGORITHM PROVIDING ANSWER

#

IN $p(\# \text{ VARIABLES})$ STEPS, p POLYNOMIAL

NP:

NOT NECESSARILY IN P BUT VERIFYING
"SOLUTION" IN P

EXAMPLE

- $A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$

TERNARY RELATIONS



CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 \cdot x_1 = x_2$

$$x_2 + x_1 = x_3$$

$$x_3 = 1$$

QUESTION

• SOLUTION IN $\mathbb{Z}$?

EXAMPLE

• $A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$

TERNARY RELATIONS



CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 \cdot x_1 = x_2$

$$x_2 + x_1 = x_3$$

$$x_3 = 1$$

QUESTION

• SOLUTION IN $\mathbb{Z}$?

UNDECIDABLE (MATYASEWIC '77)

HILBERT'S 10TH PROBLEM

EXAMPLE

$$\bullet A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$$

TERNARY RELATIONS
↙ ↘

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 + x_1 = x_2$

$$x_2 + x_1 = x_3$$

$$x_3 = 1$$

QUESTION

• SOLUTION IN $\mathbb{Z}$?

UNDECIDABLE (MATYASEWIC '77)

HILBERT'S 10TH PROBLEM

• SAME OVER $\mathbb{Z}_p$

EXAMPLE

• $A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$

TERNARY RELATIONS
↙ ↘

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 \cdot x_1 = x_2$

$$x_2 + x_1 = x_3$$

$$x_3 = 1$$

QUESTION

• SOLUTION IN $\mathbb{Z}$?

UNDECIDABLE (MATYASEWIC '77)

HILBERT'S 10TH PROBLEM

• SAME OVER $\mathbb{Z}_p$

NP-COMPLETE

↳ IF IN P $\Rightarrow$ P = NP

EXAMPLE

$$\bullet A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$$

TERNARY RELATIONS
↙ ↘

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 \cdot x_1 = x_2$

$$x_2 + x_1 = x_3$$

$$x_3 = 1$$

QUESTION

• SOLUTION IN $\mathbb{Z}$?

UNDECIDABLE (MATYASEWIC '77)

HILBERT'S 10TH PROBLEM

• SAME OVER $\mathbb{Z}_p$

NP-COMPLETE

↳ IF IN P $\Rightarrow$ P = NP

• SAME WITHOUT •

EXAMPLE

$$\bullet A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$$

TERNARY RELATIONS
↙ ↘

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 \cdot x_1 = x_2$

$$x_2 + x_1 = x_3$$

$$x_3 = 1$$

QUESTION

• SOLUTION IN $\mathbb{Z}$?

UNDECIDABLE (MATIASSEMIC '77)

HILBERT'S 10TH PROBLEM

• SAME OVER $\mathbb{Z}_p$

NP-COMPLETE

↳ IF IN P $\Rightarrow$ P = NP

• SAME WITHOUT •

IN P (GAUSS)

EXAMPLE

$$\bullet A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$$

TERNARY RELATIONS
↙ ↘

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 \cdot x_1 = x_2$
 $x_2 + x_1 = x_3$
 $x_3 = 1$

QUESTION

• SOLUTION IN $\mathbb{Z}$?

UNDECIDABLE (MATIASSEMIC '77)

HILBERT'S 10TH PROBLEM

• SAME OVER $\mathbb{Z}_p$

NP-COMPLETE

↳ IF IN P $\Rightarrow$ P = NP

• SAME WITHOUT •

IN P (GAUSS)

$$\bullet A = (\mathbb{Q}, <)$$

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 < x_2$
 $x_2 < x_3$
 $x_3 < x_1$
 $\vdots$

QUESTION

• SOLUTION IN $\mathbb{Q}$?

EXAMPLE

$$\bullet A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$$

TERNARY RELATIONS
↙ ↘

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 + x_1 = x_2$
 $x_2 + x_1 = x_3$
 $x_3 = 1$

QUESTION

• SOLUTION IN $\mathbb{Z}$?

UNDECIDABLE (MATIASSEMIC '77)

HILBERT'S 10TH PROBLEM

• SAME OVER $\mathbb{Z}_p$

NP-COMPLETE

↳ IF IN P $\Rightarrow$ P = NP

• SAME WITHOUT •

IN P (GAUSS)

$$\bullet A = (\mathbb{Q}, <)$$

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 < x_2$
 $x_2 < x_3$
 $x_3 < x_1$
 $\vdots$

QUESTION

• SOLUTION IN $\mathbb{Q}$?



EXAMPLE

$$\bullet A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$$

TERNARY RELATIONS
↙ ↘

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 \cdot x_1 = x_2$
 $x_2 + x_1 = x_3$
 $x_3 = 1$

QUESTION

• SOLUTION IN $\mathbb{Z}$?

UNDECIDABLE (MATIASSEMIC '77)
HILBERT'S 10TH PROBLEM

• SAME OVER $\mathbb{Z}_p$

NP-COMPLETE

↳ IF IN P $\Rightarrow$ P = NP

• SAME WITHOUT •

IN P (GAUSS)

$$\bullet A = (\mathbb{Q}, <)$$

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1 < x_2$
 $x_2 < x_3$
 $x_3 < x_1$
 $\vdots$

QUESTION

• SOLUTION IN $\mathbb{Q}$?



IN P

EXAMPLE

$$\bullet A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$$

TERNARY RELATIONS
 $\swarrow \searrow$

CSP(A)

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1 + x_1 = x_2$
 $x_2 + x_1 = x_3$
 $x_3 = 1$

QUESTION

- SOLUTION IN $\mathbb{Z}$?

UNDECIDABLE (MATIASSEMIC '77)
 HILBERT'S 10TH PROBLEM

- SAME OVER $\mathbb{Z}_p$

NP-COMPLETE

$\hookrightarrow$ IF IN P $\Rightarrow$ P = NP

- SAME WITHOUT

IN P (GAUSS)

$$\bullet A = (\mathbb{Q}, <)$$

CSP(A)

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1 < x_2$
 $x_2 < x_3$
 $x_3 < x_1$
 $\vdots$

QUESTION

- SOLUTION IN $\mathbb{Q}$?



IN P

$$\bullet A = \mathbb{R}_3 = \triangle_{\mathbb{Z}}^0 = (\{0, 1, 2\}, E)$$

CSP(A)

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $E(x_1, x_2)$
 $E(x_2, x_3)$
 $\vdots$

QUESTION

- SOLUTION IN $\{0, 1, 2\}$?

EXAMPLE

$$\bullet A = (\mathbb{Z}, \{0\}, \{1\}, +, =)$$

TERNARY RELATIONS
↙ ↘

CSP(A)

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1 + x_1 = x_2$
 $x_2 + x_1 = x_3$
 $x_3 = 1$

QUESTION

- SOLUTION IN $\mathbb{Z}$?

UNDECIDABLE (MATIASSEMIC '77)
HILBERT'S 10TH PROBLEM

- SAME OVER $\mathbb{Z}_p$

NP-COMPLETE

↳ IF IN P $\Rightarrow$ P = NP

- SAME WITHOUT

IN P (GAUSS)

$$\bullet A = (\mathbb{Q}, <)$$

CSP(A)

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1 < x_2$
 $x_2 < x_3$
 $x_3 < x_1$
 $\vdots$

QUESTION

- SOLUTION IN $\mathbb{Q}$?



IN P

$$\bullet A = \mathbb{R}_3 = \triangle_3^0 = (\{0, 1, 2\}, E)$$

CSP(A)

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $E(x_1, x_2)$
 $E(x_2, x_3)$
 $\vdots$

QUESTION

- SOLUTION IN $\{0, 1, 2\}$?

