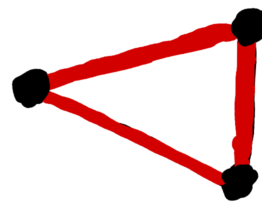
THE POWER OF POLYMORPHISMS



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TU WIEN

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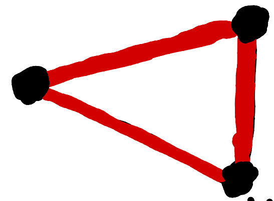
ALGEBRAS

UNIVERSAL ALGEBRA

THEORETICAL COMPUTER SCIENCE

COMPUTATIONAL PROBLEMS (CSPs)



MODEL THEORY

STRUCTURES (RELATIONAL)

PART I THE MATHEMATICS OF FINITE-DOMAIN CSPs

ALGEBRAS

RELATIONAL STRUCTURES

PART II INFINITE-DOMAIN CSPs MODELLING PROBLEMS

MATHEMATICS

PART I : THE MATHEMATICS OF FINITE-DOMAIN CSPs

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES $x_1, \ldots, x_n$
- CONSTRAINTS ON THEM

QUESTION

- $\exists$ VALUES FOR $x_1 \ldots x_n$ SATISFYING ALL CONSTRAINTS ?

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EXAMPLE

- SUDOKU
- SCHEDULING
- SOLVING EQUATIONS

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MODEL:

- SET V OF POSSIBLE VALUES (FIXED)
  E.G. $\{0,1\}, \{0,1,2\}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}, \dots$
- ALLOWED CONSTRAINTS (FIXED)
  $C_1, \dots, C_m$ RELATIONS ON V
  $C_i \subseteq V^{a_i}$

SO $\mathbb{A} := (V, C_1, \dots, C_m)$ RELATIONAL STRUCTURE = "TEMPLATE"

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1 \cdots x_n$
- CONSTRAINTS C_{i_1}(VARIABLES)
  $\vdots$
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QUESTION $\exists$ SOLUTION $x_i \mapsto a_i \in A$?

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ALTERNATIVE: GIVEN PP-SENTENCE

$\varphi \equiv \exists x_1 \dots \exists x_n \, C_{i_1}(\text{VAR}) \wedge \dots \wedge C_{i_\ell}(\text{VAR})$

QUESTION: $\mathbb{A} \models \varphi$?

PP ... PRIMITIVE POSITIVE

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PP ... PRIMITIVE POSITIVE

META-QUESTION

FOR WHAT $\mathbb{A}$ IS CSP($\mathbb{A}$) EASY / HARD ?

P: $\exists$ ALGORITHM PROVIDING ANSWER IN p(# VARIABLES) STEPS, p POLYNOMIAL

≠

NP: NOT NECESSARILY IN P BUT VERIFYING "SOLUTION" IN P

EXAMPLE

TERNARY RELATIONS

- $\mathbb{A} = (\mathbb{Z}, \{0\}, \{1\}, \dot{+}, \dot{\cdot})$

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1 \cdot x_1 = x_2$
  $x_2 + x_1 = x_3$
  $x_3 = 1$

QUESTION
- SOLUTION IN $\mathbb{Z}$?

EXAMPLE

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UNDECIDABLE (MATIYASEVIC '77)

HILBERT'S 10TH PROBLEM

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- SAME OVER $\mathbb{Z}_p$

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NP-COMPLETE
$\hookrightarrow$ IF IN P $\Rightarrow$ P = NP

EXAMPLE

TERNARY RELATIONS

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NP-COMPLETE
↳ IF IN P ⇒ P = NP

- SAME WITHOUT $\cdot$

EXAMPLE

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IN P (GAUSS)

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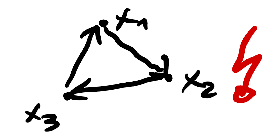
IN P (GAUSS)

- $\mathbb{A} = (\mathbb{Q}, <)$

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1 \ldots x_n$
- CONSTRAINTS $x_1 < x_2$
  $x_2 < x_3$
  $x_3 < x_1$
  $\vdots$

QUESTION
- SOLUTION IN $\mathbb{Q}$?



EXAMPLE

TERNARY RELATIONS

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- VARIABLES $x_1 \dots x_n$
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QUESTION
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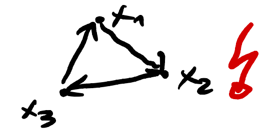

IN P

EXAMPLE

TERNARY RELATIONS

• $\mathbb{A} = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

CSP($\mathbb{A}$)

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IN P (GAUSS)

• $\mathbb{A} = (\mathbb{Q}, <)$

CSP($\mathbb{A}$)

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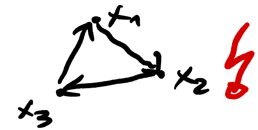
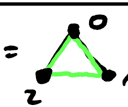
QUESTION
• SOLUTION IN $\mathbb{Q}$?



IN P

• $\mathbb{A} = K_3 =$ 

$= (\{0,1,2\}, E)$

CSP($\mathbb{A}$)

GIVEN
• VARIABLES $x_1 \dots x_n$
• CONSTRAINTS $E(x_1, x_2)$
$E(x_2, x_3)$
$\vdots$

QUESTION
• SOLUTION IN $\{0,1,2\}$?

EXAMPLE

TERNARY RELATIONS

- $\mathbb{A} = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

$\boxed{CSP(\mathbb{A})}$

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QUESTION
- SOLUTION IN $\mathbb{Z}$?

UNDECIDABLE (MATIYASEVIC '77)
HILBERT'S 10TH PROBLEM

- **SAME OVER $\mathbb{Z}_p$**

NP-COMPLETE
↳ IF IN P $\Rightarrow$ P = NP

- **SAME WITHOUT $\cdot$**

IN P (GAUSS)

- $\mathbb{A} = (\mathbb{Q}, <)$

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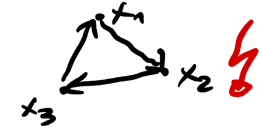
QUESTION
- SOLUTION IN $\mathbb{Q}$?



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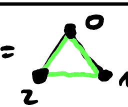
- $\mathbb{A} = K_3 =$ 

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QUESTION
- SOLUTION IN $\{0,1,2\}$?

3-COLORING PROBLEM = NP-COMPLETE

ALTERNATIVE FORMULATION OF CSP($\mathbb{A}$):

GIVEN. FINITE $\mathbb{B}$

QUESTION $\mathbb{B} \to \mathbb{A}$?, i.e.

$\exists h : \mathbb{B} \to \mathbb{A}$ HOMOMORPHISM?

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"SAME" PROBLEM:

- PP-SENTENCE $\varphi = \exists x_1 \ldots x_n \; C_{i_1}(\;) \wedge \ldots \wedge C_{i_\ell}(\;)$

  $\rightsquigarrow$ STRUCTURE: DOMAIN $\{x_1 \ldots x_n\}$
  $\mathbb{B}_\varphi$ RELATIONS C_i GIVEN BY CONSTRAINTS
  "CANONICAL DATABASE"

  THEN $\mathbb{B}_\varphi \to \mathbb{A} \iff \mathbb{A} \models \varphi$

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  $B = \{b_1 \dots b_\ell\}$

  $\rightsquigarrow$ $\varphi_{\mathbb{B}} \equiv \exists b_1 \dots b_\ell \bigwedge C_i(\;)$

  "CANONICAL QUERY OF $\mathbb{B}$"

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EXAMPLE $\mathbb{A} = K_2 =$ 

GIVEN DIGRAPH $\mathbb{B}$

QUESTION $\mathbb{B} \to K_2$?

2-COLORING PROBLEM

ALTERNATIVE FORMULATION OF CSP($\mathbb{A}$):

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2-COLORING PROBLEM

$\boxed{\text{EXERCISE}}$ CSP$(\mathbb{Q}, <) = ?$

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QUESTION $\mathbb{B} \to K_2$?

2-COLORING PROBLEM

EXERCISE $CSP(\mathbb{Q}, <) = ?$

COMPUTATIONAL COMPLEXITY OF CSP(A) CAN BE:

- UNDECIDABLE
- IN P
- NP-COMPLETE
- ANYTHING:
  EVERY COMPUTATIONAL PROBLEM IS POLYNOMIAL-TIME TURING-EQUIVALENT TO SOME CSP ($\mathbb{A}$) (BODIRSKY + GROHE '08)

 $\mathbb{A}$ FINITE $\Rightarrow$ CSP($\mathbb{A}$) IN NP

THEOREM (BULATOV, ZHUK '17)
(CONJECTURE FEDER + VARDI '93)

$\mathbb{A}$ FINITE

$\Rightarrow$ CSP($\mathbb{A}$) $\in$ P OR

NP-COMPLETE

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☆☆☆☆☆☆☆☆☆☆

(P $\neq$ NP)

CSP($\mathbb{A}$) $\in$ P $\Leftrightarrow$ $\mathbb{A}$ HAS ALGEBRAIC INVARIANT

$s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$

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ALGEBRAIC INVARIANTS = POLYMORPHISMS

$\mathbb{A} = (A;R_1,\dots,R_m)$ STRUCTURE

$f: A^\ell \to A$ POLYMORPHISM $:\Leftrightarrow$

f HOMOMORPHISM $\mathbb{A}^\ell \to \mathbb{A}$ $\Leftrightarrow$

$\forall i: \forall F_1 \dots F_\ell \in R_i$

$f(F_1, \dots, F_\ell) \in R_i$

THEOREM (BULATOV, ZHUK '17)
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f HOMOMORPHISM $\mathbb{A}^\ell \to \mathbb{A}$ $\Leftrightarrow$

$\forall i : \forall \bar{r}_1 \dots \bar{r}_\ell \in R_i$

$$f\left(\begin{pmatrix} \\ \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ \\ \\ \end{pmatrix} \right) \in R_i$$

$\boxed{\mathrm{Pol}(\mathbb{A})} := \{ f \mid f \text{ POLYMORPHISM OF } \mathbb{A} \}$

- CONTAINS PROJECTIONS $(x_1 \dots x_\ell) \mapsto x_j$
- COMPOSITION-CLOSED

$\Rightarrow$ ESSENTIALLY TERM FUNCTIONS OF AN ALGEBRA ON A !

EXAMPLE

- $\min(x,y) : \mathbb{Q}^2 \to \mathbb{Q} \quad \in \mathrm{Pol}(\mathbb{Q}, <)$

$$\begin{array}{lccc}
\min(& x_1 & y_1 &) = \\
 & \wedge & \wedge & \wedge \\
\min(& x_2 & y_2 &) =
\end{array}$$

$\boxed{\text{EXAMPLE}}$

- $\min(x,y) : \mathbb{Q}^2 \to \mathbb{Q} \quad \in \mathrm{Pol}(\mathbb{Q}, <)$

$$\begin{matrix} \min(& x_1 & y_1 &) = \\ & \wedge & \wedge & & \wedge \\ \min(& x_2 & y_2 &) = \end{matrix}$$

- $(x,y,z) \mapsto x-y+z \quad \in \mathrm{Pol}(\mathbb{Z}_r, \{0\}, \{1\}, \mp)$
  - $0-0+0 = 0$
  - $1-1+1 = 1$
  - $$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\in \mp} - \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{\in \mp} + \underbrace{\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}}_{\in \mp} = \underbrace{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}_{\in \mp}$$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

EXAMPLE

- $\min(x,y)\colon \mathbb{Q}^2 \to \mathbb{Q} \in \mathrm{Pol}(\mathbb{Q},<)$

$$\min\left(\begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} \min(x_1,y_1) \\ \wedge \\ \min(x_2,y_2) \end{pmatrix}$$

- $(x,y,z) \mapsto x-y+z \in \mathrm{Pol}(\mathbb{Z}_p, \{0\}, \{1\}, \mp)$
  - $0-0+0 = 0$
  - $1-1+1 = 1$
  - $\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\in \mp} - \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{\in \mp} + \underbrace{\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}}_{\in \mp} = \underbrace{\begin{pmatrix} \ \\ \ \\ \ \end{pmatrix}}_{\in \mp}$

  SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \mathrm{Pol}(K_3)$

NOT GREAT ☹

EXAMPLE

- $\min(x,y): \mathbb{Q}^2 \to \mathbb{Q} \quad \in \mathrm{Pol}(\mathbb{Q}, <)$

$$\begin{array}{l} \min(x_1, \; y_1) = \\ \qquad \wedge \qquad \wedge \qquad\quad \wedge \\ \min(x_2, \; y_2) = \end{array}$$

- $(x,y,z) \mapsto x - y + z \in \mathrm{Pol}(\mathbb{Z}_2, \{0\}, \{1\}, \mp)$
  - $0 - 0 + 0 = 0$
  - $1 - 1 + 1 = 1$
  - $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$$
    $$\in \mp \qquad \in \mp \qquad \in \mp \qquad \in \mp$$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \mathrm{Pol}(K_3)$

