Mixed identities in oligomorphic groups

Michael Pinsker

TU Wien

MIXED IDENTITIES IN OLIGOMORPHIC GROUPS

MICHAEL PINSKER
TU WIEN

AAA 109

LJUBLJANA

FWF Austrian Science Fund
I 5948

European Research Council
Established by the European Commission
SyG POCOCOP
GA 101071674

MIXED IDENTITIES

$\mathcal{G} = (G, \cdot, {}^{-1}, 1)$ ... GROUP

MIXED IDENTITIES

$\mathcal{G} = (G, \cdot, ^{-1}, 1)$ ... GROUP

TERM OVER $\mathcal{G}$:

EXPRESSION BUILT FROM :

$\cdot$, $^{-1}$, 1, VARIABLES

EXAMPLE $t(x, y) = x y x^{-1} y^{-1}$

MIXED IDENTITIES

$\mathcal{G} = (G, \cdot, ^{-1}, 1)$ ... GROUP

TERM OVER $\mathcal{G}$:

EXPRESSION BUILT FROM:

$\cdot$, $^{-1}$, 1, VARIABLES

EXAMPLE $t(x,y) = xyx^{-1}y^{-1}$

IDENTITY: EXPRESSION $t(x_1 \ldots x_n) = 1$

MIXED IDENTITIES

$\mathcal{G} = (G, \cdot, ^{-1}, 1)$ ... GROUP

TERM OVER $\mathcal{G}$:

EXPRESSION BUILT FROM:

$\cdot$, $^{-1}$, 1, VARIABLES

EXAMPLE $t(x,y) = xyx^{-1}y^{-1}$

IDENTITY: EXPRESSION $t(x_1 .. x_n) = 1$

$t(x_1 \dots x_n) = 1$ **IDENTITY OF $\mathcal{G}$** :$\Leftrightarrow$

$\forall f : \{x_1 .. x_n\} \to G : t(fx_1 .. fx_n) = 1$

MIXED IDENTITIES

$\mathcal{G} = (G, \cdot, ^{-1}, 1)$ ... GROUP

TERM OVER $\mathcal{G}$:

EXPRESSION BUILT FROM:

$\cdot$, $^{-1}$, 1, VARIABLES

EXAMPLE $t(x, y) = x y x^{-1} y^{-1}$

IDENTITY: EXPRESSION $t(x_1 \dots x_n) = 1$

$t(x_1 \dots x_n) = 1$ IDENTITY OF $\mathcal{G}$ $:\Leftrightarrow$

$\forall f : \{x_1 \dots x_n\} \to G : t(f x_1 \dots f x_n) = 1$

POLYNOMIAL OVER $\mathcal{G}$:

EXPRESSION BUILT FROM:

$\cdot$, $^{-1}$, 1, VARIABLES, **CONSTANTS**

MIXED IDENTITIES

$\mathcal{G} = (G, \cdot, ^{-1}, 1)$ ... GROUP

TERM OVER $\mathcal{G}$:

EXPRESSION BUILT FROM:

$\cdot$, $^{-1}$, 1, VARIABLES

EXAMPLE $t(x, y) = x y x^{-1} y^{-1}$

IDENTITY: EXPRESSION $t(x_1 .. x_n) = 1$

$t(x_1 \ldots x_n) = 1$ **IDENTITY OF $\mathcal{G}$** $:\Leftrightarrow$

$\forall f : \{x_1 .. x_n\} \to G : t(f x_1 .. f x_n) = 1$

POLYNOMIAL OVER $\mathcal{G}$:

EXPRESSION BUILT FROM:

$\cdot$, $^{-1}$, 1, VARIABLES, **CONSTANTS**

EXAMPLE $w(x; g) = x g x^{-1} g^{-1}$

x ... VARIABLE

$g \in G$

MIXED IDENTITIES

$\mathcal{G} = (G, \cdot, {}^{-1}, 1)$ ... GROUP

TERM OVER $\mathcal{G}$:

EXPRESSION BUILT FROM:

$\cdot, {}^{-1}, 1,$ VARIABLES

EXAMPLE $t(x, y) = x y x^{-1} y^{-1}$

IDENTITY: EXPRESSION $t(x_1 \dots x_n) = 1$

$t(x_1 \dots x_n) = 1$ **IDENTITY OF $\mathcal{G}$** $:\Leftrightarrow$

$\forall f : \{x_1 \dots x_n\} \to G : t(f x_1 \dots f x_n) = 1$

POLYNOMIAL OVER $\mathcal{G}$:

EXPRESSION BUILT FROM:

$\cdot, {}^{-1}, 1,$ VARIABLES, **CONSTANTS**

EXAMPLE $w(x; g) = x g x^{-1} g^{-1}$

x ... VARIABLE

$g \in G$

MIXED / GENERALIZED IDENTITY

$w(x_1, \dots, x_n, g_1, \dots, g_n) = 1$

MIXED IDENTITIES

$\mathcal{G} = (G, \cdot, {}^{-1}, 1)$ ... GROUP

TERM OVER $\mathcal{G}$:

EXPRESSION BUILT FROM:

$\cdot$, ${}^{-1}$, 1, VARIABLES

EXAMPLE $t(x,y) = xyx^{-1}y^{-1}$

IDENTITY: EXPRESSION $t(x_1 \dots x_n) = 1$

$t(x_1 \dots x_n) = 1$ **IDENTITY OF $\mathcal{G}$** :$\Leftrightarrow$

$\forall f : \{x_1 \dots x_n\} \to G : t(fx_1 \dots fx_n) = 1$

POLYNOMIAL OVER $\mathcal{G}$:

EXPRESSION BUILT FROM:

$\cdot$, ${}^{-1}$, 1, VARIABLES, **CONSTANTS**

EXAMPLE $w(x; g) = xgx^{-1}g^{-1}$

x ... VARIABLE

$g \in G$

MIXED/GENERALIZED IDENTITY

$w(x_1, \dots, x_n, g_1, \dots, g_n) = 1$

IDENTITY OF $\mathcal{G}$:$\Leftrightarrow$ $\forall f \dots$

SOLVABLE IN $\mathcal{G}$:$\Leftrightarrow$ $\exists f \dots$

MIXED IDENTITIES

$\mathcal{G} = (G, \cdot, ^{-1}, 1)$ ... GROUP

TERM OVER $\mathcal{G}$:

EXPRESSION BUILT FROM:

$\cdot$, $^{-1}$, 1, VARIABLES

EXAMPLE $t(x,y) = xyx^{-1}y^{-1}$

IDENTITY: EXPRESSION $t(x_1 .. x_n) = 1$

$t(x_1 \dots x_n) = 1$ IDENTITY OF $\mathcal{G}$:$\Leftrightarrow$

$\forall f : \{x_1 .. x_n\} \to G : t(fx_1 .. fx_n) = 1$

POLYNOMIAL OVER $\mathcal{G}$:

EXPRESSION BUILT FROM:

$\cdot$, $^{-1}$, 1, VARIABLES, **CONSTANTS**

EXAMPLE

$w(x; g) = xgx^{-1}g^{-1}$

x ... VARIABLE

$g \in G$

MIXED / GENERALIZED IDENTITY

$w(x_1, \dots, x_n, g_1, \dots, g_n) = 1$

IDENTITY OF $\mathcal{G}$: $\Leftrightarrow$ $\forall f$ ...

SOLVABLE IN $\mathcal{G}$: $\Leftrightarrow$ $\exists f$ ...

MIXED IDENTITY

$w(x_1 \dots x_n, g_1 \dots g_n) = 1$ HOLDS,

$u \in G$, $u \neq 1$

$\Rightarrow$ $uw = 1$ NOT SOLVABLE

WHAT (MIXED) IDENTITIES HOLD IN $\mathcal{G}$?

WHAT (MIXED) IDENTITIES HOLD IN G?

TERM $t(x_1, \dots, x_n)$ TRIVIAL $:\Leftrightarrow$

$t = 1$ IN THE FREE GROUP GENERATED BY $x_1, \dots, x_n$

WHAT (MIXED) IDENTITIES HOLD IN G?

TERM $t(x_1 \ldots x_n)$ TRIVIAL $:\Leftrightarrow$

$t = 1$ IN THE FREE GROUP GENERATED BY $x_1 \ldots x_n$

POLYNOMIAL $w(x_1 \ldots x_n, g_1 \ldots g_m)$ REGULAR

$:\Leftrightarrow$ $w(x_1 \ldots x_n, 1, \ldots, 1)$ NON-TRIVIAL

WHAT (MIXED) IDENTITIES HOLD IN G?

TERM $t(x_1 \dots x_n)$ **TRIVIAL** $:\Leftrightarrow$

$t = 1$ IN THE FREE GROUP GENERATED BY $x_1 \dots x_n$

POLYNOMIAL $w(x_1 \dots x_n, g_1 \dots g_m)$ **REGULAR**

$:\Leftrightarrow$ $w(x_1 \dots x_n, 1, \dots, 1)$ NON-TRIVIAL

REGULAR MIXED IDENTITIES?

WHAT (MIXED) IDENTITIES HOLD IN G?