3-COLORING PROBLEM = NP-COMPLETE

COMPUTATIONAL COMPLEXITY OF $CSP(A)$ CAN BE:

- UNDECIDABLE
- IN P
- NP-COMPLETE
- ANYTHING:
EVERY COMPUTATIONAL PROBLEM
IS POLYNOMIAL-TIME
TURING-EQUIVALENT TO SOME
 $CSP(A)$ (RUBINFELD + (RONE '08)



A FINITE $\Rightarrow CSP(A)$ IN NP

THEOREM

(BOGATOV, ZHUK '17)

(CONJECTURE FEDER+ VARDI '93)

$\mathcal{A}$ FINITE

$\Rightarrow \text{CSP}(\mathcal{A}) \in \text{P}$ OR

NP-COMPLETE

THEOREM

(BOGATOV, ZHUK '17)

(CONJECTURE FEDER+ VARDI '93)

$\mathbb{A}$ FINITE

$\Rightarrow \text{CSP}(\mathbb{A}) \in \text{P}$ OR

NP-COMPLETE



THEOREM

(BOGATOV, ZHUK '17)

(CONJECTURE FEDER+ VARDI '93)

$\mathcal{A}$ FINITE

$\Rightarrow \text{CSP}(\mathcal{A}) \in \text{P}$ OR

NP-COMPLETE



$(\text{P} \neq \text{NP})$

$\text{CSP}(\mathcal{A}) \in \text{P} \Leftrightarrow \mathcal{A}$ HAS ALGEBRAIC
INVARIANT

$$S(x, y, x, z, y, z) =$$

$$S(y, x, z, x, z, y)$$

THEOREM

(BOLATOV, ZHUK '17)

(CONJECTURE FEDER + VARDI '93)

$\mathcal{A}$ FINITE

$\Rightarrow \text{CSP}(\mathcal{A}) \in \text{P}$ OR

NP-COMPLETE



$(\text{P} \neq \text{NP})$

$\text{CSP}(\mathcal{A}) \in \text{P} \Leftrightarrow \mathcal{A}$ HAS ALGEBRAIC
INVARIANT

$$S(x, y, x, z, y, z) =$$

$$S(y, x, z, x, z, y)$$



THEOREM

(BOLATOV, ZHUK '17)
(CONJECTURE FEDER + VARDI '93)

A FINITE

$\Rightarrow \text{CSP}(A) \in \text{P}$ OR
NP-COMPLETE



$(\text{P} \neq \text{NP})$

$\text{CSP}(A) \in \text{P} \Leftrightarrow A$ HAS ALGEBRAIC
INVARIANT
 $S(x, y, x, z, y, z) =$
 $S(y, x, z, x, z, y)$



ALGEBRAIC INVARIANTS = POLYMORPHISMS

$A = (A; R_1, \dots, R_m)$ STRUCTURE

$f: A^r \rightarrow A$ POLYMORPHISM: $\Leftrightarrow$

f HOMOMORPHISM $A^r \rightarrow A \Leftrightarrow$

$\forall: \forall \bar{r}_1, \dots, \bar{r}_r \in R_i$

$$f(\overset{r_1}{\underset{|}{\text{O}}} \dots \overset{r_r}{\underset{|}{\text{O}}}) \in R_i$$

THEOREM

(BOLATOV, ZHUK '17)

(CONJECTURE FEDER + VARDI '93)

A FINITE

$\Rightarrow \text{CSP}(A) \in \text{P}$ OR
NP-COMPLETE

☆☆☆☆☆☆☆☆

$(\text{P} \neq \text{NP})$

$\text{CSP}(A) \in \text{P} \Leftrightarrow A$ HAS ALGEBRAIC
INVARIANT

$$S(x, y, x, z, y, z) = \\ S(y, x, z, x, z, y)$$

☆☆☆☆☆☆☆☆

ALGEBRAIC INVARIANTS = POLYMORPHISMS

$A = (A; R_1, \dots, R_m)$ STRUCTURE

$f: A^r \rightarrow A$ POLYMORPHISM: $\Leftrightarrow$

f HOMOMORPHISM $A^r \rightarrow A \Leftrightarrow$

$$\forall: \forall \bar{r}_1, \dots, \bar{r}_e \in R_i;$$

$$f(\overset{r_1}{\underset{|}{\parallel}} \dots \overset{r_e}{\underset{|}{\parallel}}) \in R_i$$

$\text{POL}(A) := \{f \mid f \text{ POLYMORPHISM OF } A\}$

- CONTAINS PROJECTIONS $(x_1, \dots, x_e) \mapsto x_i$
- COMPOSITION-CLOSED

$\Rightarrow$ ESSENTIALLY TERM FUNCTIONS
OF AN ALGEBRA ON A !

EXAMPLE

- $\min(x, y) : \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$

$$\min \begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix} = \begin{pmatrix} \wedge \end{pmatrix}$$

EXAMPLE

- $\min(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$

$$\min \left(\begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} \wedge \end{pmatrix}$$

- $(x, y, z) \mapsto x - y + z \in \text{Pol}(\mathbb{Z}_p, \{0, 1\}, \{0, 1\}, +)$

- $0 - 0 + 0 = 0$

- $1 - 1 + 1 = 1$

-

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{F} \sim \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{F} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{F} = \begin{pmatrix} \end{pmatrix} \in \mathbb{F}$$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

EXAMPLE

- $\min(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$

$$\min \begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix} = \begin{pmatrix} \wedge \end{pmatrix}$$

- $(x, y, z) \mapsto x - y + z \in \text{Pol}(\mathbb{Z}_p, \{0, 1\}, \{1, 1\}, +)$

- $0 - 0 + 0 = 0$

- $1 - 1 + 1 = 1$

- $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{F} \sim \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{F} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{F} = \begin{pmatrix} \end{pmatrix} \in \mathbb{F}$$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

↑
NOT GREAT 😞

EXAMPLE

- $\min(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$

$$\min \left(\begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} \wedge \end{pmatrix}$$

- $(x, y, z) \mapsto x - y + z \in \text{Pol}(\mathbb{Z}_p, \{0, 1\}, \{1, 1\}, +)$

- $0 - 0 + 0 = 0$

- $1 - 1 + 1 = 1$

- $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \sim \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \end{pmatrix}$
 $\in \bar{+} \quad \in \bar{+} \quad \in \bar{+} \quad \in \bar{+}$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

↑ NOT GREAT 😞

(BABY) THEOREM (BODNARČUK + GALUŽNIN + KOSTOV + ROMOV '04)
 GEISER '06

A, B FINITE, SAME DOMAIN

$\text{Pol}(A) \subseteq \text{Pol}(B) \xrightarrow{①} A \text{ PP-DEFINES } B$
 $\xrightarrow{②} \text{CSP}(B) \text{ REDUCES TO } \text{CSP}(A)$

EXAMPLE

- $\min(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$

$$\min \left(\begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} \wedge \end{pmatrix}$$

- $(x, y, z) \mapsto x - y + z \in \text{Pol}(\mathbb{Z}_p, \{0, 1\}, \{1, 2, \dots, p-1\})$

- $0 - 0 + 0 = 0$

- $1 - 1 + 1 = 1$

- $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \sim \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \end{pmatrix}$
 $\in \bar{\mathbb{F}} \quad \in \bar{\mathbb{F}} \quad \in \bar{\mathbb{F}} \quad \in \bar{\mathbb{F}}$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

↑ NOT GREAT 😞

(BABY) THEOREM (BODNARČUK + GALUŽNIN + KOTIGOV + ROMOV '01)
 GEISER '06

A, B FINITE, SAME DOMAIN

$\text{Pol}(A) \subseteq \text{Pol}(B) \xrightarrow{①} A \text{ PP-DEFINES } B$
 $\xrightarrow{②} \text{CSP}(B) \text{ REDUCES TO } \text{CSP}(A)$

③ PROOF

EXAMPLE

- $\min(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$

$$\min \left(\begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} \wedge \end{pmatrix}$$

- $(x, y, z) \mapsto x - y + z \in \text{Pol}(\mathbb{Z}_p, \{0, 1, 2, \dots, p-1\}, +)$

- $0 - 0 + 0 = 0$

- $1 - 1 + 1 = 1$

- $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \sim \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \end{pmatrix}$
 $\in \mathbb{F} \quad \in \mathbb{F} \quad \in \mathbb{F} \quad \in \mathbb{F}$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

↑ NOT GREAT 😞

(BABY) THEOREM

(BODNARČUK + GALUŽNIN + KOSTOV + ROMOV '01)
GEISER '06

A, B FINITE, SAME DOMAIN

$\text{Pol}(A) \subseteq \text{Pol}(B) \xrightarrow{①} A \text{ PP-DEFINES } B$

$\xrightarrow{②} \text{CSP}(B) \text{ REDUCES TO } \text{CSP}(A)$

③ PROOF:

LET $\varphi \equiv \exists x_1 \dots x_n C_{i_1}(\gamma) \wedge \dots \wedge C_{i_r}(\gamma)$