NOT GREAT ☹

(BABY) THEOREM

[BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69] GEIGER '68

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B}) \overset{①}{\Rightarrow} \mathbb{A}$ PP-DEFINES $\mathbb{B}$

$\overset{②}{\Rightarrow}$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

EXAMPLE

- $\min(x,y): \mathbb{Q}^2 \to \mathbb{Q} \quad \in \mathrm{Pol}(\mathbb{Q}, <)$

$$\begin{array}{lcccl} \min(& x_1 & & y_1 &) = \\ & \wedge & & \wedge & \quad \wedge \\ \min(& x_2 & & y_2 &) = \end{array}$$

- $(x,y,z) \mapsto x-y+z \quad \in \mathrm{Pol}(\mathbb{Z}_2, \{0\}, \{1\}, \overline{+})$
  - $0-0+0=0$
  - $1-1+1=1$
  - $$\underset{\in \overline{+}}{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}} - \underset{\in \overline{+}}{\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}} + \underset{\in \overline{+}}{\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}} = \underset{\in \overline{+}}{\begin{pmatrix}\ \\ \ \\ \ \end{pmatrix}}$$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \mathrm{Pol}(K_3)$

NOT GREAT ☹

(BABY) THEOREM

[BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69]
GEIGER '68

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B}) \overset{①}{\Longrightarrow} \mathbb{A}$ PP-DEFINES $\mathbb{B}$

$\overset{②}{\Longrightarrow}$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

---

② **PROOF**

EXAMPLE

- $\min(x,y)\colon \mathbb{Q}^2 \to \mathbb{Q} \in \mathrm{Pol}(\mathbb{Q}, <)$

$$\begin{array}{lll} \min(x_1, & y_1) = & \\ \quad\ \wedge & \ \wedge & \wedge \\ \min(x_2, & y_2) = & \end{array}$$

- $(x,y,z) \mapsto x-y+z \in \mathrm{Pol}(\mathbb{Z}_2, \{0\}, \{1\}, \neq)$
  - $0-0+0=0$
  - $1-1+1=1$
  - $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$

    $\in \neq \qquad \in \neq \qquad \in \neq \qquad \in \neq$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \mathrm{Pol}(K_3)$

NOT GREAT

(BABY) THEOREM

[BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69, GEIGER '68]

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B}) \overset{①}{\Rightarrow} \mathbb{A}$ PP-DEFINES $\mathbb{B}$

$\overset{②}{\Rightarrow}$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

---

② **PROOF**:

LET $\varphi \equiv \exists x_1 \dots x_n\, C_{i_1}(\ ) \wedge \dots \wedge C_{i_\ell}(\ )$ BE AN INSTANCE OF CSP($\mathbb{B}$)

REPLACE EACH C_{ij} BY ITS PP-DEFINITION IN $\mathbb{A}$

$\rightsquigarrow \tilde{\varphi}$!

$$\mathbb{A} \models \tilde{\varphi} \iff \mathbb{B} \models \varphi$$

EXAMPLE

- $\min(x,y): \mathbb{Q}^2 \to \mathbb{Q} \in \mathrm{Pol}(\mathbb{Q}, <)$

$$\begin{array}{l} \min(x_1 \quad y_1) = \\ \quad\;\; \wedge \qquad \wedge \qquad\quad \wedge \\ \min(x_2 \quad y_2) = \end{array}$$

- $(x,y,z) \mapsto x-y+z \in \mathrm{Pol}(\mathbb{Z}_n, \{0\}, \{1\}, \overline{+})$
  - $0-0+0 = 0$
  - $1-1+1 = 1$
  - $$\underset{\in \overline{+}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} - \underset{\in \overline{+}}{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}} + \underset{\in \overline{+}}{\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}} = \underset{\in \overline{+}}{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}$$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \mathrm{Pol}(K_3)$

NOT GREAT ☹

(BABY) THEOREM

[BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69]
GEIGER '68

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B}) \overset{①}{\Rightarrow} \mathbb{A}$ PP-DEFINES $\mathbb{B}$

$\overset{②}{\Rightarrow}$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

---

② **PROOF**:

LET $\varphi \equiv \exists x_1 \ldots x_n \, C_{i_1}(\;) \wedge \ldots \wedge C_{i_\ell}(\;)$ BE AN INSTANCE OF CSP($\mathbb{B}$)

REPLACE EACH C_{ij} BY ITS PP-DEFINITION IN $\mathbb{A}$

$\rightsquigarrow \tilde{\varphi}$!

$$\mathbb{A} \models \tilde{\varphi} \iff \mathbb{B} \models \varphi$$

① **PROOF**

EXAMPLE

- min(x,y): $\mathbb{Q}^2 \to \mathbb{Q}$ $\in \mathrm{Pol}(\mathbb{Q}, <)$

$$\begin{matrix} \min(& x_1 & y_1 &) = & \\ & \wedge & \wedge & & \wedge \\ \min(& x_2 & y_2 &) = & \end{matrix}$$

- $(x,y,z) \mapsto x-y+z$ $\in \mathrm{Pol}(\mathbb{Z}_p, \{0\}, \{1\}, \bar{+})$
  - $0-0+0=0$
  - $1-1+1=1$
  - $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$
    $$\in \bar{+} \qquad \in \bar{+} \qquad \in \bar{+} \qquad \in \bar{+}$$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in \mathrm{Pol}(K_3)$

NOT GREAT ☹

(BABY) THEOREM

[BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69, GEIGER '68]

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$$\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B}) \overset{①}{\Longrightarrow} \mathbb{A} \text{ PP-DEFINES } \mathbb{B}$$
$$\overset{②}{\Longrightarrow} \mathrm{CSP}(\mathbb{B}) \text{ REDUCES TO } \mathrm{CSP}(\mathbb{A})$$

---

② **PROOF**:

LET $\varphi \equiv \exists x_1 \dots x_n\, C_{i_1}(\ ) \wedge \dots \wedge C_{i_\ell}(\ )$ BE AN INSTANCE OF $\mathrm{CSP}(\mathbb{B})$

REPLACE EACH C_{i_j} BY ITS PP-DEFINITION IN $\mathbb{A}$ $\rightsquigarrow \tilde{\varphi}$!

$$\mathbb{A} \models \tilde{\varphi} \iff \mathbb{B} \models \varphi$$

① **PROOF**:

LET R BE RELATION

SHOW: R INVARIANT UNDER $\mathrm{Pol}(\mathbb{A})$

$\iff$

R PP-DEFINABLE IN $\mathbb{A}$

R INVARIANT UNDER $\mathrm{Pol}(\mathbb{A})$

$\Longleftrightarrow$

R PP-DEFINABLE IN $\mathbb{A}$

R INVARIANT UNDER $Pol(\mathbb{A})$

$\Leftrightarrow$

R PP-DEFINABLE IN $\mathbb{A}$

"$\Leftarrow$" EASY: $S(x_1, \dots x_l)$, $T(y_1 \dots, y_n)$ INVARIANT

$\Rightarrow$ $S \wedge T$ INVARIANT,

$\exists x_i\, S(x_1 \dots x_l)$ INVARIANT

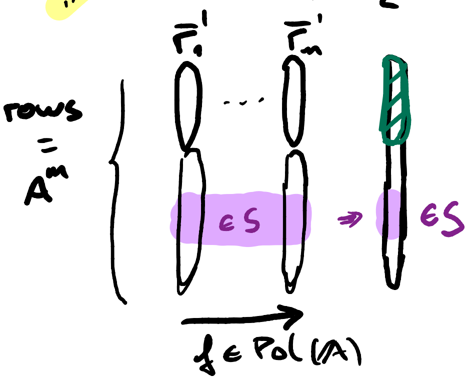
$$R \text{ invariant under } \mathrm{Pol}(\mathbb{A}) \iff R \text{ pp-definable in } \mathbb{A}$$

"$\Leftarrow$" EASY: $S(x_1, \dots x_l)$, $T(y_1 \dots, y_n)$ invariant

$\Rightarrow$ $S \wedge T$ invariant,

$\exists x_i\, S(x_1 \dots x_l)$ invariant

"$\Rightarrow$" LET $R = \{\bar{t}_1 \dots \bar{t}_m\}$ ($\mathbb{A}$ finite!)



$R' := \{f(\bar{t}_1' \dots \bar{t}_m') \mid f \in \mathrm{Pol}(\mathbb{A})\}$

R' is $\{\wedge, =\}$-DEFINABLE!

R = PROJECTION OF R' ONTO FIRST COORDINATES!

$\square$

R INVARIANT UNDER $\mathrm{Pol}(\mathbb{A})$

$\Leftrightarrow$

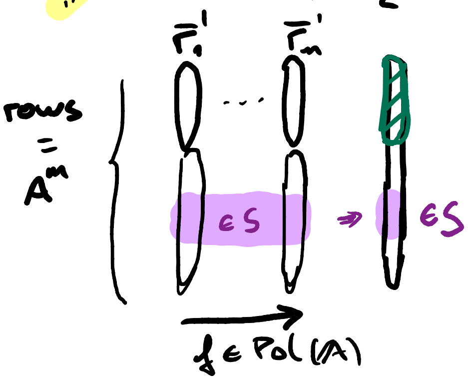
R PP-DEFINABLE IN $\mathbb{A}$

"$\Leftarrow$" EASY: $S(x_1,\dots,x_\ell)$, $T(y_1,\dots,y_n)$ INVARIANT

$\Rightarrow$ $S \wedge T$ INVARIANT,

$\exists x_i\ S(x_1 \dots x_\ell)$ INVARIANT

"$\Rightarrow$" LET $R = \{\bar{r}_1 \dots \bar{r}_m\}$ ($\mathbb{A}$ FINITE!)



$R' := \{f(\bar{r}_1' \dots \bar{r}_m') \mid f \in \mathrm{Pol}(\mathbb{A})\}$

R' IS $\{\wedge, =\}$-DEFINABLE!

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$\square$

(BABY) THEOREM (BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69, GEIGER '68)

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

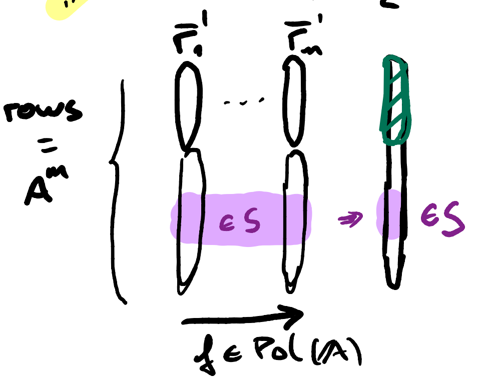
$\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B}) \overset{①}{\Rightarrow} \mathbb{A}$ PP-DEFINES $\mathbb{B}$

$\overset{②}{\Rightarrow}$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

$\boxed{\begin{array}{l} R \text{ INVARIANT UNDER } \mathrm{Pol}(\mathbb{A}) \\ \Leftrightarrow \\ R \text{ PP-DEFINABLE IN } \mathbb{A} \end{array}}$

"$\Leftarrow$" EASY: $S(x_1,\dots,x_\ell)$, $T(y_1,\dots,y_n)$ INVARIANT
$\Rightarrow$ $S \wedge T$ INVARIANT,
$\exists x_i\, S(x_1 \dots x_\ell)$ INVARIANT

"$\Rightarrow$" LET $R = \{\bar{t}_1 \dots \bar{t}_m\}$ ($\mathbb{A}$ FINITE!)



$R' := \{f(\bar{t}_1' \dots \bar{t}_m') \mid f \in \mathrm{Pol}(\mathbb{A})\}$

R' IS $\{\wedge, =\}$-DEFINABLE!

R = PROJECTION OF R' ONTO FIRST COORDINATES!

$\square$

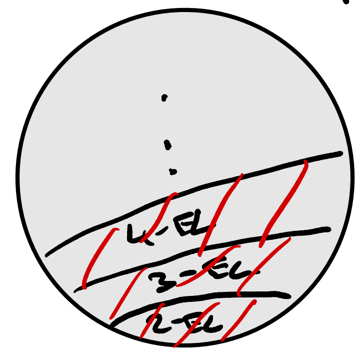
(BABY) THEOREM

(BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69)
GEIGER '68

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B}) \overset{①}{\Rightarrow} \mathbb{A}$ PP-DEFINES $\mathbb{B}$
$\overset{②}{\Rightarrow}$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

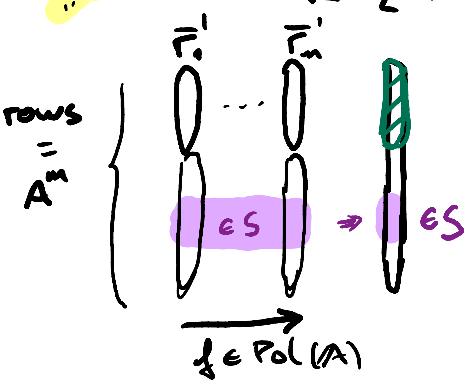
POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-DEFINABILITY



R INVARIANT UNDER $\mathrm{Pol}(\mathbb{A})$
$\Leftrightarrow$
R PP-DEFINABLE IN $\mathbb{A}$

"$\Leftarrow$" EASY: $S(x_1, \dots x_\ell)$, $T(y_1 \cdots, y_n)$ INVARIANT
$\Rightarrow$ $S \wedge T$ INVARIANT,
$\exists x_i\ S(x_1 \dots x_\ell)$ INVARIANT

"$\Rightarrow$" LET $R = \{\bar{r}_1 \dots \bar{r}_m\}$ ($\mathbb{A}$ FINITE!)