TERM $t(x_1 \dots x_n)$ TRIVIAL $:\Leftrightarrow$

$t = 1$ IN THE FREE GROUP GENERATED BY $x_1 \dots x_n$

POLYNOMIAL $w(x_1 \dots x_n, g_1 \dots g_m)$ REGULAR

$:\Leftrightarrow$ $w(x_1 \dots x_n, 1, \dots, 1)$ NON-TRIVIAL

REGULAR MIXED IDENTITIES?

G FINITE $\Rightarrow$ $w^{|G|} = 1$ HOLDS

WHAT (MIXED) IDENTITIES HOLD IN G?

TERM $t(x_1 \dots x_n)$ **TRIVIAL** $:\Leftrightarrow$

$t = 1$ IN THE FREE GROUP GENERATED BY $x_1 \dots x_n$

POLYNOMIAL $w(x_1 \dots x_n, g_1 \dots g_m)$ **REGULAR**

$:\Leftrightarrow$ $w(x_1 \dots x_n, 1, \dots, 1)$ NON-TRIVIAL

REGULAR MIXED IDENTITIES?

G FINITE $\Rightarrow$ $w^{|G|} = 1$ HOLDS

CONJECTURE (KLYACHKO, THOM '16)

w REGULAR $\Rightarrow$ $\exists H \geq G$

$w = 1$ SOLVABLE IN H

G FINITE $\Rightarrow$ H FINITE

WHAT (MIXED) IDENTITIES HOLD IN G?

TERM $t(x_1 \dots x_n)$ TRIVIAL $:\Leftrightarrow$

$t = 1$ IN THE FREE GROUP GENERATED BY $x_1 \dots x_n$

POLYNOMIAL $w(x_1 \dots x_n, g_1 \dots g_m)$ REGULAR

$:\Leftrightarrow$ $w(x_1 \dots x_n, 1, \dots, 1)$ NON-TRIVIAL

REGULAR MIXED IDENTITIES?

G FINITE $\Rightarrow$ $w^{|G|} = 1$ HOLDS

CONJECTURE (KLYATCHKO, THOM '16)

w REGULAR $\Rightarrow$ $\exists H \geq G$

$w = 1$ SOLVABLE IN H

G FINITE $\Rightarrow$ H FINITE

ONE VARIABLE:

KERVAIRE–LAUDENBACH CONJECTURE

REGULARITY:

$G = \langle a, b \mid a^2, b^3 \rangle$

$x a x^{-1} b^{-1} = 1$

WHAT (MIXED) IDENTITIES HOLD IN G?

TERM $t(x_1 \dots x_n)$ TRIVIAL $:\Leftrightarrow$

$t = 1$ IN THE FREE GROUP GENERATED BY $x_1 \dots x_n$

POLYNOMIAL $w(x_1 \dots x_n, g_1 \dots g_m)$ REGULAR $:\Leftrightarrow$ $w(x_1, \dots, x_n, 1, \dots, 1)$ NON-TRIVIAL

REGULAR MIXED IDENTITIES?

G FINITE $\Rightarrow$ $w^{|G|} = 1$ HOLDS

CONJECTURE (KLYACHKO, THOM '16)

w REGULAR $\Rightarrow$ $\exists H \geq G$

$w = 1$ SOLVABLE IN H

G FINITE $\Rightarrow$ H FINITE

ONE VARIABLE:
KERVAIRE–LAUDENBACH CONJECTURE

REGULARITY:

$G = \langle a, b \mid a^2, b^3 \rangle$

$xax^{-1}b^{-1} = 1$

PROVED FOR:

- G FINITE, ONE VARIABLE (GERSTENHABER, ROTHAUS '62)
- HYPERLINEAR G (SEE PESTOV '08)
- TORSION-FREE G: SOME CASES (KLYACHKO '93)

WHAT (MIXED) IDENTITIES HOLD IN G?

TERM $t(x_1 \dots x_n)$ **TRIVIAL** $:\Leftrightarrow$

$t = 1$ IN THE FREE GROUP GENERATED BY $x_1 \dots x_n$

POLYNOMIAL $w(x_1 \dots x_n, g_1 \dots g_m)$ **REGULAR**

$:\Leftrightarrow$ $w(x_1, \dots, x_n, 1, \dots, 1)$ NON-TRIVIAL

REGULAR MIXED IDENTITIES?

G FINITE $\Rightarrow$ $w^{|G|} = 1$ HOLDS

CONJECTURE (KLYACHKO, THOM '16)

w REGULAR $\Rightarrow$ $\exists H \geq G$

$w = 1$ SOLVABLE IN H

G FINITE $\Rightarrow$ H FINITE

ONE VARIABLE:
KERVAIRE–LAUDENBACH CONJECTURE

REGULARITY:

$G = \langle a, b \mid a^2, b^3 \rangle$

$x a x^{-1} b^{-1} = 1$

PROVED FOR:

- G FINITE, ONE VARIABLE (GERSTENHABER, ROTHAUS '62)
- HYPERLINEAR G (SEE PESTOV '08)
- TORSION-FREE G: SOME CASES (KLYACHKO '93)

THEOREM (ADELEKE, HOLLAND '94)

ALL $t(x_1 \dots x_n) = g$ SOLVABLE IN $\mathrm{Aut}(\mathbb{Q}, <)$

WHAT (MIXED) IDENTITIES HOLD IN G?