BE AN INSTANCE OF $\text{CSP}(B)$

REPLACE EACH C_{i_j} BY ITS PP-DEFINITION

IN A
 $\leadsto \tilde{\varphi}!$

$$(A \models \tilde{\varphi} \Leftrightarrow B \models \varphi)$$

EXAMPLE

- $\min(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$

$$\min \left(\begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} \wedge \end{pmatrix}$$

- $(x, y, z) \mapsto x - y + z \in \text{Pol}(\mathbb{Z}_p, \{0, 1\}, \{1, 1\}, \{1\})$

- $0 - 0 + 0 = 0$

- $1 - 1 + 1 = 1$

- $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \sim \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \end{pmatrix}$
 $\in \mathbb{F} \quad \in \mathbb{F} \quad \in \mathbb{F} \quad \in \mathbb{F}$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

↑ NOT GREAT 😞

(BABY) THEOREM (BODNARČUK + GALUŽNIN + KOSTOV + ROMOV '01)
GEISER '06

A, B FINITE, SAME DOMAIN

$\text{Pol}(A) \subseteq \text{Pol}(B) \xrightarrow{①} A \text{ PP-DEFINES } B$
 $\xrightarrow{②} \text{CSP}(B) \text{ REDUCES TO } \text{CSP}(A)$

③ PROOF:

LET $\varphi \equiv \exists x_1 \dots x_n C_{i_1}(\gamma) \wedge \dots \wedge C_{i_r}(\gamma)$

BE AN INSTANCE OF $\text{CSP}(B)$

REPLACE EACH C_{i_j} BY ITS PP-DEFINITION
 IN A
 $\leadsto \tilde{\varphi}$!

$$(A \models \tilde{\varphi} \Leftrightarrow B \models \varphi)$$

④ PROOF

EXAMPLE

- $\min(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$

$$\min\left(\begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} \wedge \end{pmatrix}$$

- $(x, y, z) \mapsto x - y + z \in \text{Pol}(\mathbb{Z}_p, \{0, 1\}, \mathbb{Z}_p)$

- $0 - 0 + 0 = 0$

- $1 - 1 + 1 = 1$

- $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{F} \sim \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{F} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{F} = \begin{pmatrix} \end{pmatrix} \in \mathbb{F}$$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

↑ NOT GREAT 😞

"MORE POLYMORPHISMS
 $\Rightarrow$ EASIER CSP"

POLYMORPHISMS GIVE:

- REDUCTIONS BETWEEN CSPs
- ALGORITHMS FOR CSPs

(BABY) THEOREM

(BODNARČUK + KALUŽNIN + KOSTOV +
ROMOV '91)
GEIGER '96

A, B FINITE, SAME DOMAIN

$\text{Pol}(A) \subseteq \text{Pol}(B) \xrightarrow{①} A \text{ PP-DEFINES } B$

$\xrightarrow{②} \text{CSP}(B) \text{ REDUCES TO } \text{CSP}(A)$

③ PROOF:

LET $\varphi \equiv \exists x_1 \dots x_n C_{i_1}(\dots) \wedge \dots \wedge C_{i_r}(\dots)$

BE AN INSTANCE OF $\text{CSP}(B)$

REPLACE EACH C_{i_j} BY ITS PP-DEFINITION
IN A
 $\leadsto \tilde{\varphi}!$

$$(A \models \tilde{\varphi} \Leftrightarrow B \models \varphi)$$

④ PROOF:

LET R BE RELATION. SHOW:

R INVARIANT UNDER $\text{Pol}(A)$

$\Leftrightarrow$

R PP-DEFINABLE IN A

(BABY) THEOREM

(BODNARČUK + KALUŽNIN + KOSTOV +
ROMOV '91)
GEIGER '98

A, B FINITE, SAME DOMAIN

$\text{Pol}(A) \subseteq \text{Pol}(B) \stackrel{①}{\Rightarrow} A \text{ PP-DEFINES } B$

$\stackrel{②}{\Rightarrow} \text{CSP}(B) \text{ REDUCES TO } \text{CSP}(A)$

③ PROOF:

LET $\varphi \equiv \exists x_1 \dots x_n C_{i_1}(\gamma_1 \dots \gamma_n) \wedge C_{i_2}(\gamma_1 \dots \gamma_n)$

BE AN INSTANCE OF $\text{CSP}(B)$

REPLACE EACH C_{i_j} BY ITS PP-DEFINITION
IN A
 $\leadsto \tilde{\varphi} !$

$(A \models \tilde{\varphi} \Leftrightarrow B \models \varphi)$

④ PROOF:

LET R BE RELATION. SHOW:

R INVARIANT UNDER $\text{Pol}(A)$

$\Leftrightarrow$

R PP-DEFINABLE IN A

←

EASY: $S(x_1, \dots, x_k), T(y_1, \dots, y_n)$ INVARIANT

$\Rightarrow S \wedge T$ INVARIANT,

$\exists x; S(x_1, \dots, x_k)$ INVARIANT

(BABY) THEOREM

(BODNARČUK + KALUŽNIN + KOSTOV
ROMOV '91)
GEIGER '98

A, B FINITE, SAME DOMAIN

$\text{Pol}(A) \subseteq \text{Pol}(B) \stackrel{①}{\Rightarrow} A \text{ PP-DEFINES } B$

$\stackrel{②}{\Rightarrow} \text{CSP}(B) \text{ REDUCES TO } \text{CSP}(A)$

③ PROOF:

LET $\varphi \equiv \exists x_1 \dots x_n C_{i_1}(\gamma_1 \dots \gamma_n) \wedge C_{i_2}(\gamma_1 \dots \gamma_n)$

BE AN INSTANCE OF $\text{CSP}(B)$

REPLACE EACH C_{i_j} BY ITS PP-DEFINITION
IN A
 $\leadsto \tilde{\varphi}$!

$(A \models \tilde{\varphi} \Leftrightarrow B \models \varphi)$

④ PROOF:

LET R BE RELATION. SHOW:

R INVARIANT UNDER $\text{Pol}(A)$

$\Leftrightarrow$

R PP-DEFINABLE IN A

$\Leftarrow$

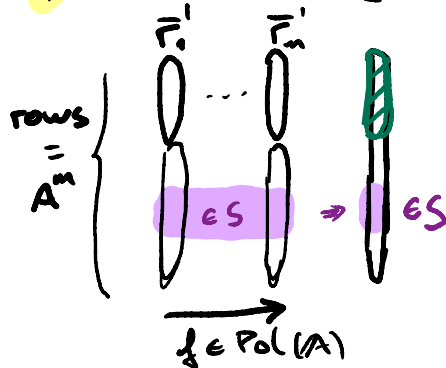
EASY: $S(x_1, \dots, x_k), T(y_1, \dots, y_n)$ INVARIANT

$\Rightarrow S \wedge T$ INVARIANT,

$\exists x_i, S(x_1, \dots, x_k)$ INVARIANT

$\rightarrow$

LET $R = \{F_1, \dots, F_m\}$ (A FINITE!)



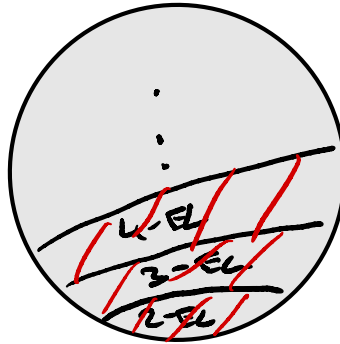
$R' := \{f(F_1, \dots, F_m) \mid f \in \text{Pol}(A)\}$

R' IS $\{ \wedge, \exists \}$ -DEFINABLE!

R = PROJECTION OF R' ONTO FIRST COORDINATES!