$R' := \{f(\bar{r}_1' \dots \bar{r}_m') \mid f \in \mathrm{Pol}(\mathbb{A})\}$

R' IS $\{\wedge, =\}$-DEFINABLE!

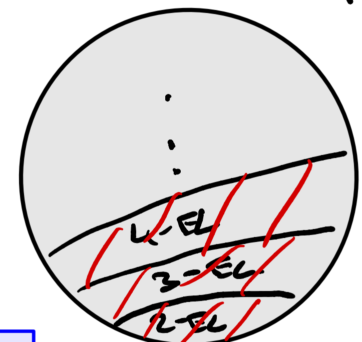
R = PROJECTION OF R' ONTO FIRST COORDINATES!

$\square$

(BABY) THEOREM [BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69, GEIGER '68]

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN
$\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B}) \overset{①}{\Rightarrow} \mathbb{A}$ PP-DEFINES $\mathbb{B}$
$\overset{②}{\Rightarrow}$ $\mathrm{CSP}(\mathbb{B})$ REDUCES TO $\mathrm{CSP}(\mathbb{A})$

POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-DEFINABILITY



THEOREM (POST '41)

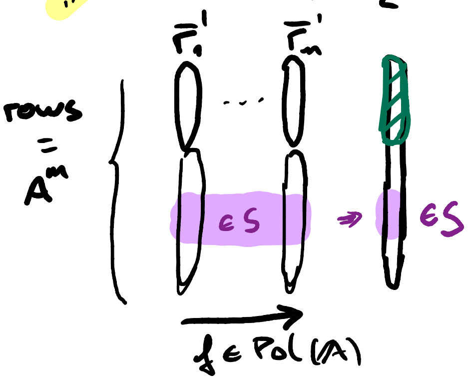
$\mathbb{A}$ BOOLEAN $\Rightarrow$ $\mathrm{Pol}(\mathbb{A})$ CONTAINS:
- CONSTANT
- max
- min
- maj
- minority

OR $\mathrm{Pol}(\mathbb{A}) = \{$"UNARY" INJECTIONS$\}$

R INVARIANT UNDER $\mathrm{Pol}(\mathbb{A})$
$\Leftrightarrow$
R PP-DEFINABLE IN $\mathbb{A}$

"$\Leftarrow$" EASY: $S(x_1, \dots x_\ell)$, $T(y_1 \cdots, y_n)$ INVARIANT
$\Rightarrow$ $S \wedge T$ INVARIANT,
$\exists x_i\; S(x_1 \dots x_\ell)$ INVARIANT

"$\Rightarrow$" LET $R = \{\bar{F}_1 \dots \bar{F}_m\}$ ($\mathbb{A}$ FINITE!)



$R' := \{f(\bar{F}_1' \dots \bar{F}_m') \mid f \in \mathrm{Pol}(\mathbb{A})\}$

R' IS $\{\wedge, =\}$-DEFINABLE!

R = PROJECTION OF R' ONTO FIRST COORDINATES!

$\square$

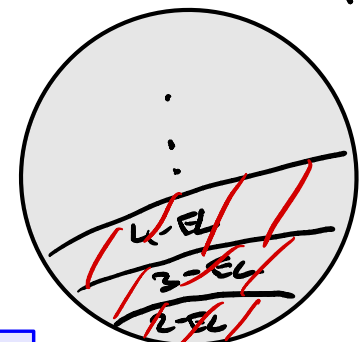
(BABY) THEOREM (BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69) GEIGER '68

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B}) \overset{①}{\Rightarrow} \mathbb{A}$ PP-DEFINES $\mathbb{B}$

$\overset{②}{\Rightarrow}$ $\mathrm{CSP}(\mathbb{B})$ REDUCES TO $\mathrm{CSP}(\mathbb{A})$

POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-DEFINABILITY



THEOREM (POST '41)

$\mathbb{A}$ BOOLEAN $\Rightarrow$ $\mathrm{Pol}(\mathbb{A})$ CONTAINS:

- CONSTANT
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- min
- maj
- minority

OR $\mathrm{Pol}(\mathbb{A}) = \{$"UNARY" INJECTIONS$\}$

THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

$\boxed{\text{THEOREM}}$ (POST '41)

$\mathbb{A}$ BOOLEAN $\Rightarrow$ $\mathrm{Pol}(\mathbb{A})$ CONTAINS:

- CONSTANT
- max
- min
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OR $\mathrm{Pol}(\mathbb{A}) = \{$ "UNARY" INJECTIONS $\}$

$\boxed{\text{THEOREM}}$ (SCHAEFER '78) COMPLEXITY DICHOTOMY

THEOREM (POST '41)

$\mathbb{A}$ BOOLEAN $\Rightarrow$ POL($\mathbb{A}$) CONTAINS:

- CONSTANT
- max
- min
- majority
- minority

OR POL($\mathbb{A}$) = { "UNARY" INJECTIONS }

THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS NAE = $\{(x_1, x_2, x_3) \mid$ NOT ALL EQUAL $\}$

$\boxed{\text{THEOREM}}$ (POST '41)

$\mathbb{A}$ BOOLEAN $\Rightarrow$ Pol($\mathbb{A}$) CONTAINS:

- CONSTANT
- max
- min
- majority
- minority

OR Pol($\mathbb{A}$) = { "UNARY" INJECTIONS }

$\boxed{\text{THEOREM}}$ (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS NAE = { (x_1, x_2, x_3) | NOT ALL EQUAL }

ALGORITHMS

THEOREM (POST '41)

$\mathbb{A}$ BOOLEAN $\Rightarrow$ Pol($\mathbb{A}$) CONTAINS:

- CONSTANT
- max
- min
- majority
- minority

OR Pol($\mathbb{A}$) = {"UNARY" INJECTIONS}

THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS NAE = {(x_1, x_2, x_3) | NOT ALL EQUAL}

ALGORITHMS

— CONSTANT WLOG 0:

SET ALL VARIABLES OF INSTANCE TO 0

THEOREM (POST '41)

$\mathbb{A}$ BOOLEAN $\Rightarrow$ $\mathrm{Pol}(\mathbb{A})$ CONTAINS:

- CONSTANT
- max
- min
- majority
- minority

OR $\mathrm{Pol}(\mathbb{A}) = \{$"UNARY" INJECTIONS$\}$

THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS $\quad NAE = \{(x_1, x_2, x_3) \mid$ NOT ALL EQUAL$\}$

ALGORITHMS

- CONSTANT WLOG 0:

  SET ALL VARIABLES OF INSTANCE TO 0

- MINORITY: $m(x,x,y) = m(x,y,x) = m(y,x,x) = y$

  $x - y + z$!

  INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$

  GAUSS ALGORITHM

THEOREM (POST '41)

$\mathbb{A}$ BOOLEAN $\Rightarrow$ $Pol(\mathbb{A})$ CONTAINS:

- CONSTANT
- max
- min
- majority
- minority

OR $Pol(\mathbb{A}) = \{$"UNARY" INJECTIONS$\}$

THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS NAE $= \{(x_1, x_2, x_3) \mid$ NOT ALL EQUAL$\}$

ALGORITHMS

- CONSTANT WLOG 0:
  SET ALL VARIABLES OF INSTANCE TO 0
- MINORITY: $m(x,x,y) = m(x,y,x) = m(y,x,x) = y$
  $x - y + z$!
  INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$
  GAUSS ALGORITHM
- min (AND DUALLY max)
  INVARIANT REL. HAVE SMALLEST TUPLE
  SET EACH VARIABLE TO 0 UNLESS
  SOME CONSTRAINT REQUIRES 1

THEOREM (POST '41)

A BOOLEAN $\Rightarrow$ Pol(A) CONTAINS:

- CONSTANT
- max
- min
- majority
- minority

OR Pol(A) = {"UNARY" INJECTIONS}

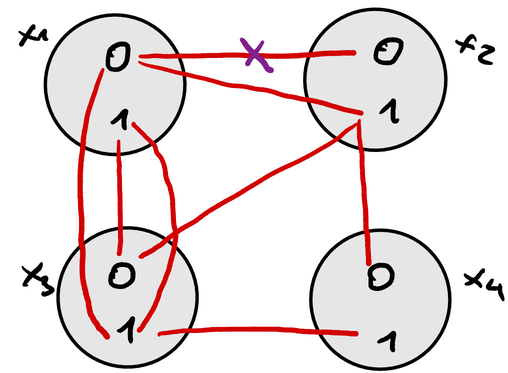
THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS NAE = $\{(x_1, x_2, x_3) \mid$ NOT ALL EQUAL$\}$

ALGORITHMS

- CONSTANT WLOG 0:
  SET ALL VARIABLES OF INSTANCE TO 0
- MINORITY: $m(x,x,y) = m(x,y,x) = m(y,x,x) = y$
  $x - y + z$!
  INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$
  GAUSS ALGORITHM
- min (AND DUALLY max)
  INVARIANT REL. HAVE SMALLEST TUPLE
  SET EACH VARIABLE TO 0 UNLESS SOME CONSTRAINT REQUIRES 1
- MAJORITY $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$
  LOCAL CONSISTENCY ALGORITHM
  (2,3)-MINIMALITY "

- KEEP LISTS OF POSSIBLE VALUES FOR PAIRS OF VARIABLES (EDGES)
- REMOVE EDGES LOOKING AT CONSISTENCY OF THE LISTS FOR 3 VARIABLES AT A TIME



THEOREM (POST '41)

A BOOLEAN ⇒ POL(A) CONTAINS:

- CONSTANT
- max
- min
- majority
- minority

OR POL(A) = { "UNARY" INJECTIONS }

THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

HARDNESS NAE = $\{(x_1,x_2,x_3) \mid$ NOT ALL EQUAL $\}$

ALGORITHMS

– CONSTANT WLOG 0: SET ALL VARIABLES OF INSTANCE TO 0

– MINORITY: $m(x,x,y) = m(x,y,x) = m(y,x,x) = y$

$x - y + z$!

INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$

GAUSS ALGORITHM

– min (AND DUALLY max)

INVARIANT REL. HAVE SMALLEST TUPLE

SET EACH VARIABLE TO 0 UNLESS SOME CONSTRAINT REQUIRES 1

– MAJORITY $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$

LOCAL CONSISTENCY ALGORITHM

(2,3)– MINIMALITY "

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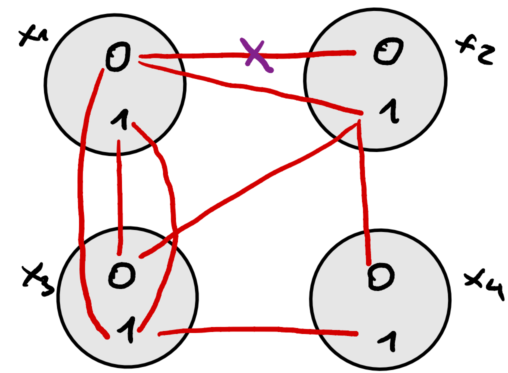

EXERCISE

SHOW LOCAL CONSISTENCY SOLVES CSP CORRECTLY IF SOLUTIONS INVARIANT UNDER MAJORITY

PP-DEFINABILITY REQUIRES EQUAL DOMAIN!