TERM $t(x_1 \dots x_n)$ TRIVIAL $:\Leftrightarrow$

$t = 1$ IN THE FREE GROUP GENERATED BY $x_1 \dots x_n$

POLYNOMIAL $w(x_1 \dots x_n, g_1 \dots g_m)$ REGULAR

$:\Leftrightarrow$ $w(x_1 \dots x_n, 1, \dots, 1)$ NON-TRIVIAL

REGULAR MIXED IDENTITIES?

G FINITE $\Rightarrow$ $w^{|G|} = 1$ HOLDS

CONJECTURE (KLYATCHKO, THOM '16)

w REGULAR $\Rightarrow$ $\exists H \geq G$

$w = 1$ SOLVABLE IN H

G FINITE $\Rightarrow$ H FINITE

ONE VARIABLE:

KERVAIRE–LAUDENBACH CONJECTURE

REGULARITY:

$G = \langle a, b \mid a^2, b^3 \rangle$

$xax^{-1}b^{-1} = 1$

PROVED FOR:

- G FINITE, ONE VARIABLE

(GERSTENHABER, ROTHAUS '62)

- HYPERLINEAR G (SEE PESTOV '08)
- TORSION-FREE G: SOME CASES

(KLYATCHKO '93)

THEOREM (ADELEKE, HOLLAND '94)

ALL $t(x_1 \dots x_n) = g$ SOLVABLE IN $\mathrm{Aut}(\mathbb{Q}, <)$

THEOREM (SCHNEIDER, THOM '22)

$\mathrm{Sym}(\mathbb{N})$ NO REGULAR MIXED IDENTITIES

WHAT (MIXED) IDENTITIES HOLD IN G?

TERM $t(x_1 \dots x_n)$ **TRIVIAL** $:\Leftrightarrow$

$t = 1$ IN THE FREE GROUP GENERATED BY $x_1 \dots x_n$

POLYNOMIAL $w(x_1 \dots x_n, g_1 \dots g_m)$ **REGULAR**

$:\Leftrightarrow$ $w(x_1 \dots x_n, 1, \dots, 1)$ NON-TRIVIAL

REGULAR MIXED IDENTITIES ?

G FINITE $\Rightarrow$ $w^{|G|} = 1$ HOLDS

CONJECTURE (KLYACHKO, THOM '16)

w REGULAR $\Rightarrow$ $\exists H \geq G$

$w = 1$ SOLVABLE IN H

G FINITE $\Rightarrow$ H FINITE

ONE VARIABLE:
KERVAIRE–LAUDENBACH CONJECTURE

REGULARITY:

$G = \langle a, b \mid a^2, b^3 \rangle$

$x a x^{-1} b^{-1} = 1$

PROVED FOR:

- G FINITE, ONE VARIABLE (GERSTENHABER, ROTHAUS '62)
- HYPERLINEAR G (SEE PESTOV '08)
- TORSION-FREE G: SOME CASES (KLYACHKO '93)

THEOREM (ADELEKE, HOLLAND '94)

ALL $t(x_1 \dots x_n) = g$ SOLVABLE IN $\mathrm{Aut}(\mathbb{Q}, <)$

THEOREM (SCHNEIDER, THOM '22)

$\mathrm{Sym}(\mathbb{N})$ NO REGULAR MIXED IDENTITIES

THEOREM (GOLUBCHIK, MIKHALEV '82)

V VECTOR SPACE OVER INFINITE FIELD

$\Rightarrow$ $GL(V)$ NO MIXED IDENTITIES

X ... COUNTABLY INFINITE SET

$\mathrm{Sym}(X)$ ... FULL SYMMETRIC GROUP

$G \leq \mathrm{Sym}(X)$ OLIGOMORPHIC $:\Longleftrightarrow$

$\forall n \geq 1 \quad G \curvearrowright X^n$ FINITELY MANY ORBITS

X ... COUNTABLY INFINITE SET

$\mathrm{Sym}(X)$ ... FULL SYMMETRIC GROUP

$G \leq \mathrm{Sym}(X)$ OLIGOMORPHIC $:\Leftrightarrow$

$\forall n \geq 1$ $G \curvearrowright X^n$ FINITELY MANY ORBITS

<u>THEOREM</u> (RYLL-NARDZEWSKI, ENGELER, SVENONIUS)