□

POLYMORPHISMS ALLOW FACTORING OF
THE FINITE WORLD BY PP-DEFINABILITY



THEOREM (POST '41)

A BOOLEAN $\Rightarrow \text{POL}(A)$ CONTAINS:

- CONSTANT
- MAX
- MAJORITY
- MIN
- MINORITY

OR $\text{POL}(A) = \{ \text{"UNARY" INJECTIONS} \}$

THEOREM (POST '41)

1A BOOLEAN $\Rightarrow$ $\text{POL}(A)$ CONTAINS:

- CONSTANT
- MAX
- MAJORITY
- MIN
- MINORITY

OR $\text{POL}(A) = \{ \text{"UNARY" INJECTIONS} \}$

THEOREM (SCHAEFER '78) COMPLEXITY DICOTOMY

THEOREM (POST '41)

1A BOOLEAN $\Rightarrow$ $\text{POL}(A)$ CONTAINS:

- CONSTANT
- MAX
- MAJORITY
- MIN
- MINORITY

OR $\text{POL}(A) = \{ \text{"UNARY" INJECTIONS} \}$

THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS $\text{NAE} = \{ (x_1, x_2, x_3) \mid \text{NOT ALL EQUAL} \}$

THEOREM (POST '41)

1A BOOLEAN $\Rightarrow$ $\text{POL}(A)$ CONTAINS:

- CONSTANT
- MAX
- MAJORITY
- MIN
- MINORITY

OR $\text{POL}(A) = \{ \text{"UNARY" INJECTIONS} \}$

THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS $\text{NAE} = \{ (x_1, x_2, x_3) \mid \text{NOT ALL EQUAL} \}$

ALGORITHMS

THEOREM (POST '41)

1A BOOLEAN $\Rightarrow$ $POL(A)$ CONTAINS:

- CONSTANT
- MAX
- MAJORITY
- MIN
- MINORITY

OR $POL(A) = \{ \text{"UNARY" INJECTIONS} \}$

THEOREM (SCHAEFER '78) COMPLEXITY DICOTOMY

HARDNESS $NAE = \{ (x_1, x_2, x_3) \mid \text{NOT ALL EQUAL} \}$

ALGORITHMS

- CONSTANT WLOG 0:

SET ALL VARIABLES OF INSTANCE TO 0

THEOREM (POST '41)

IF A BOOLEAN $\Rightarrow$ $POL(A)$ CONTAINS:

- CONSTANT
- MAX
- MAJORITY
- MIN
- MINORITY

OR $POL(A) = \{ \text{"UNARY" INJECTIONS} \}$

THEOREM (SCHAEFER '78) COMPLEXITY DICOTOMY

HARDNESS $NAE = \{ (x_1, x_2, x_3) \mid \text{NOT ALL EQUAL} \}$

ALGORITHMS

- CONSTANT WLOG 0:

SET ALL VARIABLES OF INSTANCE TO 0

- MINORITY: $m(x, x, y) = m(x, y, x) = m(y, x, x) = y$

$x - y + z$!

INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$

GAUSS ALGORITHM

THEOREM (POST '41)

1A BOOLEAN $\Rightarrow$ POL(A) CONTAINS:

- CONSTANT
- MAX
- MAJORITY
- MIN
- MINORITY

OR $\text{POL}(A) = \{ \text{"UNARY" INJECTIONS} \}$

THEOREM (SCHAEFER '78) COMPLEXITY DICOTOMY

HARDNESS $\text{NAE} = \{ (x_1, x_2, x_3) \mid \text{NOT ALL EQUAL} \}$

ALGORITHMS

- CONSTANT WLOG 0:

SET ALL VARIABLES OF INSTANCE TO 0

- MINORITY: $m(x, x, y) = m(x, y, x) = m(y, x, x) = y$

$x - y + z$!

INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$

GAUSS ALGORITHM

- MIN (AND DUALY MAX)

INVARIANT REL. HAVE SMALLEST TUPLE

SET EACH VARIABLE TO 0 UNLESS
SOME CONSTRAINT REQUIRES 1

THEOREM (POST '41)

1A BOOLEAN $\Rightarrow$ POL(A) CONTAINS:

- CONSTANT
- MAX
- MAJORITY
- MIN
- MINORITY

OR $\text{POL}(A) = \frac{1}{2}$ "UNARY" INJECTIONS

THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS $\text{NAE} = \frac{1}{2}(x_1, x_2, x_3)$ (NOT ALL EQUAL)

ALGORITHMS

- CONSTANT WLOG 0:

SET ALL VARIABLES OF INSTANCE TO 0

- MINORITY: $m(x, x, y) = m(x, y, x) = m(y, x, x) = y$

$x - y + z$!

INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$

GAUSS ALGORITHM

- MIN (AND DUALY MAX)

INVARIANT REL. HAVE SMALLEST TUPLE

SET EACH VARIABLE TO 0 UNLESS
SOME CONSTRAINT REQUIRES 1

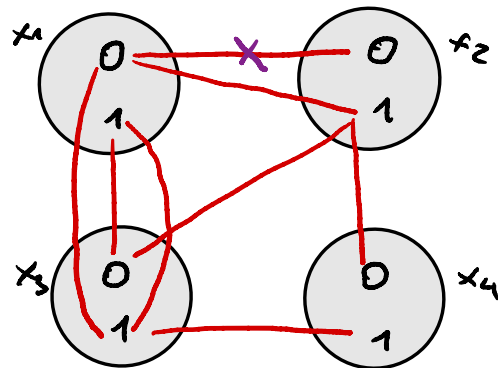
- MAJORITY $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$

LOCAL CONSISTENCY ALGORITHM

(2,3)-MINIMALITY

• KEEP LISTS OF POSSIBLE VALUES FOR
PAIRS OF VARIABLES (EDGES)

• REMOVE EDGES LOOKING AT CONSISTENCY
OF THE LISTS FOR 3 VARIABLES AT A
TIME



THEOREM (POST '41)

1A BOOLEAN $\Rightarrow$ POL(A) CONTAINS:

- CONSTANT
- MAX
- MAJORITY
- MIN
- MINORITY

OR $\text{POL}(A) = \frac{1}{2}$ "UNARY" INJECTIONS

THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS $\text{NAE} = \frac{1}{2}(x_1, x_2, x_3) \text{ (NOT ALL EQUAL)}$

ALGORITHMS

- CONSTANT WLOG 0:

SET ALL VARIABLES OF INSTANCE TO 0

- MINORITY: $m(x, y, z) = m(x, z, y) = m(y, x, z) = y$

$x - y + z$!

INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$

GAUSS ALGORITHM

- MIN (AND DUALY MAX)

INVARIANT REL. HAVE SMALLEST TUPLE

SET EACH VARIABLE TO 0 UNLESS
SOME CONSTRAINT REQUIRES 1

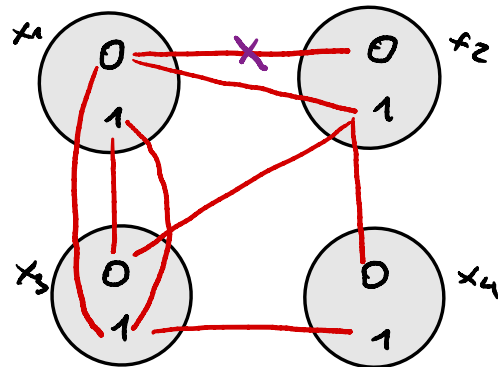
- MAJORITY $m(x, y, z) = m(x, z, y) = m(y, x, z) = x$

LOCAL CONSISTENCY ALGORITHM

(2,3)-MINIMALITY

• KEEP LISTS OF POSSIBLE VALUES FOR
PAIRS OF VARIABLES (EDGES)