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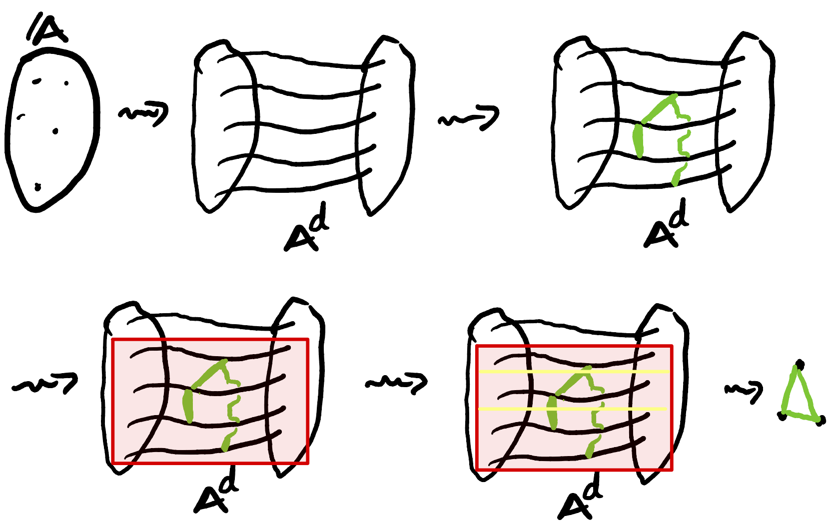
DEFINITION

$\mathbb{A}$ PP-INTERPRETS $\mathbb{B}$ $\Leftrightarrow$

$\mathbb{B}$ IS A PP-DEFINABLE FACTOR OF PP-DEFINABLE SUBSTRUCTURE OF PP-DEFINABLE STRUCTURE ON FINITE POWER OF $\mathbb{A}$



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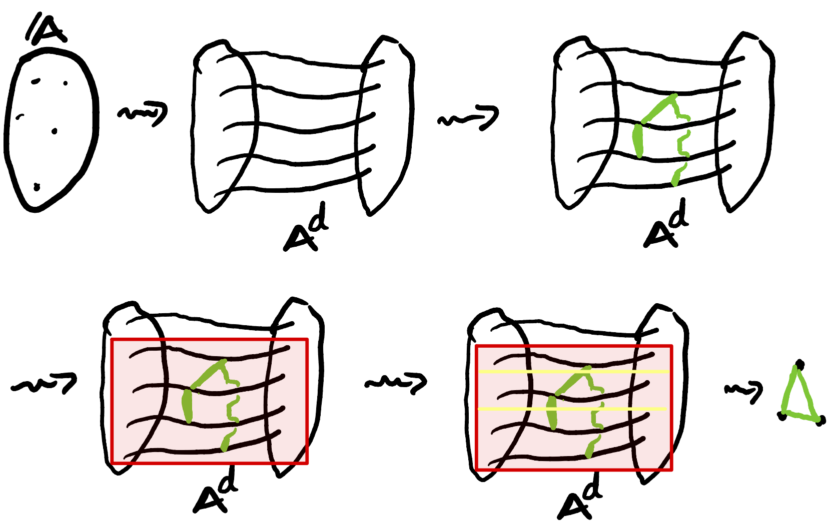

EXAMPLE

$$(\mathbb{Z},+,\cdot,0,1) \rightsquigarrow (\mathbb{Z}^2,\tilde{+},\cdot,(0,1),(1,1)) \rightsquigarrow$$

$$\rightsquigarrow (\mathbb{Z}\times(\mathbb{Z}\setminus\{0\}),\tilde{+},\cdot,(0,1),(1,1))$$

$$\rightsquigarrow (\mathbb{Q},+,\cdot,0,1)$$

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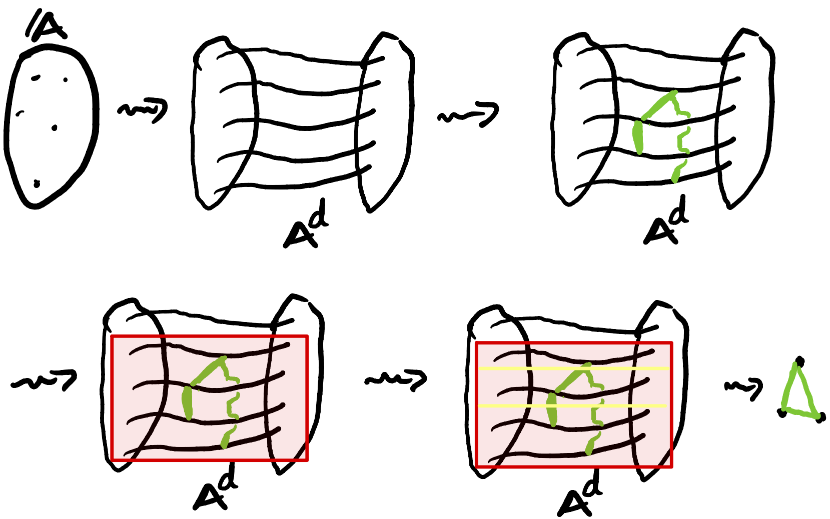

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EXERCISE

- SHOW THIS IS A PP-INTERPRETATION
- PP-INTERPRET $(\mathbb{Q},<)$

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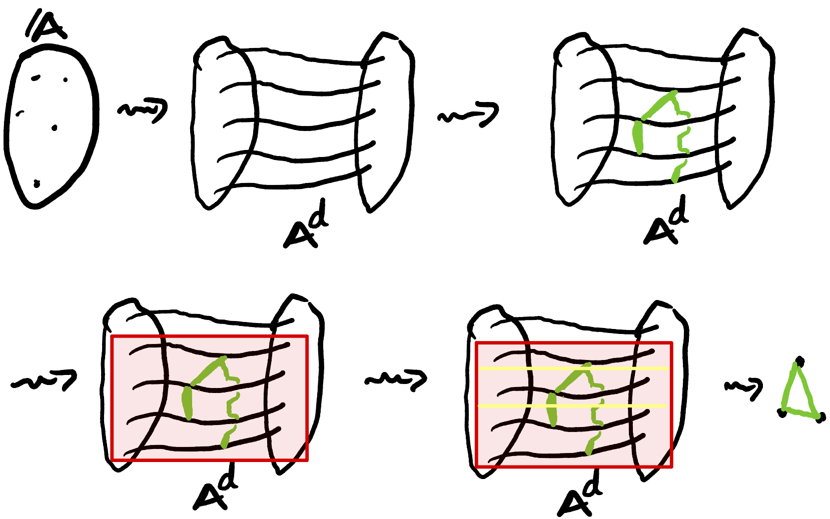

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POLYMORPHISMS & PP-DEFINITIONS ✓
PP-INTERPRETATIONS ?

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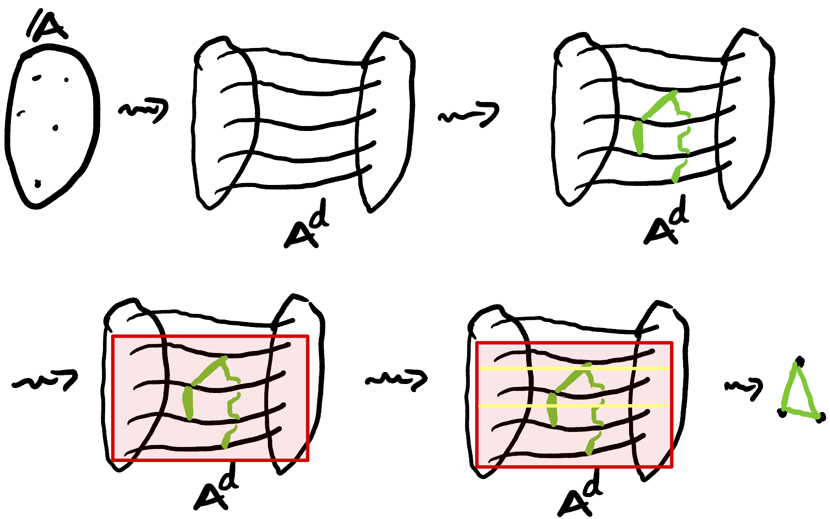

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EXERCISE

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POLYMORPHISMS & PP-DEFINITIONS ✓

PP-INTERPRETATIONS ?

EQUATIONAL CONDITIONS:

- $\exists f \in \mathrm{Pol}(\mathbb{Q},<)\ \forall x,y\quad f(x,y)=f(y,x)$
- $\exists f \in \mathrm{Pol}(\mathbb{Z}_1,\{0\},\{1\},+)$ $\forall x,y,z\quad f(x,x,y)=f(y,x,x)=y$
- $\exists f \in \mathrm{Pol}(K_3):\ \forall x\quad fff(x)=x$

DEFINITION

- EQUATIONAL ("STRONG MALCEV") CONDITION: SENTENCE

$$\Psi \equiv \exists f_1 \exists f_2 \cdots \forall x_1 \forall x_2 \cdots$$
$$s_1(\text{variables}) = t_1(\text{variables})$$
$$\wedge$$
$$\vdots$$
$$\wedge \quad s_k(\text{variables}) = t_k(\text{variables})$$

WHERE s_i, t_i ... TERMS OVER $f_1, f_2, \ldots$

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$\Leftrightarrow$ Ψ SATISFIABLE BY PROJECTIONS

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(ADOLESCENT) THEOREM

(BULATOV + JEAVONS + KROKHIN '00)

$$\mathrm{EQ}(\mathrm{Pol}(\mathbb{A})) \subseteq \mathrm{EQ}(\mathrm{Pol}(\mathbb{B}))$$

$\overset{1}{\Rightarrow}$ $\mathbb{A}$ PP-INTERPRETS $\mathbb{B}$

$\overset{2}{\Rightarrow}$ $\mathrm{CSP}(\mathbb{B})$ REDUCES TO $\mathrm{CSP}(\mathbb{A})$

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---

② PROOF:

DEFINITION

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② PROOF: AS WITH PP-DEFINITIONS

GIVEN INSTANCE OF $CSP(\mathbb{B})$:

$$\varphi = \exists x_1 \dots x_n\, C_{i_1}(\ ) \wedge \dots \wedge C_{i_n}(\ )$$

- FOR EACH OCCURENCE OF x_i CREATE NEW VARIABLES $x_i^1 \dots x_i^d$ (d ... POWER)
- ADD CONSTRAINTS THAT ALL TUPLES $(x_i^1 \dots x_i^d)$ BE EQUIVALENT
- ADD CONSTRAINTS RESTRICTING $(x_i^1 \dots x_i^d)$ TO SUBSET
- TRANSLATE CONSTRAINTS ON x_i INTO CONSTRAINTS ON x_i^d

DEFINITION

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EXERCISE CHECK THIS WORKS

① **PROOF SKETCH** OF:

$$\mathrm{EQ}(\mathrm{Pol}(\mathbb{A})) \subseteq \mathrm{EQ}(\mathrm{Pol}(\mathbb{B}))$$

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$\boxed{\text{DEFINITION}}$

$\mathbb{A}, \mathbb{B}$ RELATIONAL STRUCTURES

$\varphi : \mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$

CLONE HOMOMORPHISM $:\Leftrightarrow$

- φ PRESERVES ARITIES
- φ PROJECTIONS
- φ COMPOSITION

PROOF SKETCH OF:

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$\boxed{\text{(ADOLESCENT) THEOREM}}$ (BULATOV + JEAVONS + KROKHIN '00)

$\mathbb{A}, \mathbb{B}$ FINITE

$\exists\, \varphi : \mathrm{Pol}(\mathbb{A}) \xrightarrow[\text{HOM}]{} \mathrm{Pol}(\mathbb{B})$

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EXERCISE: SHOW THE CONVERSE

POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-DEF~~INA~~BILITY

INTERPRETABILITY

POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP- ~~DEFINABILITY~~ INTERPRETABILITY

$\boxed{\text{EXAMPLE}}$ ON THE BOOLEAN DOMAIN $\{0,1\}$

CONSIDER

$$R = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$CSP(\{0,1\}, R)$ IS 1-IN-3SAT (NP-COMPLETE)

$Pol(\{0,1\}, R) = \{ f : f \text{ PROJECTION} \}$

$\boxed{\text{EXERCISE}}$ SHOW IT.

$\downarrow$ HOM

$Pol(\mathbb{A})$ FOR ANY $\mathbb{A}$!

STRONGEST WRT PP-INTERPRETATIONS!

POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP- ~~DEFINABILITY~~ INTERPRETABILITY

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(NP-COMPLETE)

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STRONGEST WRT PP-INTERPRETATIONS!

EXAMPLE

K_3

$CSP(K_3)$ IS 3-COLORING

POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-~~DEFINABILITY~~ INTERPRETABILITY

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STRONGEST WRT PP-INTERPRETATIONS!

EXAMPLE

K_3

$CSP(K_3)$ IS 3-COLORING

EXERCISE $\forall f(x_1, \dots, x_n) \in \mathrm{Pol}(K_3)$

$\exists i \ \exists g \in \mathrm{Aut}(K_3)$

$f(x_1, \dots, x_n) = g(x_i)$

POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-~~DEFINABILITY~~ INTERPRETABILITY

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EXERCISE SHOW IT.

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STRONGEST WRT PP-INTERPRETATIONS!