$G = \mathrm{Aut}(\mathbb{X}) \Rightarrow$

G OLIGOMORPHIC $\Leftrightarrow$ $\mathbb{X}$ ω-CATEGORICAL

X ... COUNTABLY INFINITE SET

$\mathrm{Sym}(X)$ ... FULL SYMMETRIC GROUP

$G \leq \mathrm{Sym}(X)$ OLIGOMORPHIC $:\Longleftrightarrow$

$\forall n \geq 1$ $G \curvearrowright X^n$ FINITELY MANY ORBITS

THEOREM (RYLL-NARDZEWSKI, ENGELER, SVENONIUS)

$G = \mathrm{Aut}(\mathbb{X}) \Rightarrow$

G OLIGOMORPHIC $\Leftrightarrow$ $\mathbb{X}$ ω-CATEGORICAL

EXAMPLE

- $\mathrm{Sym}(X)$
- $GL(V)$, V VECTOR SPACE OVER FINITE FIELD
- $\mathrm{Aut}(\mathbb{Q}, <)$
- Aut(Random graph / digraph / poset ...)

X ... COUNTABLY INFINITE SET

$\mathrm{Sym}(X)$ ... FULL SYMMETRIC GROUP

$G \leq \mathrm{Sym}(X)$ OLIGOMORPHIC $:\Longleftrightarrow$

$\forall n \geq 1$ $G \curvearrowright X^n$ FINITELY MANY ORBITS

THEOREM (RYLL-NARDZEWSKI, ENGELER, SVENONIUS)

$G = \mathrm{Aut}(\mathbb{X}) \Longrightarrow$

G OLIGOMORPHIC $\Longleftrightarrow$ $\mathbb{X}$ ω-CATEGORICAL

EXAMPLE

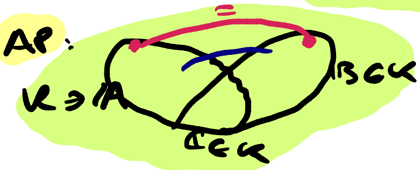
- $\mathrm{Sym}(X)$
- $GL(V)$, V VECTOR SPACE OVER FINITE FIELD
- $\mathrm{Aut}(\mathbb{Q},<)$
- $\mathrm{Aut}($Random graph / digraph / poset ...$)$

GENERAL: FRAÏSSÉ-LIMIT OF CLASS K OF FINITE STRUCTURES WITH:

HP:

JEP:

AP:



X ... COUNTABLY INFINITE SET

$\mathrm{Sym}(X)$ ... FULL SYMMETRIC GROUP

$G \leq \mathrm{Sym}(X)$ OLIGOMORPHIC $:\Leftrightarrow$

$\forall n \geq 1 \quad G \curvearrowright X^n$ FINITELY MANY ORBITS

THEOREM (RYLL-NARDZEWSKI, ENGELER, SVENONIUS)

$G = \mathrm{Aut}(\mathbb{X}) \Rightarrow$

G OLIGOMORPHIC $\Leftrightarrow \mathbb{X}$ ω-CATEGORICAL

EXAMPLE

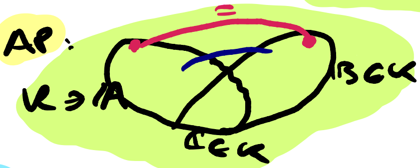
- $\mathrm{Sym}(X)$
- $GL(V)$, V VECTOR SPACE OVER FINITE FIELD
- $\mathrm{Aut}(\mathbb{Q}, <)$
- Aut(Random graph / digraph / poset ...)

<u>GENERAL</u>: FRAÏSSÉ-LIMIT OF CLASS K OF FINITE STRUCTURES WITH:

HP:

JEP:

AP:



THEOREM (MACPHERSON '85)

G OLIGOMORPHIC $\Rightarrow$ NO IDENTITIES WITHOUT CONSTANTS

X ... COUNTABLY INFINITE SET

$\mathrm{Sym}(X)$ ... FULL SYMMETRIC GROUP

$G \leq \mathrm{Sym}(X)$ OLIGOMORPHIC $:\Longleftrightarrow$

$\forall n \geq 1 \quad G \curvearrowright X^n$ FINITELY MANY ORBITS

THEOREM (RYLL-NARDZEWSKI, ENGELER, SVENONIUS)

$G = \mathrm{Aut}(\mathbb{X}) \Longrightarrow$

G OLIGOMORPHIC $\Longleftrightarrow$ $\mathbb{X}$ ω-CATEGORICAL

EXAMPLE

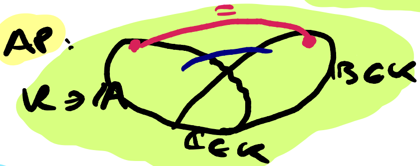
- $\mathrm{Sym}(X)$
- $GL(V)$, V VECTOR SPACE OVER FINITE FIELD
- $\mathrm{Aut}(\mathbb{Q}, <)$
- Aut(Random graph / digraph / poset ...)