• REMOVE EDGES LOOKING AT CONSISTENCY
OF THE LISTS FOR 3 VARIABLES AT A
TIME



EXERCISE

SHOW LOCAL CONSISTENCY SOLVES CSP
CORRECTLY IF SOLUTIONS INVARIANT
UNDER MAJORITY

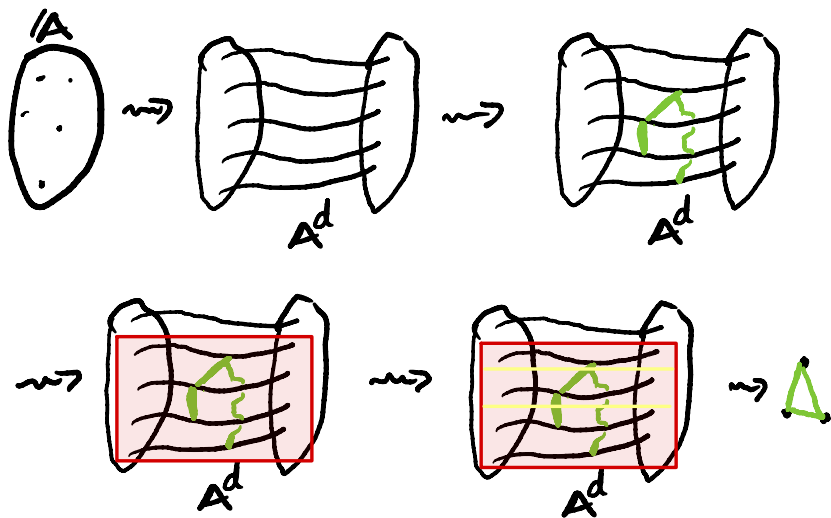
 PP-DEFINABILITY REQUIRES EQUAL
DOMAINS



DEFINITION

1A PP-INTERPRETS 1B $\Leftrightarrow$

**1B IS A PP-DEFINABLE FACTOR
OF PP-DEFINABLE SUBSTRUCTURE
OF PP-DEFINABLE STRUCTURE
ON FINITE POWER OF A**

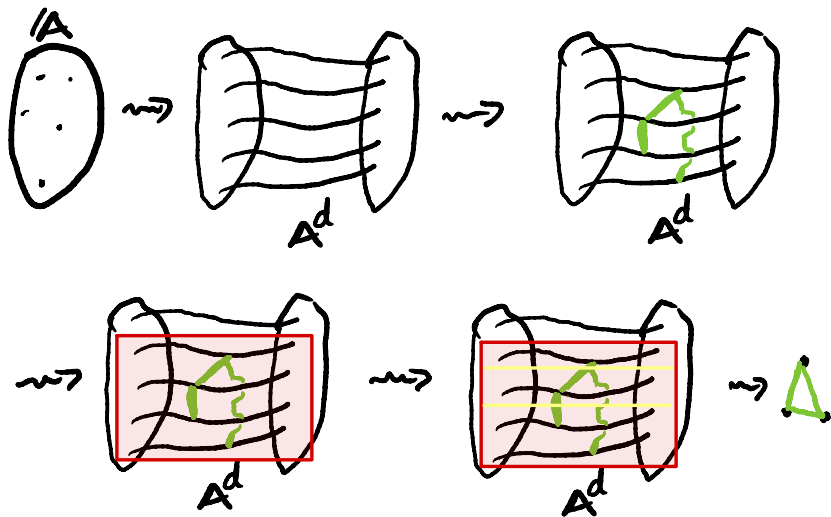


PP-DEFINABILITY REQUIRES EQUAL DOMAINS!

DEFINITION

$\mathcal{A}$ PP-INTERPRETS $\mathcal{B}$ $\Leftrightarrow$

$\mathcal{B}$ IS A PP-DEFINABLE FACTOR
OF PP-DEFINABLE SUBSTRUCTURE
OF PP-DEFINABLE STRUCTURE
ON FINITE POWER OF $\mathcal{A}$



EXAMPLE

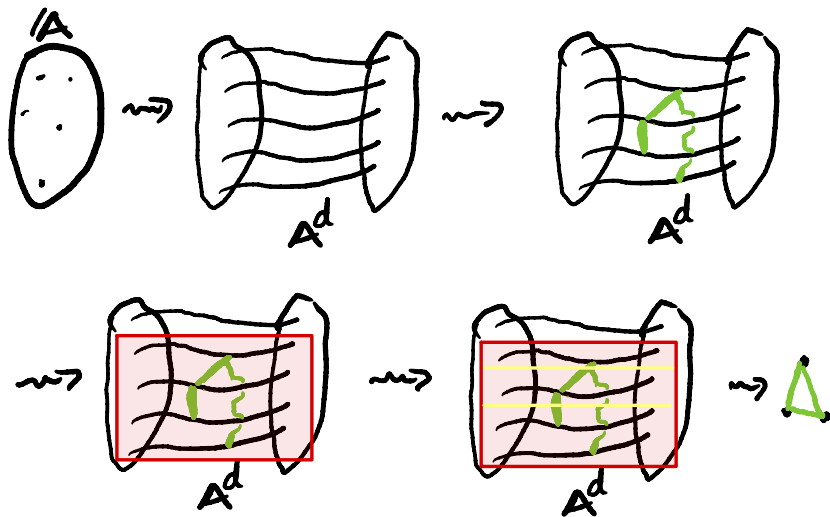
$$\begin{aligned}
 (\mathbb{Z}, +, \cdot, 0, 1) &\rightsquigarrow (\mathbb{Z}^2, \tilde{+}, \cdot, (0, 0), (1, 1)) \rightsquigarrow \\
 &\rightsquigarrow (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), \tilde{+}, \cdot, (0, 1), (1, 1)) \\
 &\rightsquigarrow (\mathbb{Q}, +, \cdot, 0, 1)
 \end{aligned}$$

 PP-DEFINABILITY REQUIRES EQUAL DOMAIN!

DEFINITION

$\mathcal{A}$ PP-INTERPRETS $\mathcal{B} \Leftrightarrow$

$\mathcal{B}$ IS A PP-DEFINABLE FACTOR
OF PP-DEFINABLE SUBSTRUCTURE
OF PP-DEFINABLE STRUCTURE
ON FINITE POWER OF $\mathcal{A}$



EXAMPLE

$$\begin{aligned} (\mathbb{Z}, +, \cdot, 0, 1) &\rightsquigarrow (\mathbb{Z}^2, \tilde{+}, \cdot, (0, 0), (1, 1)) \rightsquigarrow \\ &\rightsquigarrow (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), \tilde{+}, \cdot, (0, 1), (1, 1)) \\ &\rightsquigarrow (\mathbb{Q}, +, \cdot, 0, 1) \end{aligned}$$

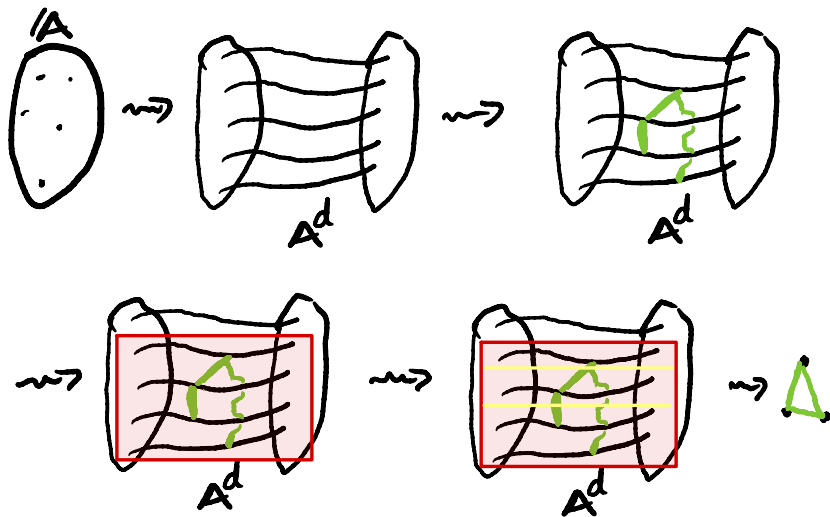
EXERCISE • SHOW THIS IS A PP-INTERPRETATION
• PP-INTERPRET $(\mathbb{Q}, <)$



DEFINITION

1A PP-INTERPRETS 1B $\Leftrightarrow$

B IS A PP-DEFINABLE FACTOR
OF PP-DEFINABLE SUBSTRUCTURE
OF PP-DEFINABLE STRUCTURE
ON FINITE POWER OF A



EXAMPLE

$$\begin{aligned} (\mathbb{Z}, +, \cdot, 0, 1) &\hookrightarrow (\mathbb{Z}^2, \tilde{+}, \cdot, (0, 1), (1, 1)) \hookrightarrow \\ &\hookrightarrow (\mathbb{Z} \times (\mathbb{Z} \setminus \{0, 1\}), \tilde{+}, \cdot, (0, 1), (1, 1)) \\ &\hookrightarrow (\mathbb{Q}, +, \cdot, 0, 1) \end{aligned}$$

EXERCISE • SHOW THIS IS A PP-INTERPRETATION

- PP-INTERPRET (Q. 2)

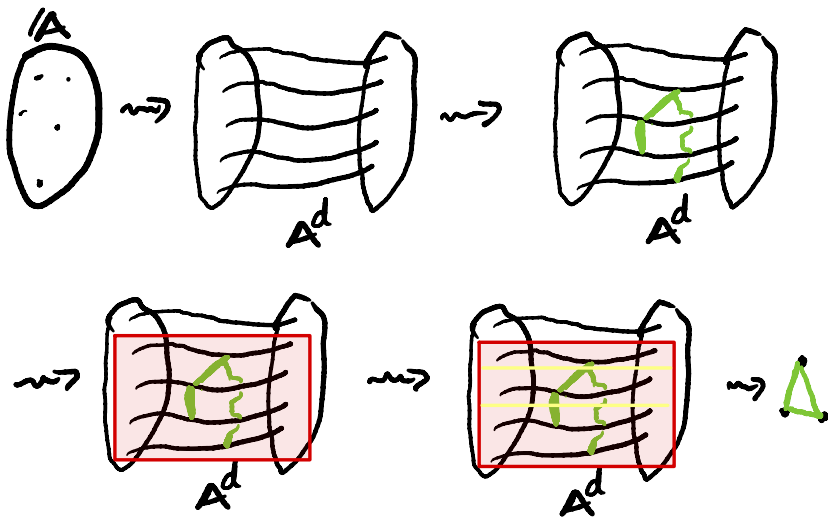
POLYMORPHISMS & PP-DEFINITIONS ✓
PP-INTERPRETATIONS ?