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K_3

$CSP(K_3)$ IS 3-COLORING

EXERCISE $\forall f(x_1 \dots x_n) \in Pol(K_3)$
$\exists i \; \exists g \in Aut(K_3)$
$f(x_1 \dots x_n) = g(x_i)$

COROLLARY

$\varphi : Pol(K_3) \to Pol(\{0,1\}, R)$

$f \mapsto$ i-th PROJ

CLONE HOMOMORPHISM

$\Rightarrow$ K_3 PP-INTERPRETS $(\{0,1\},R)$ AND VICE-VERSA

$\Rightarrow$ 3-COLORING IS NP-COMPLETE

PP-INTERPRETATIONS INSUFFICIENT

... TO EXPLAIN ALL NP-HARDNESS

BY 1-IN-3SAT

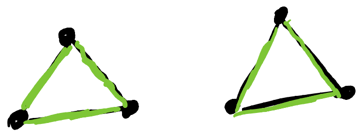
PP-INTERPRETATIONS INSUFFICIENT

... TO EXPLAIN ALL NP-HARDNESS

BY 1-IN-3SAT

EXAMPLE

$\mathbb{A} = K_3 \dot\cup K_3$

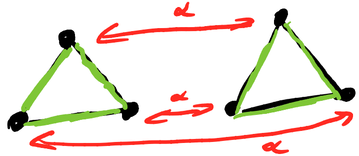


PP-INTERPRETATIONS INSUFFICIENT

... TO EXPLAIN ALL NP-HARDNESS BY 1-IN-3SAT

EXAMPLE

$\mathbb{A} = K_3 \dot\cup K_3$



LET $s(x,y,z) = \begin{cases} z, & x,y \text{ IN SAME } K_3 \\ x, & \text{OTHERWISE} \end{cases}$

LET $\alpha(x)$ FLIP THE TWO COPIES

THEN $\alpha, s \in \mathrm{Pol}(\mathbb{A})$

AND $s(x,x,y) = y = s(y, \alpha y, x)$

$\Rightarrow \mathrm{Pol}(\mathbb{A}) \not\to \mathrm{Pol}(K_3)$

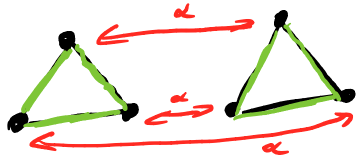
PP-INTERPRETATIONS INSUFFICIENT

... TO EXPLAIN ALL NP-HARDNESS

BY 1-IN-3SAT

EXAMPLE

$\mathbb{A} = K_3 \dot\cup K_3$



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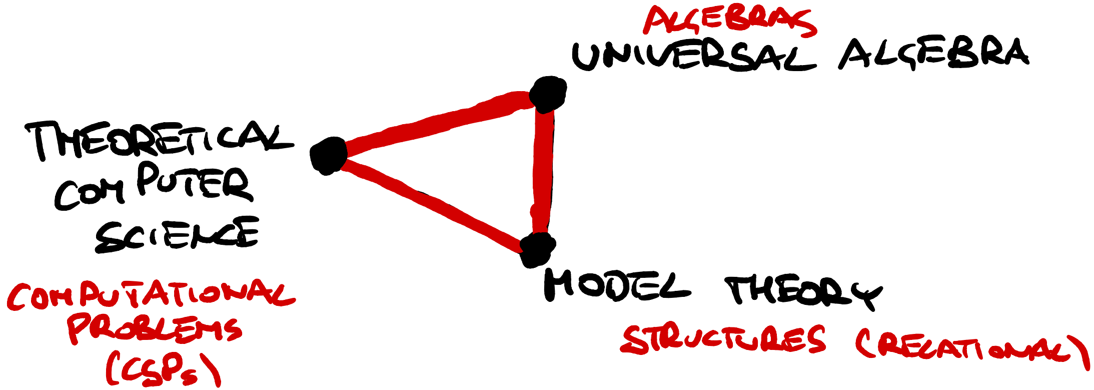
BUT $\mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(K_3)$!

SINCE $\mathbb{A} \xrightarrow{\text{HOM}} K_3$, $K_3 \xrightarrow{\text{HOM}} \mathbb{A}$



PART I THE MATHEMATICS OF FINITE-DOMAIN CSPs

ALGEBRAS

RELATIONAL STRUCTURES

PART II INFINITE-DOMAIN CSPs MODELLING PROBLEMS

MATHEMATICS

SUMMARY

- CSP($\mathbb{A}$) : DECIDING PRIMITIVE POSITIVE THEORY OF $\mathbb{A}$

  EXAMPLE: SOLVING EQUATIONS
  SAT
  n-COLORING

  THEORETICAL COMPUTER SCIENCE

- PP-DEFINITIONS (BABY) & PP-INTERPRETATIONS (ADOLESCENT)

  $\Rightarrow$ REDUCTIONS

  "MODEL THEORY"

- CHARACTERIZED BY:
  POLYMORPHISMS / EQUATIONS

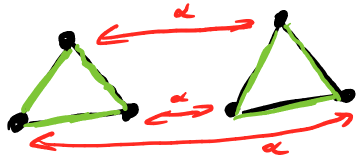
  UNIVERSAL ALGEBRA

PP-INTERPRETATIONS INSUFFICIENT

... TO EXPLAIN ALL NP-HARDNESS BY $CSP(K_3)$

EXAMPLE

$\mathbb{A} = K_3 \dot\cup K_3$



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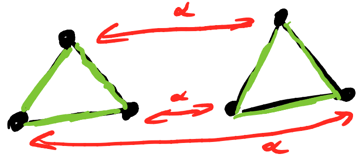

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DEFINITION

$\mathbb{A}, \mathbb{B}$ RELATIONAL STRUCTURES

HOMOMORPHICALLY EQUIVALENT $:\Leftrightarrow$

$\exists h: \mathbb{A} \to \mathbb{B}$ HOMOMORPHISM & VICE-VERSA

PP-INTERPRETATIONS INSUFFICIENT

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EXAMPLE

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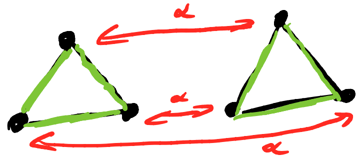

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PROOF

"$\Rightarrow$"

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ \mathbb{A} \arrow[rr, leftrightarrow] & & \mathbb{B} \end{tikzcd}$$

"$\Leftarrow$" $\quad \mathbb{A} \in CSP(\mathbb{A}) \Rightarrow \mathbb{A} \to \mathbb{B}$

PP-INTERPRETATIONS INSUFFICIENT

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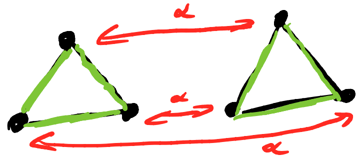
EXAMPLE

$\mathbb{A} = K_3 \,\dot\cup\, K_3$



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PROOF

"$\Rightarrow$"

$$\begin{tikzcd} & \mathbb{X} \arrow[dl] \arrow[dr] & \\ \mathbb{A} \arrow[rr, leftrightarrow] & & \mathbb{B} \end{tikzcd}$$

"$\Leftarrow$" $\mathbb{A} \in CSP(\mathbb{A}) \Rightarrow \mathbb{A} \to \mathbb{B}$

- H.E. ALLOWS A DIFFERENT FACTORING OF THE WORLD

PP-INTERPRETATIONS INSUFFICIENT

... TO EXPLAIN ALL NP-HARDNESS BY $CSP(K_3)$

EXAMPLE

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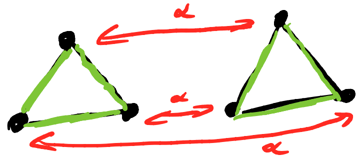

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$\mathbb{A}, \mathbb{B}$ H.E. $\Rightarrow CSP(\mathbb{A}) = CSP(\mathbb{B})$

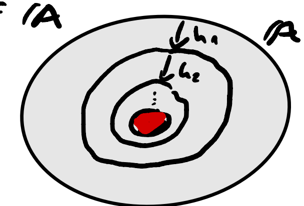
PROOF

$$\begin{tikzcd} & \mathbb{X} \arrow[dl] \arrow[dr] & \\ \mathbb{A} \arrow[rr, leftrightarrow] & & \mathbb{B} \end{tikzcd}$$

"$\to$"

"$\Leftarrow$" $\quad \mathbb{A} \in CSP(\mathbb{A}) \Rightarrow \mathbb{A} \to \mathbb{B}$

- H.E. ALLOWS A DIFFERENT FACTORING OF THE WORLD
- EACH EQUIVALENCE CLASS HAS UNIQUE SMALLEST REPRESENTATIVE: THE CORE OF $\mathbb{A}$



DEFINITION

$\mathbb{A}$ **PP-CONSTRUCTS** $\mathbb{B}$:$\iff$

$$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathbb{C}_2 \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_3 \rightsquigarrow \dots \rightsquigarrow \mathbb{B}$$

DEFINITION

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(ADULT) THEOREM

(BARTO + OPRŠAL + P. 16)

EQ^1 ... HEIGHT 1 - CONDITIONS

$f_i(\text{variables}) = f_j(\text{variables})$

$\cancel{s_i}$ $\cancel{t_j}$

$$EQ^1(Pol(\mathbb{A})) \subseteq EQ^1(Pol(\mathbb{B}))$$

DEFINITION

$\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$ $:\Leftrightarrow$

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PRESERVING H1-EQUATIONS

"MINION HOMOMORPHISM"

$\Leftrightarrow$ $\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$

DEFINITION

$\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$ $:\Longleftrightarrow$

$$\mathbb{A} \xrightarrow{\text{PP-INT}} \mathbb{C}_1 \xrightarrow{\text{H.E}} \mathbb{C}_2 \xrightarrow{\text{PP-INT}} \mathbb{C}_3 \rightsquigarrow \dots \rightsquigarrow \mathbb{B}$$

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$\Longleftrightarrow$ $\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$

$\Longrightarrow$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

DEFINITION

$\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$:$\Leftrightarrow$

$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathbb{C}_2 \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_3 \rightsquigarrow \cdots \rightsquigarrow \mathbb{B}$

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EXAMPLE

HEIGHT 1 :

- $m(x,x,y) = m(x,y,x) = m(y,x,x)$
- $c(x_1, \dots, x_n) = c(x_2, \dots, x_n, x_1)$

NOT HEIGHT 1 :

- $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$
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DEFINITION

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EXAMPLE

$\mathbb{A} = K_3 \,\dot\cup\, K_3$ PP-CONSTRUCTS K_3

PP-~~INTERPRETS~~

DEFINITION

$\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$ $:\Leftrightarrow$

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EXAMPLE

$\mathbb{A} = K_3 \dot\cup K_3$ PP-CONSTRUCTS K_3
PP-~~INTERPRETS~~

$f \in Pol(\mathbb{A}) \Rightarrow f|_{K_3}$ IS ESSENTIALLY UNARY

DEFINITION

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PP-~~INTERPRETS~~

$f \in Pol(\mathbb{A}) \Rightarrow f|_{K_3}$ IS ESSENTIALLY UNARY

$\Rightarrow$ $EQ^1(Pol(\mathbb{A}))$ TRIVIAL

COROLLARY

$EQ^{\wedge}(Pol(\mathbb{A}))$ TRIVIAL

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS <u>EVERY</u> FINITE STRUCTURE

$\Rightarrow$ $CSP(\mathbb{A})$ NP-COMPLETE

COROLLARY

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THEOREM $EQ^{\wedge}(Pol(\mathbb{A}))$ NON-TRIVIAL $\Rightarrow$

COROLLARY

$EQ^\wedge(\mathrm{Pol}(\mathbb{A}))$ TRIVIAL

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERY FINITE STRUCTURE

$\Rightarrow$ $CSP(\mathbb{A})$ NP-COMPLETE

THEOREM $EQ^\wedge(\mathrm{Pol}(\mathbb{A}))$ NON-TRIVIAL $\Rightarrow$

$\mathrm{Pol}(\mathbb{A}) \models \exists f \, \forall x,y,z \quad f(x,y,x,z,y,z) = f(y,x,z,x,z,y)$

6-ARY SIGGERS (SIGGERS '10)

COROLLARY

$EQ^\wedge(Pol(\mathbb{A}))$ TRIVIAL

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6-ARY SIGGERS (SIGGERS '10)

$\models \exists f \; \forall a,r,e \quad f(a,r,e,a) = f(r,a,r,e)$

4-ARY SIGGERS (KEARNES + MARKOVIĆ + MCKENZIE '14)

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4-ARY SIGGERS (KEARNES + MARKOVIĆ + MCKENZIE '14)

$\models \exists f \, \forall x_1 \dots x_n$

$f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$

$\forall n \geq |A|$ PRIME

CYCLIC (BARTO + KOZIK '11)

COROLLARY

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$\models \exists f\ \forall x,y\quad f(x \dots xy) = \dots = f(yx \dots x)$

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WNU (MARÓTI + MCKENZIE '08)

COROLLARY

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$\models \exists f \; \forall x_1 \dots x_n \quad f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$

$\forall n > |A|$ PRIME

CYCLIC (BARTO + KOZIK '11)

$\models \exists f \; \forall x,y \quad f(x \dots xy) = \dots = f(yx \dots x)$

$\forall n > |A|$ PRIME

WNU (MARÓTI + MCKENZIE '08)

$\Rightarrow$ $CSP(\mathbb{A}) \in P$ (BULATOV, ZHUK '17)

THEOREM $\mathrm{EQ}^1(\mathrm{Pol}(\mathbb{A}))$ NON-TRIVIAL

$\Rightarrow \mathrm{CSP}(\mathbb{A}) \in \mathrm{P}$ (BULATOV, ZHUK '17)

THEOREM $EQ^{\omega}(Pol(\mathbb{A}))$ NON-TRIVIAL

$\implies CSP(\mathbb{A}) \in P$ (BULATOV, ZHUK '17)

ALGORITHM

REDUCES TO SMALLER STRUCTURE
BY SOLVING LINEAR EQUATIONS
OR LOCAL CONSISTENCY CHECKING

PART II: INFINITE DOMAIN CSPs: MODELLING PROBLEMS MATHEMATICS

FINITE-DOMAIN CSPs: RATHER LIMITED

FINITE-DOMAIN CSPs : RATHER LIMITED

∈NP

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD

FINITE-DOMAIN CSPs: RATHER LIMITED

$\in NP$

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

$$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$$

SATISFYING ASSIGNMENT?