GENERAL: FRAÏSSÉ-LIMIT OF CLASS K OF FINITE STRUCTURES WITH:

HP:

JEP:

AP:



THEOREM (MACPHERSON '85)

G OLIGOMORPHIC $\Longrightarrow$ NO IDENTITIES WITHOUT CONSTANTS

THEOREM AUTOMORPHISM GROUPS OF:

- RANDOM GRAPH — ETEDADIALIABADI, GAO, LE MAÎTRE, MELLERAY '21
- DIGRAPH — DE LA NUEZ, GHADERNEZHAD '19
- POSET
- PERMUTATION $\}$ BODIRSKY, THOM, SCHNEIDER '24
- VECTOR SPACE OVER FINITE FIELD — BRADFORD, THOM, SCHNEIDER
- HALL'S UNIVERSAL GROUP
- * URYSOHN SPACE — ETEDADIALIABADI, GAO, LE MAÎTRE, MELLERAY '21

NO REGULAR MIXED IDENTITIES!

CONJECTURE (BODIRSKY, THOM, SCHNEIDER '24)

$\mathcal{G}$ OLIGOMORPHIC

$\Rightarrow$ NO REGULAR MIXED IDENTITIES

CONJECTURE (BODIRSKY, THOM, SCHNEIDER '24)

G OLIGOMORPHIC

$\Rightarrow$ NO REGULAR MIXED IDENTITIES

$\mathrm{Aut}(\mathbb{Q}, <)$ OPEN!

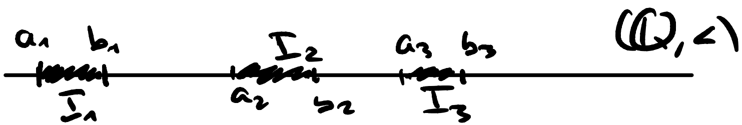
CONJECTURE (BODIRSKY, THOM, SCHNEIDER '24)

G OLIGOMORPHIC

$\Longrightarrow$ NO REGULAR MIXED IDENTITIES

$\mathrm{Aut}(\mathbb{Q},<)$ OPEN!

EXAMPLE



$\mathrm{Supp}\, g_i \subseteq I_i$

$$w = [[g_1^x, g_3], [g_2^x, g_2]]$$

- $x(I_1) \cap I_3 = \emptyset \Longrightarrow w = 1$
- $ \neq \emptyset \Longrightarrow x[I_2] \cap I_2 = \emptyset$

$\Longrightarrow w = 1$

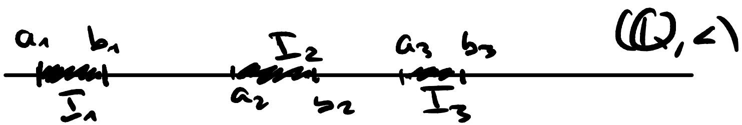
CONJECTURE (BODIRSKY, THOM, SCHNEIDER '24)

G OLIGOMORPHIC

$\Rightarrow$ NO REGULAR MIXED IDENTITIES

$\mathrm{Aut}(\mathbb{Q}, <)$ OPEN!

EXAMPLE



$\mathrm{Supp}\, g_i \subseteq I_i$

$$w = [[g_1^x, g_3], [g_2^x, g_2]]$$

- $x(I_1) \cap I_3 = \emptyset \Rightarrow w = 1$
- $\qquad\qquad \neq \emptyset \Rightarrow x[I_2] \cap I_2 = \emptyset$
  $\Rightarrow w = 1$

ALGEBRAIC CLOSURE

$F \subseteq X$ FINITE

$\mathrm{acl}(F) = \{x \in X \mid O(x)$ FINITE UNDER $G_F\}$

NO ALGEBRAICITY: $\forall F$ $\mathrm{acl}(F) = F$

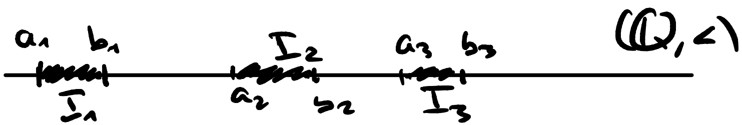
CONJECTURE (Bodirsky, Thom, Schneider '24)

G oligomorphic

$\Rightarrow$ no regular mixed identities

$\mathrm{Aut}(\mathbb{Q},<)$ open!