 PP-DEFINABILITY REQUIRES EQUAL DOMAINS!

DEFINITION

1A PP-INTERPRETS **1B** $\Leftrightarrow$

1B IS A PP-DEFINABLE FACTOR
OF PP-DEFINABLE SUBSTRUCTURE
OF PP-DEFINABLE STRUCTURE
ON FINITE POWER OF **A**



EXAMPLE

$$\begin{aligned} (\mathbb{Z}, +, \cdot, 0, 1) &\leadsto (\mathbb{Z}^2, \tilde{+}, \cdot, (0, 0), (1, 1)) \leadsto \\ &\leadsto (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), \tilde{+}, \cdot, (0, 1), (1, 1)) \\ &\leadsto (\mathbb{Q}, +, \cdot, 0, 1) \end{aligned}$$

EXERCISE • SHOW THIS IS A PP-INTERPRETATION
• PP-INTERPRET $(\mathbb{Q}, <)$

POLYMORPHISMS & PP-DEFINITIONS ✓
PP-INTERPRETATIONS ?

EQUATIONAL CONDITIONS:

- $\exists f \in \text{Pol}(\mathbb{Q}, <) \forall x, y \quad f(x, y) = f(y, x)$
- $\exists f \in \text{Pol}(\mathbb{Z}, \{0\}, \{1\}, +)$
 $\forall x, y, z \quad f(x, x, y) = f(y, x, x) = y$
- $\exists f \in \text{Pol}(K_3) : \forall x \quad f f f(x) = x$

DEFINITION

- EQUATIONAL (STRONG MAL'CEV) CONDITION:
SENTENCE

$$\Psi \equiv \exists f_1 \exists f_2 \dots \forall x_1 \forall x_2 \dots$$

$$s_1(\text{variables}) = t_1(\text{variables})$$

$\wedge$

$\vdots$

$$\wedge s_u(\text{variables}) = t_u(\text{variables})$$

WHERE $s_i, t_i \dots$ TERMS OVER $f_1, f_2, \dots$

DEFINITION

- EQUATIONAL (STRONG MAL'CEV) CONDITION:
SENTENCE

$$\Psi \equiv \exists f_1 \exists f_2 \dots \forall x_1 \forall x_2 \dots$$

$$\begin{aligned} & S_1(\text{variables}) = t_1(\text{variables}) \\ & \wedge \\ & \vdots \\ & \wedge S_k(\text{variables}) = t_k(\text{variables}) \end{aligned}$$

WHERE $S_i, t_i \dots$ TERMS OVER $f_1, f_2, \dots$

- Ψ TRIVIAL: $\Leftrightarrow$ FOR (ANY STRUCTURE) $\models \Psi$
 $\Leftrightarrow \Psi$ SATISFIABLE BY
PROJECTIONS

DEFINITION

- **EQUATIONAL (STRONG MAL'CEV) CONDITION:**

SENTENCE

$$\Psi \equiv \exists f_1 \exists f_2 \dots \forall x_1 \forall x_2 \dots$$

$$s_1(\text{variables}) = t_1(\text{variables})$$

$\wedge$

$\vdots$

$$\wedge s_k(\text{variables}) = t_k(\text{variables})$$

WHERE $s_i, t_i, \dots$ TERMS OVER $f_1, f_2, \dots$

- Ψ TRIVIAL: $\Leftrightarrow \text{POL}(\text{ANY STRUCTURE}) \models \Psi$
 $\Leftrightarrow \Psi$ SATISFIABLE BY PROJECTIONS
- $\text{EQ}(\text{POL}(A)) := \{\Psi \mid \text{POL}(A) \models \Psi\}$

DEFINITION

- **EQUATIONAL (STRONG MAL'CEV) CONDITION:**
SENTENCE

$$\Psi \equiv \exists f_1 \exists f_2 \dots \forall x_1 \forall x_2 \dots$$

$$\begin{aligned} & s_1(\text{variables}) = t_1(\text{variables}) \\ & \wedge \\ & \vdots \\ & \wedge s_k(\text{variables}) = t_k(\text{variables}) \end{aligned}$$

WHERE $s_i, t_i, \dots$ TERMS OVER $f_1, f_2, \dots$

- Ψ TRIVIAL: $\Leftrightarrow \text{Pol}(\text{ANY STRUCTURE}) \models \Psi$
 $\Leftrightarrow \Psi$ SATISFIABLE BY PROJECTIONS
- $\text{EQ}(\text{Pol}(A)) := \{\Psi \mid \text{Pol}(A) \models \Psi\}$

(ADOLESCENT) THEOREM (BULATOV + JEAVONS + KROKHIN 1995)

$$\text{EQ}(\text{Pol}(A)) \subseteq \text{EQ}(\text{Pol}(B))$$

$$\stackrel{Q}{\Rightarrow} A \text{ PP-INTERPRETS } B$$

$$\stackrel{Q}{\Rightarrow} \text{CSP}(B) \text{ REDUCES TO } \text{CSP}(A)$$

POLYMORPHISMS ALLOW FACTORING OF
THE FINITE WORLD BY PP-DEFINABILITY

INTERPRETABILITY



POLYMORPHISMS ALLOW FACTORING OF
THE FINITE WORLD BY PP-DEFINABILITY

INTERPRETABILITY



EXAMPLE ON THE BOOLEAN DOMAIN $\{0,1\}$
CONSIDER

$$R = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$\text{CSP}(\{0,1\}, R)$ IS 1-IN-3SAT
(NP-COMPLETE)

$$\text{Pol}(\{0,1\}, R) = \{f: f \text{ PROJECTION}\}$$

$\downarrow$ HOM

$\text{Pol}(A)$ FOR ANY A !

STRONGEST WRT PP-INTERPRETATIONS!

EXERCISE SHOW IT.

POLYMORPHISMS ALLOW FACTORING OF
THE FINITE WORLD BY PP-DEFINABILITY
INTERPRETABILITY



EXAMPLE ON THE BOOLEAN DOMAIN $\{0,1\}$
CONSIDER

$$R = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$CSP(\{0,1\}, R)$ IS 1-IN-3SAT
(NP-COMPLETE)

$$Pol(\{0,1\}, R) = \{f : f \text{ PROJECTION}\}$$

$\downarrow$ HOM

$Pol(A)$ FOR ANY A !

STRONGEST WRT PP-INTERPRETATIONS!

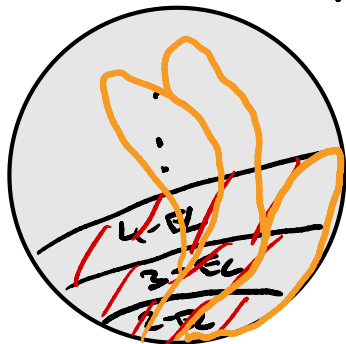
EXAMPLE



$CSP(K_3)$ IS 3-COLORING

EXERCISE SHOW IT.

POLYMORPHISMS ALLOW FACTORING OF
THE FINITE WORLD BY PP-DEFINABILITY
INTERPRETABILITY



EXAMPLE ON THE BOOLEAN DOMAIN $\{0,1\}$
CONSIDER

$$R = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$\text{CSP}(\{0,1\}, R)$ IS 1-IN-3SAT
(NP-COMPLETE)

$$\text{Pol}(\{0,1\}, R) = \{f: f \text{ PROJECTION}\}$$

↓ HOM

$\text{Pol}(A)$ FOR ANY A !