FINITE-DOMAIN CSPs: RATHER LIMITED

$\in$ NP:

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

  $$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$$

  SATISFYING ASSIGNMENT?

- n-COLORING PROBLEM

  GENERALIZATION: H-COLORING PROBLEM

  H GRAPH, GIVEN GRAPH G, DECIDE IF $G \to H$ (HELL + NEŠETŘIL '91)

  FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

FINITE-DOMAIN CSPs: RATHER LIMITED

$\in$ NP

- SOLVING **(LINEAR) EQUATIONS** / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: **RESTRICTIONS OF SAT**

  $$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$$

  SATISFYING ASSIGNMENT?

- **n-COLORING PROBLEM**

  GENERALIZATION: **H-COLORING PROBLEM**

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INFINITE-DOMAIN CSPs

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  FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

INFINITE-DOMAIN CSPs

- (LINEAR) EQUATIONS OVER $\mathbb{Q}$ / $\mathbb{Z}$

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FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

INFINITE-DOMAIN CSPs

- (LINEAR) EQUATIONS OVER $\mathbb{Q}$ / $\mathbb{Z}$
- $CSP(\mathbb{Z}, <)$ AND FO-DEFINABLE STRUCTURES E.G. $CSP(\mathbb{Z}, \{(x,y) \mid |x-y| \in \{1,3\}\})$

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- (LINEAR) EQUATIONS OVER $\mathbb{Q}$ / $\mathbb{Z}$
- CSP($\mathbb{Z}$, <) AND FO-DEFINABLE STRUCTURES E.G. CSP($\mathbb{Z}$, $\{(x,y) \mid |x-y| \in \{1,3\}\}$)
- CSP($\mathbb{Q}$, <) AND —"— E.G. CSP($\mathbb{Q}$, BETWEEN(x,y,z))

$\boxed{\text{EXERCISE}}$

COMPLEXITY?

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- "FORBIDDEN PATTERN PROBLEMS"

$\boxed{\text{EXERCISE}}$

COMPLEXITY?

GRAPH ORIENTATION PROBLEM

$\mathcal{F}$ ... FINITE SET OF TOURNAMENTS

GIVEN UNDIRECTED GRAPH G: CAN IT BE ORIENTED $\mathcal{F}$-FREE?

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EXAMPLE

- $\mathcal{F} = \{$  ,  $\}$ $\rightsquigarrow$ PROBLEM TRIVIAL
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EXERCISE: COMPLEXITY?

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$\exists \mathbb{A}_{\mathcal{F}}$ "NICE": $\mathrm{CSP}(\mathbb{A}_{\mathcal{F}}) = \{ G \mid G$ HAS $\mathcal{F}$-FREE ORIENTATION$\}$

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MMSNP: τ ... RELATIONAL SIGNATURE, σ ... UNARY PREDICATES

$\mathcal{F}$ ... FINITE SET OF $\tau \cup \sigma$-STRUCTURES

- GIVEN τ-STRUCTURE $\mathbb{X}$, CAN IT BE σ-COLORED SO THAT IT IS $\mathcal{F}$-FREE?

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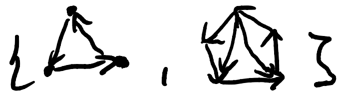
- $\mathcal{F} =$  $\rightsquigarrow$ PROBLEM TRIVIAL
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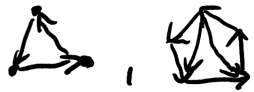
- $\mathcal{F} = \{$  $\}$ $\rightsquigarrow$ PROBLEM TRIVIAL

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EXAMPLE

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GMSNP: $\mathcal{F} = \{$  $\}$ —— " —— (COLORING EDGES)

GRAPH ORIENTATION, MMSNP, GMSNP:

WHY ALL THOSE FORBIDDEN PATTERN PROBLEMS?

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NP = ESO: EXISTENTIAL SECOND ORDER LOGIC (FAGIN '73)

τ ... SIGNATURE, K ... ISOMORPHISM-CLOSED CLASS OF FINITE τ-STRUCTURES

$\Rightarrow$ $K \in NP$ $\Leftrightarrow$ K DEFINABLE BY ESO-SENTENCE:

$$\exists X_1 \ldots X_n \, \Phi$$ FIRST-ORDER

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SNP = STRICT NP: $\exists X_1 \dots X_n \, \forall \bar{x} \, \overbrace{\Phi(\bar{x}, X_1 \dots X_n)}^{\text{QUANTIFIER-FREE}}$

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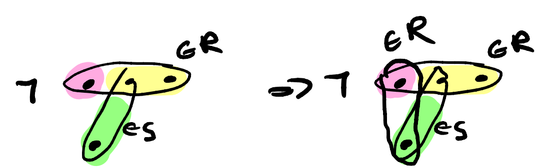
$$\bigwedge \neg (\dots \wedge \dots \wedge \dots)$$ FORBIDDEN PATTERNS

PATTERNS CONNECTED, MONOTONE $\Rightarrow$ K IS A CSP

= CLOSED UNDER HOMOMORPHISMS: τ-ATOMS ONLY APPEAR POSITIVELY



THE GRAPH ORIENTATION PROBLEM AS CSP

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$\mathcal{F}$... FINITE SET OF FINITE TOURNAMENTS

$\mathcal{K} = \{ \mathbb{D} \mid \mathbb{D}$ $\mathcal{F}$-FREE ORIENTED FINITE GRAPH $\}$

$\mathcal{K}$ CLOSED UNDER SUBSTRUCTURES
AMALGAMATION PROPERTY

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$\mathbb{A} = (A, \to) :=$ Fraïssé-Limit $(\mathcal{K})$

$\mathbb{G} := (A, E) \qquad E(x,y) :\Leftrightarrow x \to y \vee y \to x$

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$\mathbb{H} \in CSP(\mathbb{G}) \Leftrightarrow \mathbb{H} \overset{HOM}{\Longrightarrow} \mathbb{G}$

$\overset{Ex.}{\Longleftrightarrow} \mathbb{H} \overset{EMB}{\Longrightarrow} \mathbb{G}$

$\Leftrightarrow$ $\mathbb{H}$ HAS $\mathcal{F}$-FREE ORIENTATION

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$\boxed{\text{THEOREM}}$ (BODIRSKY + GUZMÁN-PRO '23)
(FELLER + P. '24)
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$\mathrm{CSP}(\mathbb{G}) \in \mathrm{P}$ UNLESS $\mathbb{G}$ PP-CONSTRUCTS $\mathbb{K}_3$

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$CSP(\mathbb{G}) \in P$ UNLESS $\mathbb{G}$ PP-CONSTRUCTS K_3

WHAT IF THE CLASS OF SOLUTIONS TO A PROBLEM DOES NOT HAVE AMALGAMATION?

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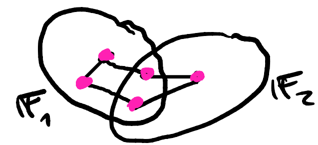
$\boxed{\text{EXAMPLE}}$

COLOURED GRAPHS WITHOUT PINK 5-CYCLE



(C_5)

OFTEN WE CAN EXTEND SIGNATURE TO OBTAIN AMALGAMATION

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THEOREM

(BODIRSKY + GUZMÁN-PRO '23)
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$\mathrm{CSP}(\mathbb{G}) \in \mathrm{P}$ UNLESS $\mathbb{G}$ PP-CONSTRUCTS K_3

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EXAMPLE

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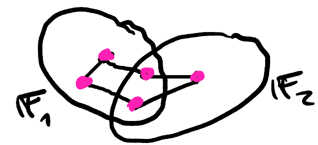

(C_5)

OFTEN WE CAN EXTEND SIGNATURE TO OBTAIN AMALGAMATION

THEOREM

(CHERLIN-SHELAH-SHI)

$\mathcal{F}$... FINITE SET OF STRUCTURES, CONNECTED, HOM-CLOSED

$\Rightarrow \exists \mathbb{A}$ REDUCT OF HOMOGENEOUS STRUCTURE GIVEN BY FINITE SET OF FORBIDDEN STRUCTURES

$\mathrm{CSP}(\mathbb{A}) = \{ \mathbb{X} \mid \exists \mathbb{X}^*$ EXPANSION AVOIDING $\mathcal{F} \}$

CONJECTURE [Bodirsky + P. '11]

$\mathbb{A}$ (FIRST-ORDER) REDUCT OF HOMOGENEOUS STRUCTURE $\mathbb{B}$ GIVEN BY FINITE SET OF FORBIDDEN PATTERNS

$\Rightarrow$ CSP($\mathbb{A}$) IN P UNLESS

$\mathbb{A}$ PP-CONSTRUCTS K_3

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EXAMPLE

- TRUE FOR GRAPH ORIENTATION w/ FORBIDDEN TOURNAMENTS

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- TRUE FOR MMSNP

$\mathcal{F} = \{$  $\}$ NO-MONOCHROMATIC TRIANGLE

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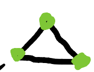
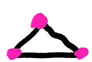
$\mathcal{F} = \{$  ,  $\}$ NO-MONOCHROMATIC TRIANGLE

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FIRST-ORDER REDUCTS OF $(\mathbb{Q},<)$

CONJECTURE [BODIRSKY + P. '11]

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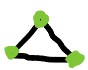
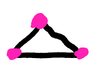
  $\mathcal{F} = \{$  ,  $\}$ NO-MONOCHROMATIC TRIANGLE

- OPEN FOR GMSNP

  $\mathcal{F} = \{$  ,  $\}$

FIRST-ORDER REDUCTS OF $(\mathbb{Q},<)$

- CSP$(\mathbb{Q},<)$

  GIVEN FINITE DIGRAPH $\mathbb{D}$, $\mathbb{D} \xrightarrow{?} (\mathbb{Q},<)$

CONJECTURE

[BODIRSKY + P. '11]

$\mathbb{A}$ (FIRST-ORDER) REDUCT OF HOMOGENEOUS STRUCTURE IS GIVEN BY FINITE SET OF FORBIDDEN PATTERNS

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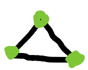
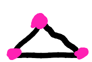
$\mathcal{F} = \{$  ,  $\}$ NO-MONOCHROMATIC TRIANGLE

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GIVEN FINITE DIGRAPH $\mathbb{D}$, $\mathbb{D} \overset{?}{\to} (\mathbb{Q},<)$

- CSP$(\mathbb{Q}, B(x,y,z))$

$B(x,y,z) :\Leftrightarrow (x<y \wedge y<z) \vee$

$(z<y \wedge y<x)$

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  $\mathcal{F} = \{$  $\}$ NO-MONOCHROMATIC TRIANGLE
- OPEN FOR GMSNP

  $\mathcal{F} = \{$  $\}$

FIRST-ORDER REDUCTS OF $(\mathbb{Q}, <)$

- CSP$(\mathbb{Q}, <)$

  GIVEN FINITE DIGRAPH $\mathbb{D}$, $\mathbb{D} \overset{?}{\rightarrow} (\mathbb{Q}, <)$

- CSP$(\mathbb{Q}, B(x,y,z))$

  $$B(x,y,z) :\Leftrightarrow (x<y \wedge y<z) \vee (z<y \wedge y<x)$$

- CSP$(\mathbb{Q}, R)$

  $$R(x,y,z) \Leftrightarrow (x=y<z) \vee (x=z<y) \vee (y=z<x)$$

  EXERCISE: COMPLEXITY?