EXAMPLE



$\mathrm{Supp}\, g_i \subseteq I_i$

$w = [[g_1^x, g_3], [g_2^x, g_2]]$

- $x(I_1) \cap I_3 = \emptyset \Rightarrow w = 1$
- $\neq \emptyset \Rightarrow x[I_2] \cap I_2 = \emptyset$

$\Rightarrow w = 1$

ALGEBRAIC CLOSURE

$F \subseteq X$ finite

$\mathrm{acl}(F) = \{x \in X \mid O(x)$ finite under $G_F\}$

NO ALGEBRAICITY: $\forall F$ $\mathrm{acl}(F) = F$

EXAMPLE

- $\mathrm{Sym}(X)$
- $\mathrm{Aut}(\mathbb{Q},<)$
- $\mathrm{Aut}(\text{random graph})$
- $GL(V)$

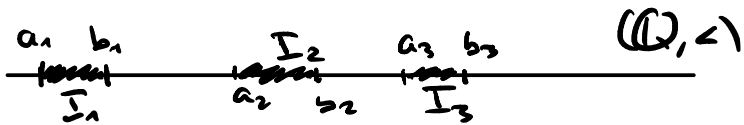
CONJECTURE (BODIRSKY, THOM, SCHNEIDER '24)

G OLIGOMORPHIC

$\Rightarrow$ NO REGULAR MIXED IDENTITIES

$\mathrm{Aut}(\mathbb{Q},<)$ OPEN!

EXAMPLE



$\mathrm{Supp}\ g_i \subseteq I_i$

$$w = [[g_1^x, g_3], [g_2^x, g_2]]$$

- $x(I_1) \cap I_3 = \emptyset \Rightarrow w = 1$
- $\neq \emptyset \Rightarrow x[I_2] \cap I_2 = \emptyset$
  $\Rightarrow w = 1$

ALGEBRAIC CLOSURE

$F \subseteq X$ FINITE

$\mathrm{acl}(F) = \{x \in X \mid O(x)$ FINITE UNDER $G_F\}$

NO ALGEBRAICITY: $\forall F$ $\mathrm{acl}(F) = F$

EXAMPLE

- $\mathrm{Sym}(X)$ ✓
- $\mathrm{Aut}(\mathbb{Q},<)$
- $\mathrm{Aut}(\text{random graph})$
- $GL(V)$

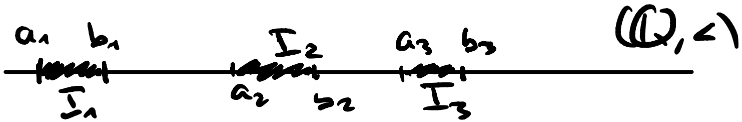
CONJECTURE (Bodirsky, Thom, Schneider '24)

G oligomorphic

$\Rightarrow$ no regular mixed identities

$\mathrm{Aut}(\mathbb{Q},<)$ OPEN!

EXAMPLE



$\mathrm{Supp}\, g_i \subseteq I_i$

$$w = [[g_1^x, g_3], [g_2^x, g_2]]$$

- $x(I_1) \cap I_3 = \emptyset \Rightarrow w = 1$
- $\neq \emptyset \Rightarrow x[I_2] \cap I_2 = \emptyset$

  $\Rightarrow w = 1$

ALGEBRAIC CLOSURE

$F \subseteq X$ finite

$\mathrm{acl}(F) = \{x \in X \mid O(x)$ finite under $G_F\}$

NO ALGEBRAICITY: $\forall F$ $\mathrm{acl}(F) = F$

EXAMPLE

- $\mathrm{Sym}(X)$ ✓
- $\mathrm{Aut}(\mathbb{Q},<)$ ✓
- $\mathrm{Aut}(\text{random graph})$
- $GL(V)$

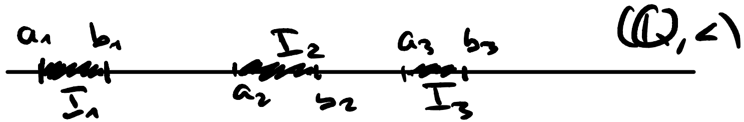
CONJECTURE (BODIRSKY, THOM, SCHNEIDER '24)

G OLIGOMORPHIC

$\Longrightarrow$ NO REGULAR MIXED IDENTITIES

$\mathrm{Aut}(\mathbb{Q},<)$ OPEN!