STRONGEST WRT PP-INTERPRETATIONS!

EXAMPLE



$\text{CSP}(K_3)$ IS 3-COLORING

EXERCISE

$$\forall f(x_1, \dots, x_n) \in \text{Pol}(K_3)$$

$$\exists i \exists g \in \text{Aut}(K_3)$$

$$f(x_1, \dots, x_n) = g(x_i)$$

POLYMORPHISMS ALLOW FACTORING OF
THE FINITE WORLD BY PP-DEFINABILITY
INTERPRETABILITY



EXAMPLE ON THE BOOLEAN DOMAIN $\{0,1\}$
CONSIDER

$$R = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$\text{CSP}(\{0,1\}, R)$ IS 1-IN-3SAT
(NP-COMPLETE)

$$\text{Pol}(\{0,1\}, R) = \{f: f \text{ PROJECTION}\}$$

$\downarrow$ HOM

$\text{Pol}(A)$ FOR ANY A !

STRONGEST WRT PP-INTERPRETATIONS!

EXAMPLE



$\text{CSP}(K_3)$ IS 3-COLORING

EXERCISE

$$\forall f(x_1, \dots, x_n) \in \text{Pol}(K_3)$$

$$\exists i \exists g \in \text{Aut}(K_3)$$

$$f(x_1, \dots, x_n) = g(x_i)$$

COROLLARY

$$\varphi: \text{Pol}(K_3) \rightarrow \text{Pol}(\{0,1\}, R)$$

$$f \mapsto i\text{-th PROJ}$$

PRESERVES EQUATIONS

$\Rightarrow K_3$ PP-INTERPRETS $(\{0,1\}, R)$
AND VICE-VERSA

$\Rightarrow$ 3-COLORING IS NP-COMPLETE

PP-INTERPRETATIONS INSUFFICIENT

... TO EXPLAIN ALL NP-HARDNESS

BY 1-IN-3SAT

PP-INTERPRETATIONS INSUFFICIENT
... TO EXPLAIN ALL NP-HARDNESS
BY 1-IN-3SAT

EXAMPLE

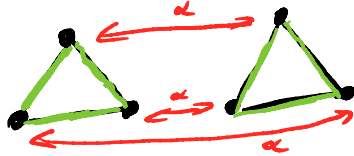
$$I_A = K_3 \dot{\cup} K_3$$



PP-INTERPRETATIONS INSUFFICIENT
... TO EXPLAIN ALL NP-HARDNESS
BY 1-IN-3SAT

EXAMPLE

$$A = K_3 \dot{\cup} K_3$$



$$\text{LET } S(x, y, z) = \begin{cases} z, & x, y \text{ IN SAME } K_3 \\ x, & \text{OTHERWISE} \end{cases}$$

LET $\alpha(x)$ FLIP THE TWO COPIES

THEN $\alpha, S \in \text{POL}(A)$

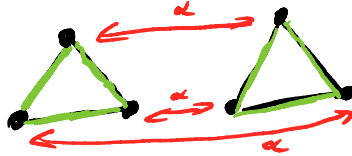
$$\text{AND } S(x, x, y) = y = S(y, \alpha y, x)$$

$$\Rightarrow \text{EQ}(\text{POL}(A)) \not\equiv \text{EQ}(\text{POL}(K_3))$$

PP-INTERPRETATIONS INSUFFICIENT
 ... TO EXPLAIN ALL NP-HARDNESS
 BY 1-IN-3SAT

EXAMPLE

$$A = K_3 \dot{\cup} K_3$$



$$\text{LET } S(x, y, z) = \begin{cases} z, & x, y \text{ IN SAME } K_3 \\ x, & \text{OTHERWISE} \end{cases}$$

LET $\alpha(x)$ FLIP THE TWO COPIES

THEN $\alpha, s \in \text{POL}(A)$

$$\text{AND } S(x, \alpha(x), y) = y = S(y, \alpha(y), x)$$

$$\Rightarrow \text{EQ}(\text{POL}(A)) \not\equiv \text{EQ}(\text{POL}(K_3))$$

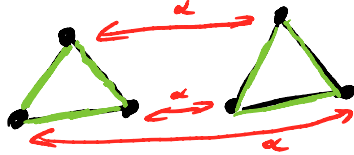
$$\text{BUT } \text{CSP}(A) = \text{CSP}(K_3) !$$

$$\text{SINCE } A \xrightarrow{\text{hom}} K_3, K_3 \xrightarrow{\text{hom}} A$$

PP-INTERPRETATIONS INSUFFICIENT
... TO EXPLAIN ALL NP-HARDNESS
BY 1-IN-3SAT

EXAMPLE

$$A = K_3 \dot{\cup} K_3$$



$$\text{LET } S(x, y, z) = \begin{cases} z, & x, y \text{ IN SAME } K_3 \\ x, & \text{OTHERWISE} \end{cases}$$

LET $\alpha(x)$ FLIP THE TWO COPIES

THEN $\alpha, s \in \text{POL}(A)$

$$\text{AND } S(x, \alpha(y), \gamma) = \gamma = S(\gamma, \alpha(y), x)$$

$$\Rightarrow \text{EQ}(\text{POL}(A)) \not\equiv \text{EQ}(\text{POL}(K_3))$$

$$\text{BUT } \text{CSP}(A) = \text{CSP}(K_3) !$$

$$\text{SINCE } A \xrightarrow{\text{Hom}} K_3, K_3 \xrightarrow{\text{Hom}} A$$

A AND K_3 ARE
HOMOMORPHICALLY EQUIVALENT
(H.E.)

DEFINITION

$\text{A PP-CONSTRUCTS B} : \Leftrightarrow$

$$\text{A} \xrightarrow{\text{PP-INT}} \mathcal{C}_1 \xrightarrow{\text{H.E}} \mathcal{C}_2 \xrightarrow{\text{PP-INT}} \mathcal{C}_3 \rightarrow \dots \rightarrow \text{B}$$

DEFINITION

A PP-CONSTRUCTS $B : \Leftrightarrow$

$$A \xrightarrow{\text{PP-INT}} C_1 \xrightarrow{\text{H.E}} C_2 \xrightarrow{\text{PP-INT}} C_3 \rightarrow \dots \rightarrow B$$

(ADULT) THEOREM

(BARTO + UPRŠAL + P. 16)

$EQ^1 \dots$ HEIGHT 1 - CONDITIONS

$$f_i(\text{variables}) = f_j(\text{variables})$$

~~f_i~~

~~f_j~~

$$EQ^1(\text{POL}(A)) \leq EQ^1(\text{POL}(B))$$

$\Leftrightarrow A$ PP-CONSTRUCTS B

$\Rightarrow \text{CSP}(B)$ REDUCES TO $\text{CSP}(A)$

DEFINITION

A PP-CONSTRUCTS $B : \Leftrightarrow$

$$A \xrightarrow{\text{PP-INT}} C_1 \xrightarrow{\text{H.E}} C_2 \xrightarrow{\text{PP-INT}} C_3 \rightarrow \dots \rightarrow B$$

(ADULT) THEOREM

(BARTO + UPRŠAL + P. 16)

$EQ^1 \dots$ HEIGHT 1 - CONDITIONS

$$f_i(\text{variables}) = f_j(\text{variables})$$

~~f_i~~

~~f_j~~

$$EQ^1(POL(A)) \leq EQ^1(POL(B))$$

$\Leftrightarrow A$ PP-CONSTRUCTS B

$\Rightarrow CSP(B)$ REDUCES TO $CSP(A)$

EXAMPLE

HEIGHT 1 :

- $m(k, x, y) = m(x, y, x) = m(y, x, x)$
- $c(x_1 \dots x_n) = c(x_2 \dots x_n, x_1)$

NOT HEIGHT 1 :

- $m(x, x, y) = m(k, y, x) = m(y, k, k) = x$
- $s(x, x, y) = s(y, xy, x)$

DEFINITION

A PP-CONSTRUCTS $B : \Leftrightarrow$

$$A \xrightarrow{\text{PP-INT}} C_1 \xrightarrow{\text{H.E}} C_2 \xrightarrow{\text{PP-INT}} C_3 \rightarrow \dots \rightarrow B$$