CONJECTURE [BODIRSKY + P. '11]

$\mathbb{A}$ (FIRST-ORDER) REDUCT OF HOMOGENEOUS STRUCTURE IS GIVEN BY FINITE SET OF FORBIDDEN PATTERNS

$\Rightarrow$ CSP($\mathbb{A}$) IN P UNLESS

$\mathbb{A}$ PP-CONSTRUCTS K_3

EXAMPLE

- TRUE FOR GRAPH ORIENTATION W/ FORBIDDEN TOURNAMENTS
- TRUE FOR MMSNP

$\mathcal{F} = \{$  $\}$ NO-MONOCHROMATIC TRIANGLE

- OPEN FOR GMSNP

$\mathcal{F} = \{$  $\}$

FIRST-ORDER REDUCTS OF $(\mathbb{Q},<)$

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EXERCISE: COMPLEXITY?

CSPs OF FO-REDUCTS OF $(\mathbb{Q},<)$

"LINEAR-ORDER-SAT", "TEMPORAL CSPs"

THEOREM

- LINEAR-ORDER-SAT ($\mathbb{A}$) IN P UNLESS $\mathbb{A}$ PP-CONSTRUCTS K_3 (BODIRSKY + KÁRA '07)
- GRAPH-SAT (BODIRSKY + P. '11)
- POSET-SAT (VAN PHAM + KOMPATSCHER '16)
- TOURNAMENT-SAT (MOTTET + P. '21)
- HYPERGRAPH-SAT (MOTTET + NAGY + P. '23)

$\vdots$

THEOREM

- LINEAR-ORDER-SAT($\mathbb{A}$) IN P UNLESS $\mathbb{A}$ PP-CONSTRUCTS K_3 (BODIRSKY + KÁRA '07)
- GRAPH-SAT (BODIRSKY + P. '11)
- POSET-SAT (VAN PHAM + KOMPATSCHER '16)
- TOURNAMENT-SAT (MOTTET + P. '21)
- HYPERGRAPH-SAT (MOTTET + NAGY + P. 23)

$\vdots$

METHOD: POLYMORPHISMS!

IN A NUTSHELL: FOR ω-CAT. $\mathcal{A}$:

BASIC THEORY ... AS FOR FINITE STRUCTURES

ADVANCED THEORY ... MUCH HARDER / FALSE

IN A NUTSHELL: FOR ω-CAT. $\mathbb{A}$:

BASIC THEORY ... AS FOR FINITE STRUCTURES

ADVANCED THEORY ... MUCH HARDER / FALSE

(BABY) THEOREM (BODIRSKY + NEŠETŘIL '03)

$\mathbb{A}$ ω-CATEGORICAL, $R \subseteq A^m$ RELATION

$\Rightarrow$ R PP-DEFINABLE IN $\mathbb{A}$ $\Leftrightarrow$

R INVARIANT UNDER $\mathrm{Pol}(\mathbb{A})$

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PROOF

"$\rightarrow$" TRIVIAL

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PROOF

"$\Rightarrow$" TRIVIAL

"$\Leftarrow$"

R IS A UNION OF ORBITS OF

$\mathrm{Aut}\,\mathbb{A} \curvearrowright A^m$

LET $\bar{r}_1, \dots, \bar{r}_l \in R$ BE REPRESENTATIVES OF THESE ORBITS

LET $(\bar{a}_0, \bar{a}_1, \dots)$ ENUMERATION OF A^l

IN A NUTSHELL: FOR ω-CAT. $\mathbb{A}$:

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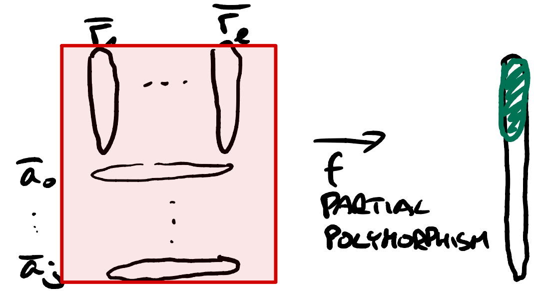
LET $(\bar{a}_0, \bar{a}_1, \dots)$ ENUMERATION OF A^ℓ

PP-DEFINE R_j THIS WAY:



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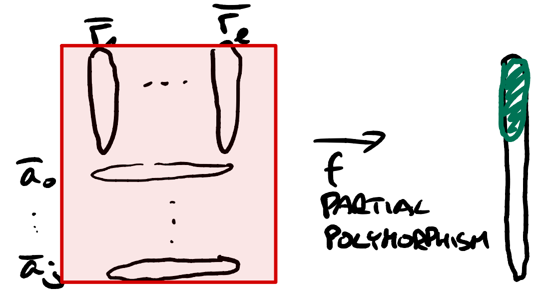

- $R_j \supseteq R$ (PROJECTIONS)

IN A NUTSHELL: FOR ω-CAT. $\mathbb{A}$:

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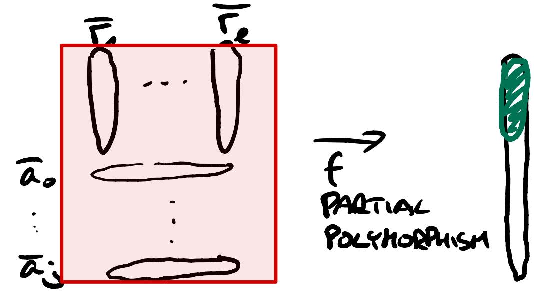

- $R_j \supseteq R$ (PROJECTIONS)
- $R_j \supseteq R_{j+1}$, EVENTUALLY CONSTANT

  (R_j UNION OF ORBITS)

$\Rightarrow$ $\bigcap_j R_j =: R_\infty \supseteq R$ PP-DEFINABLE

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PROOF

"$\Rightarrow$" TRIVIAL

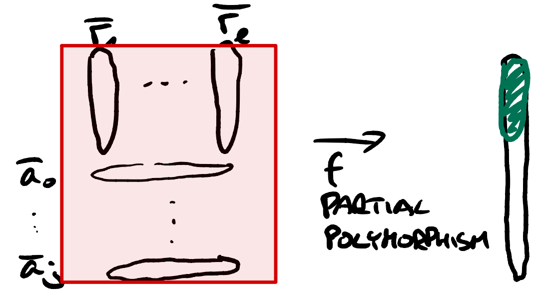
"$\Leftarrow$"

R IS A UNION OF ORBITS OF $\mathrm{Aut}\,\mathbb{A} \curvearrowright A^m$

LET $\bar t_1, \dots, \bar t_\ell \in R$ BE REPRESENTATIVES OF THESE ORBITS

LET $(\bar a_0, \bar a_1, \dots)$ ENUMERATION OF A^ℓ

PP-DEFINE R_j THIS WAY:



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- $R_j \supseteq R_{j+1}$, EVENTUALLY CONSTANT

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  $\Rightarrow$ $\bigcap_j R_j =: R_\infty$ PP-DEFINABLE, $R_\infty \supseteq R$
- $\bar t \in R_\infty$

  $\Rightarrow \exists f_j$ PART. POLYMORPHISMS WITNESSES

IN A NUTSHELL: FOR ω-CAT. $\mathbb{A}$:

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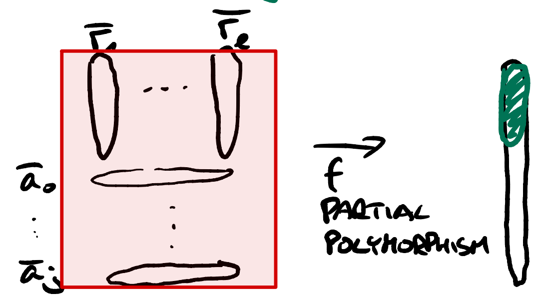

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  $\therefore \exists$ SUBSEQUENCE SUCH THAT f_j EXTENDS f_i, $i<j$ MOD. ORBITS

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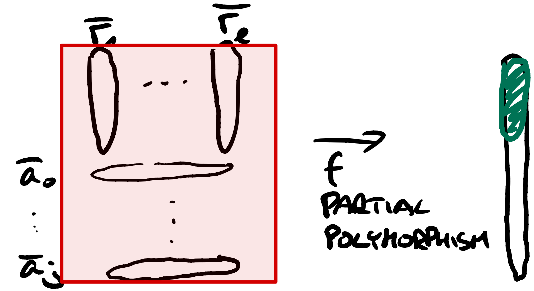

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  $\therefore$ $\exists$ SUBSEQUENCE SUCH THAT f_j EXTENDS f_i, $i<j$ MOD. ORBIS

  $\therefore$ COMPOSING WITH AUTOMORPHISMS, f_j EXTENDS f_i $\quad f := \bigcup f_i$

  $\Rightarrow$ $\bar{r} = f(\bar{r}_1, \dots \bar{r}_j) \in R$

ADOLESCENT / ADULT THEOREMS:

CHARACTERISE

- PP-INTERPRETATIONS BY EQUATIONS / CLONE HOMOMORPHISMS
- PP-CONSTRUCTIONS BY H1 EQUATIONS / MINION HOMOMORPHISMS

?

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TOPOLOGY

ADOLESCENT / ADULT THEOREMS:

CHARACTERISE

- PP-INTERPRETATIONS BY EQUATIONS / CLONE HOMOMORPHISMS
- PP-CONSTRUCTIONS BY H1 EQUATIONS / MINION HOMOMORPHISMS

?

TOPOLOGY

$A^{A^n} = \{f : A^n \to A\}$ CARRIES

TOPOLOGY OF POINTWISE CONVERGENCE:

$(f_i)_{i \in \omega} \to f \iff$

$\forall F \subseteq A^n$ FINITE

$f_i|_F = f|_F$ EVENTUALLY

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COMPLETE METRIC SPACE / **POLISH SPACE**

$\bigcup_n A^{A^n}$ ... SUM SPACE

(EACH A^{A^n} CLOPEN)

ADOLESCENT / ADULT THEOREMS:

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COMPLETE METRIC SPACE / **POLISH SPACE**

$\bigcup_n A^{A^n}$ ... SUM SPACE (EACH A^{A^n} CLOPEN)

Pol(A) CLOSED SUBSPACE:

$f \notin Pol(A) \Rightarrow \exists F \subseteq A$ FINITE:

$O_{f|_F} = \{g \mid g|_F = f|_F\} \cap Pol(A) = \emptyset$

THEOREM (ADOLESCENT + ADULT)

BODIRSKY P. 45

BARTO + OPRŠAL + P. 17

$\mathbb{A}$ ω-CATEGORICAL, $\mathbb{B}$ FINITE $\Rightarrow$

THEOREM (ADOLESCENT + ADULT)

BODIRSKY-P. 45

BARTO + OPRŠAL + P. 17

$\mathbb{A}$ ω-CATEGORICAL, $\mathbb{B}$ FINITE $\Rightarrow$

① $\mathrm{Pol}(\mathbb{A}) \xrightarrow[\text{CLONE HOMOMORPHISM}]{\text{u.c}} \mathrm{Pol}(\mathbb{B})$ $\Leftrightarrow$ $\mathbb{A}$ PP-INTERPRETS $\mathbb{B}$

$\Leftrightarrow$ $\exists F \subseteq A$ FINITE :

$\mathrm{EQ}(\mathrm{Pol}(\mathbb{A})|_F) \subseteq \mathrm{EQ}(\mathrm{Pol}(\mathbb{B}))$

$\rightarrow$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

THEOREM (ADOLESCENT + ADULT)

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CLONE HOMOMORPHISM

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② $\mathrm{Pol}(\mathbb{A}) \xrightarrow{u.c.} \mathrm{Pol}(\mathbb{B})$ $\Leftrightarrow$ $\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$

MINION HOMOMORPHISM

$\Leftrightarrow$ $\exists F \subseteq A$ FINITE:

$$EQ^1(\mathrm{Pol}(\mathbb{A})|_F) \subseteq EQ^1(\mathrm{Pol}(\mathbb{B}))$$

$\rightarrow$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

THEOREM (ADOLESCENT + ADULT)

BODIRSKY + P. 45

BARTO + OPRŠAL + P. 17

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---

② $\mathrm{Pol}(\mathbb{A}) \xrightarrow[\text{MINION HOMOMORPHISM}]{\text{u.c.}} \mathrm{Pol}(\mathbb{B}) \Leftrightarrow \mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$

$\Leftrightarrow$ $\exists F \subseteq A$ FINITE:

$EQ^1(\mathrm{Pol}(\mathbb{A})|_F) \subseteq EQ^1(\mathrm{Pol}(\mathbb{B}))$

$\rightarrow$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

WHAT IF $\mathrm{Pol}(\mathbb{A}) \cancel{\xrightarrow[\text{MINION}]{\text{u.c}}} \mathrm{Pol}(K_3)$?

BODIRSKY + P. 45
BARTO + OPRŠAL + P. 17

THEOREM (ADOLESCENT + ADULT)

$\mathbb{A}$ ω-CATEGORICAL, $\mathbb{B}$ FINITE $\Rightarrow$

1. $\mathrm{Pol}(\mathbb{A}) \xrightarrow{\text{u.c}} \mathrm{Pol}(\mathbb{B})$ $\Leftrightarrow$ $\mathbb{A}$ PP-INTERPRETS $\mathbb{B}$

   CLONE HOMOMORPHISM

   $\Leftrightarrow$ $\exists F \subseteq A$ FINITE :

   $$EQ(\mathrm{Pol}(\mathbb{A})|_F) \subseteq EQ(\mathrm{Pol}(\mathbb{B}))$$

   $\to$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

---

2. $\mathrm{Pol}(\mathbb{A}) \xrightarrow{\text{u.c.}} \mathrm{Pol}(\mathbb{B})$ $\Leftrightarrow$ $\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$

   MINION HOMOMORPHISM

   $\Leftrightarrow$ $\exists F \subseteq A$ FINITE:

   $$EQ^1(\mathrm{Pol}(\mathbb{A})|_F) \subseteq EQ^1(\mathrm{Pol}(\mathbb{B}))$$

   $\to$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

WHAT IF $\mathrm{Pol}(\mathbb{A}) \cancel{\xrightarrow[\text{MINION}]{\text{u.c}}} \mathrm{Pol}(K_3)$?

RECALL $\mathbb{A}$ FINITE $\Rightarrow$

$\mathrm{Pol}(\mathbb{A})$ HAS

- CYCLIC OPERATION $c(x_1 \dots x_n) = c(x_2 \dots x_n x_1)$
- WNU OPERATION $w(x \dots x y) = \dots = w(y x \dots x)$
- 4-ARY SIGGERS OPERATION $s(a,r,e,a) = s(r,a,r,e)$
- 6-ARY SIGGERS OPERATION $s(xyxzyz) = s(yxzxzy)$

...

FALSE FOR ω-CATEGORICAL $\mathcal{A}$!

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

$\boxed{\text{EXAMPLE}}$

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL

CSP(A) $\in$ P

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

EXAMPLE

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL

CSP$(\mathbb{A}) \in$ P

- DOES Pol$(\mathbb{A})$ HAVE CYCLIC $c(x_1, \ldots, x_n)$?

<u>FALSE</u> FOR ω-CATEGORICAL $\mathbb{A}$!

EXAMPLE

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL

$CSP(\mathbb{A}) \in P$

- DOES $Pol(\mathbb{A})$ HAVE CYCLIC $c(x_1 \dots x_n)$?

  PICK $a_1, \dots, a_n$ DISTINCT

$$
\begin{array}{lllllll}
c(& a_1 & \dots & a_n &) & = & a \\
 & | & | \quad | & | & & & \| \\
c(& a_2 & \dots & a_n & a_1) & = & a
\end{array}
\qquad \lightning
$$

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

EXAMPLE

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL
$\mathrm{CSP}(\mathbb{A}) \in P$

- DOES $\mathrm{Pol}(\mathbb{A})$ HAVE CYCLIC $c(x_1 \dots x_n)$?

  PICK $a_1, \dots, a_n$ DISTINCT

$$
\begin{array}{cccccccc}
c(& a_1 & a_2 & \cdots & a_{n-1} & a_n &) = & a \\
 & \neq & \neq & \neq & \neq & \neq & & \| \\
c(& a_2 & \cdots & & a_n & a_1 &) = & a
\end{array}
\qquad \lightning
$$

- $s(x, y, x, z, y, z) = s(y, x, z, x, z, y)$ $\lightning$

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

$\boxed{\text{EXAMPLE}}$

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ $\quad$ ω-CATEGORICAL

$\quad$ $CSP(\mathbb{A}) \in P$

- DOES $Pol(\mathbb{A})$ HAVE CYCLIC $c(x_1 \dots x_n)$?

  PICK $a_1, \dots, a_n$ DISTINCT

$$c(a_1 \dots a_n) = a$$
$$\|$$
$$c(a_2 \dots a_n a_1) = a$$

↯

- $s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$ ↯

- ANY SINGLE NON-TRIVIAL H1-EQUATION ↯

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

EXAMPLE

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL

CSP($\mathbb{A}$) $\in$ P

- DOES Pol($\mathbb{A}$) HAVE CYCLIC $c(x_1 \dots x_n)$?

  PICK $a_1, \dots, a_n$ DISTINCT

$$\begin{array}{l} c(a_1 \dots a_n) = a \\ \neq \; \neq \; \neq \; \neq \quad \neq \quad \| \\ c(a_2 \dots a_{n-1} a_1) = a \end{array} \quad ⚡$$

- $s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$ ⚡

- ANY SINGLE NON-TRIVIAL H1-EQUATION ⚡

EXERCISE

WHY DOES K_ω NOT PP-CONSTRUCT K_3?

PSEUDO-H1-EQUATIONS

EQUIVALENCE RELATION ON A^{A^n} :

$$f \sim g \Leftrightarrow g \in \overline{\{\alpha f \mid \alpha \in \mathrm{Aut}(A)\}}$$

$$\Leftrightarrow \forall F \subseteq A \text{ FINITE}$$

$$\exists \alpha \in \mathrm{Aut}(A)$$

$$g|_F = \alpha f|_F$$

"f, g ARE EQUAL MODULO ORBITS"

PSEUDO-H1-EQUATIONS

EQUIVALENCE RELATION ON A^{A^n}:

$$f \sim g \iff g \in \overline{\{\alpha f \mid \alpha \in \mathrm{Aut}(\mathbb{A})\}}$$

$$\iff \forall F \subseteq A \text{ FINITE } \exists \alpha \in \mathrm{Aut}\,\mathbb{A} \quad g|_F = \alpha f|_F$$

"f, g ARE EQUAL MODULO ORBITS"

EXERCISE $A^{A^n}/\sim \;=: A^{A^n}/\mathrm{Aut}\,\mathbb{A}$ COMPACT (POLISH)

PSEUDO-M1-EQUATIONS

EQUIVALENCE RELATION ON A^{A^n}:

$$f \sim g \iff g \in \overline{\{\alpha f \mid \alpha \in \mathrm{Aut}(A)\}}$$

$$\iff \forall F \subseteq A \text{ FINITE}$$

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$$g|_F = \alpha f|_F$$

"f, g ARE EQUAL MODULO ORBITS"

$\boxed{\text{EXERCISE}}$ $A^{A^n}/\sim \;=:\; A^{A^n}/\mathrm{Aut}\,A$

COMPACT (POLISH)

IDEA: TRY TO MIMIC PROOFS FOR FINITE STRUCTURES IN CASE OF

$\mathrm{Pol}(A)/\mathrm{Aut}\,A$

PSEUDO-H1-EQUATIONS

EQUIVALENCE RELATION ON A^{A^n}:

$$f \sim g \iff g \in \overline{\{\alpha f \mid \alpha \in \mathrm{Aut}(\mathbb{A})\}}$$

$$\iff \forall F \subseteq A \text{ FINITE } \exists \alpha \in \mathrm{Aut}\,\mathbb{A} \quad g|_F = \alpha f|_F$$

"f, g ARE EQUAL MODULO ORBITS"

EXERCISE $A^{A^n}/\sim \;=:\; A^{A^n}/\mathrm{Aut}\,\mathbb{A}$ COMPACT (POLISH)

IDEA: TRY TO MIMIC PROOFS FOR FINITE STRUCTURES IN CASE OF $\mathrm{Pol}(\mathbb{A})/\mathrm{Aut}\,\mathbb{A}$

THEOREM (BARTO+P. '17)

$\mathbb{A}$ ω-CATEGORICAL, $\mathrm{Pol}(\mathbb{A})$ NON-TRIVIAL:

$$\mathrm{Pol}(\mathbb{A}) \;\cancel{\xrightarrow[\text{MINION HOMOMORPHISM}]{\text{u.c.}}}\; \mathrm{Pol}(K_3)$$

$\Rightarrow \mathrm{Pol}(\mathbb{A}) \models$

$$\exists u, v, s : \forall x, y, z$$

$$u \circ s(x,y,x,z,y,z) = v \circ s(y,x,z,x,z,y).$$

PSEUDO-H1-EQUATIONS

EQUIVALENCE RELATION ON A^{A^n}:

$$f \sim g \Leftrightarrow g \in \overline{\{\alpha f \mid \alpha \in \mathrm{Aut}(\mathbb{A})\}}$$

$\Leftrightarrow$ $\forall F \subseteq A$ FINITE

$\exists \alpha \in \mathrm{Aut}\,\mathbb{A}$

$$g|_F = \alpha f|_F$$

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EXAMPLE

$\mathbb{A} = (\mathbb{Q}, <, x=y \rightarrow u=v)$: ω-CATEGORICAL

$\mathrm{CSP}(\mathbb{A})$ IN P

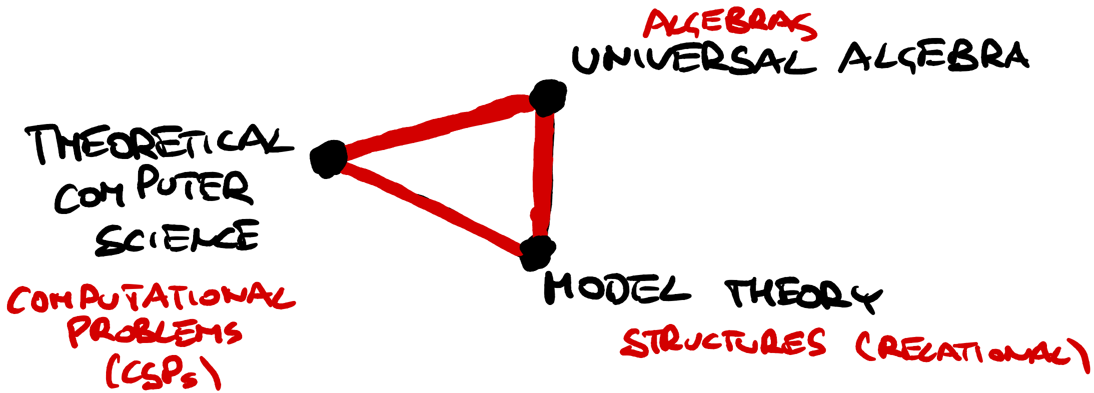
NO PSEUDO-CYCLIC POLYMORPHISM

EXERCISE: SHOW IT!

SUMMARY

- PP-CONSTRUCTIONS COMBINE PP-INTERPRETATIONS + HOMOMORPHIC EQUIVALENCE
- CHARACTERIZED BY H1-EQUATIONS OF POLYMORPHISMS
- WORKS ALSO FOR ω-CATEGORICAL STRUCTURES (W/ TOPOLOGY)
- $EQ^1(Pol(\mathbb{A}))$ TRIVIAL $\Rightarrow$ NP-HARD $CSP(\mathbb{A})$
- NON-TRIVIAL $\Rightarrow$ MANY CONCRETE H1-EQUATIONS FOR FINITE $\mathbb{A}$

  SOME (BUT NOT ALL) PSEUDO-EQUATIONS FOR ω-CATEGORICAL $\mathbb{A}$
- FORBIDDEN PATTERN PROBLEMS $\rightsquigarrow$ NATURAL ω-CATEGORICAL CSPs IN NP
- DICHOTOMY CONJECTURE FOR SUCH PROBLEMS



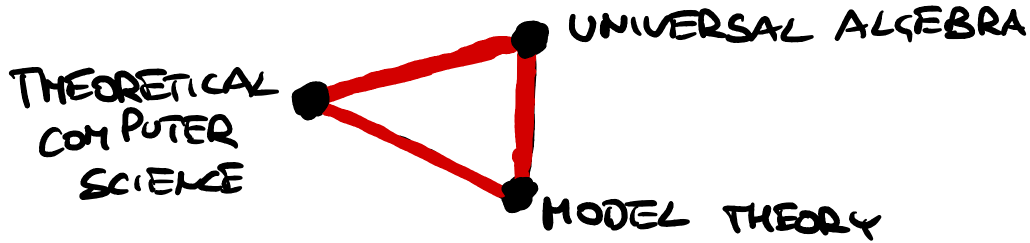
PART I THE MATHEMATICS OF FINITE-DOMAIN CSPs

ALGEBRAS

RELATIONAL STRUCTURES

PART II INFINITE-DOMAIN CSPs MODELLING PROBLEMS

MATHEMATICS


THEORETICAL COMPUTER SCIENCE
UNIVERSAL ALGEBRA
MODEL THEORY

Thank you!