(ADULT) THEOREM

(BARTO + UPRŠAL + P. 16)

$EQ^1 \dots$ HEIGHT 1 - CONDITIONS

$$f_i(\text{variables}) = g_j(\text{variables})$$

~~f_i~~

~~g_j~~

$$EQ^1(\text{Pol}(A)) \leq EQ^1(\text{Pol}(B))$$

$\Leftrightarrow A$ PP-CONSTRUCTS B

$\Rightarrow \text{CSP}(B)$ REDUCES TO $\text{CSP}(A)$

EXAMPLE

HEIGHT 1 :

- $m(k, x, y) = m(x, y, x) = m(y, x, x)$
- $c(x_1 \dots x_n) = c(x_2 \dots x_n, x_1)$

NOT HEIGHT 1 :

- $m(k, x, y) = m(k, y, x) = m(y, k, k) = x$
- $s(x, x, y) = s(y, xy, x)$

EXAMPLE

$A = U_3 \cup U_3$ PP-CONSTRUCTS U_3

~~PP-INTERPRETS~~

$f \in \text{Pol}(A) \Rightarrow f|_{U_3}$ IS ESSENTIALLY UNARY

COROLLARY

$EQ^*(P_0(A))$ TRIVIAL

$\Rightarrow A$ PP-CONSTRUCTS EVERY
FINITE STRUCTURE

$\Rightarrow CSP(A)$ NP-COMPLETE

COROLLARY

$EQ^*(P_0(A))$ TRIVIAL

$\Rightarrow$ A PP-CONSTRUCTS EVERY
FINITE STRUCTURE

$\Rightarrow CSP(A)$ NP-COMPLETE

THEOREM

$EQ^*(P_0(A))$ NON-TRIVIAL $\Rightarrow$

COROLLARY

$EQ^*(PO(A))$ TRIVIAL

$\Rightarrow$ A PP-CONSTRUCTS EVERY
FINITE STRUCTURE

$\Rightarrow CSP(A)$ NP-COMPLETE

THEOREM

$EQ^*(PO(A))$ NON-TRIVIAL $\Rightarrow$

$PO(A) \models \exists R \forall x, y, z$

$R(x, y, x, z, y, z) =$

$R(y, x, z, x, z, y)$

6-ARY
SIGGERS

(SIGGERS '0)

COROLLARY

$EQ^*(PO(A))$ TRIVIAL

$\Rightarrow$ $\exists$ A PP-CONSTRUCTS EVERY
FINITE STRUCTURE

$\Rightarrow$ $CSP(A)$ NP-COMPLETE

THEOREM

$EQ^*(PO(A))$ NON-TRIVIAL $\Rightarrow$

$PO(A) \models \exists f \forall x, y, z$

6-ARY
SIGGERS

$f(x, y, x, z, y, z) =$

$f(y, x, z, x, z, y)$

(SIGGERS '0)

$\models \exists f \forall a, r, e$

4-ARY
SIGGERS

$f(a, r, e, a) =$

$f(r, a, r, e)$

(KEARNES + MARLOWE
& MCKENZIE '14)

COROLLARY

$EQ^*(Po(A))$ TRIVIAL

$\Rightarrow$ A PP-CONSTRUCTS EVERY
FINITE STRUCTURE

$\Rightarrow CSP(A)$ NP-COMPLETE

THEOREM

$EQ^*(Po(A))$ NON-TRIVIAL $\Rightarrow$

$$Po(A) \models \exists f \forall x, y, z \quad f(x, y, x, z, y, z) = f(y, x, z, x, z, y)$$

6-ARY (SIGGERS '0)

SIGGERS

$$\models \exists f \forall a, r, e \quad f(a, r, e, a) = f(r, a, r, e)$$

4-ARY (KEARNES + MARLOWE + MCKENZIE '14)

SIGGERS

$$\models \exists f \forall x_1 \dots x_n \quad f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$$

$\forall n \geq |A|$ PRIME

CYCLIC (BARTO & HODGSON '11)

COROLLARY

$EQ^*(Po(A))$ TRIVIAL

$\Rightarrow$ A PP-CONSTRUCTS EVERY
FINITE STRUCTURE

$\Rightarrow CSP(A)$ NP-COMPLETE

THEOREM

$EQ^*(Po(A))$ NON-TRIVIAL $\Rightarrow$

$$Po(A) \models \exists f \forall x, y, z \quad f(x, y, x, z, y, z) = f(y, x, z, x, z, y)$$

6-ARY SIGGERS (SIGGERS '0)

$$\models \exists f \forall a, r, e, a \quad f(a, r, e, a) = f(r, a, r, e)$$

4-ARY SIGGERS (KEARNES + MARLOWE + MCKENZIE '14)

$$\models \exists f \forall x_1 \dots x_n \quad f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$$

$\forall n \geq |A|$ PRIME
CYCLIC (BARTO & KLEIN '11)

$$\models \exists f \forall x, y \quad f(x \dots x, y) = \dots = f(y, x \dots x)$$

WNU
 $\forall n \geq |A|$ PRIME (MARSTON + MCKENZIE '08)

COROLLARY

$EQ^*(Po(A))$ TRIVIAL

$\Rightarrow$ A PP-CONSTRUCTS EVERY
FINITE STRUCTURE

$\Rightarrow CSP(A)$ NP-COMPLETE

THEOREM

$EQ^*(Po(A))$ NON-TRIVIAL $\Rightarrow$

$$Po(A) \models \exists f \forall x, y, z \quad f(x, y, x, z, y, z) = f(y, x, z, x, z, y)$$

6-ARY (Siggers '0)

SIGGERS

$$\models \exists f \forall a, r, e \quad f(a, r, e, a) = f(r, a, r, e)$$

4-ARY (KEARNS + MARCOWICZ + MCKENZIE '14)

SIGGERS

$$\models \exists f \forall x_1 \dots x_n \quad f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$$

$\forall n \geq |A|$ PRIME

CYCLIC (BARTO & HODGSON '11)

$$\models \exists f \forall x, y \quad f(x \dots x, y) = \dots = f(y, x \dots x)$$

$\forall n \geq |A|$ PRIME

WNU (MARSTI + MCKENZIE '08)

$\Rightarrow CSP(A) \in P$ (BUKATOV, ZHUK '17)

THEOREM $EQ'(Pol(A))$ NON-TRIVIAL

$\Rightarrow CSP(A) \in P$ (Bulatov, Zhuk '17)

THEOREM $EQ'(POL(A))$ NON-TRIVIAL
 $\Rightarrow CSP(A) \in P$ (BUGAYEV, ZHUK '17)

ALGORITHM

REDUCES TO SMALLER STRUCTURE
BY SOLVING LINEAR EQUATIONS
OR LOCAL CONSISTENCY CHECKING

SUMMARY

- $CSP(A)$: DECIDING PRIMITIVE POSITIVE THEORY OF A

EXAMPLE: SOLVING EQUATIONS

SAT

n -COLORING

THEORETICAL COMPUTER SCIENCE

- PP-DEFINITIONS (BABY), PP-INTERPRETATIONS (ADOLESCENT),
PP-CONSTRUCTIONS (ADULT) $\Rightarrow$ REDUCTIONS

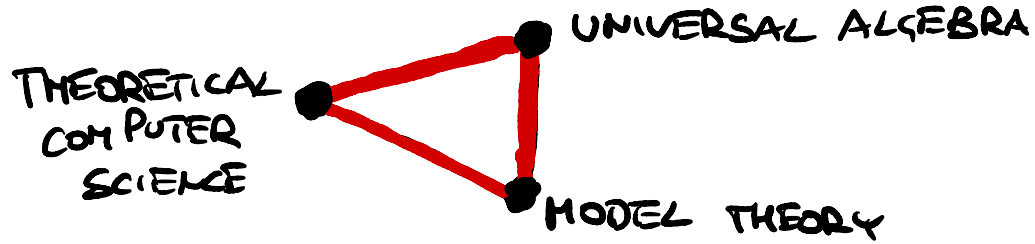
"MODEL THEORY"

- CHARACTERIZED BY :

POLYMORPHISMS / EQUATIONS / HA EQUATIONS

UNIVERSAL ALGEBRA

THESE ALSO GIVE ALGORITHMS.



Thank you!