EXAMPLE



$\mathrm{Supp}\ g_i \subseteq I_i$

$$w = [[g_1^x, g_3],[g_2^x, g_2]]$$

- $x(I_1) \cap I_3 = \emptyset \Rightarrow w = 1$
- $\neq \emptyset \Rightarrow x[I_2] \cap I_2 = \emptyset$
  $\Rightarrow w = 1$

ALGEBRAIC CLOSURE

$F \subseteq X$ FINITE

$\mathrm{acl}(F) = \{x \in X \mid O(x) \text{ FINITE UNDER } G_F\}$

NO ALGEBRAICITY: $\forall F$ $\mathrm{acl}(F) = F$

EXAMPLE

- $\mathrm{Sym}(X)$ ✓
- $\mathrm{Aut}(\mathbb{Q},<)$ ✓
- $\mathrm{Aut}(\text{random graph})$ ✓
- $GL(V)$

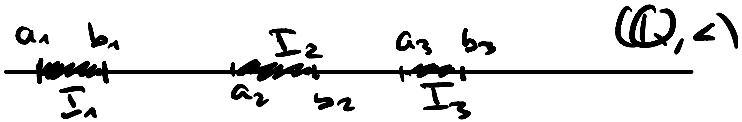
CONJECTURE (BODIRSKY, THOM, SCHNEIDER '24)

G OLIGOMORPHIC

$\Rightarrow$ NO REGULAR MIXED IDENTITIES

$\mathrm{Aut}(\mathbb{Q},<)$ OPEN!

EXAMPLE



$\mathrm{Supp}\, g_i \subseteq I_i$

$$w = [[g_1^x, g_3], [g_2^x, g_2]]$$

- $x(I_1) \cap I_3 = \emptyset \Rightarrow w = 1$
- $\neq \emptyset \Rightarrow x[I_2] \cap I_2 = \emptyset$

  $\Rightarrow w = 1$

ALGEBRAIC CLOSURE

$F \subseteq X$ FINITE

$\mathrm{acl}(F) = \{x \in X \mid O(x)$ FINITE UNDER $G_F\}$

NO ALGEBRAICITY: $\forall F$ $\mathrm{acl}(F) = F$

EXAMPLE

- $\mathrm{Sym}(X)$ ✓
- $\mathrm{Aut}(\mathbb{Q},<)$ ✓
- $\mathrm{Aut}(\text{random graph})$ ✓
- $GL(V)$ ✗

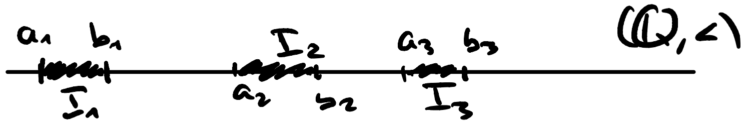
CONJECTURE (Bodirsky, Thom, Schneider '2?)

G oligomorphic

$\Rightarrow$ no regular mixed identities

$\mathrm{Aut}(\mathbb{Q},<)$ OPEN!

EXAMPLE



$\mathrm{Supp}\ g_i \subseteq I_i$

$w = [[g_1^x, g_3], [g_2^x, g_2]]$

- $x(I_1) \cap I_3 = \emptyset \Rightarrow w = 1$
- $\neq \emptyset \Rightarrow x[I_2] \cap I_2 = \emptyset$
  $\Rightarrow w = 1$

ALGEBRAIC CLOSURE

$F \subseteq X$ finite

$\mathrm{acl}(F) = \{x \in X \mid O(x) \text{ finite under } G_F\}$

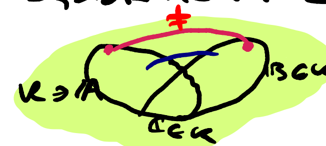
NO ALGEBRAICITY: $\forall F\ \mathrm{acl}(F) = F$

EXAMPLE

- $\mathrm{Sym}(X)$ ✓
- $\mathrm{Aut}(\mathbb{Q},<)$ ✓
- $\mathrm{Aut}(\text{random graph})$ ✓
- $GL(V)$ ✗

FRAÏSSÉ-LIMIT $\mathcal{K}$

NO ALGEBRAICITY $\Leftrightarrow$ SAP



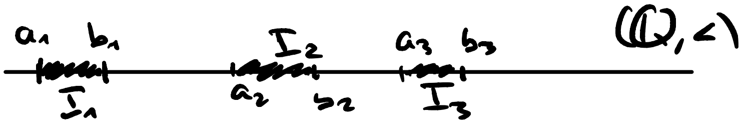
CONJECTURE (BODIRSKY, THOM, SCHNEIDER '24)

G OLIGOMORPHIC

$\Rightarrow$ NO REGULAR MIXED IDENTITIES

$\mathrm{Aut}(\mathbb{Q},<)$ OPEN!

EXAMPLE



$\mathrm{Supp}\ g_i \subseteq I_i$

$w = [[g_1^x, g_3], [g_2^x, g_2]]$

- $x(I_1) \cap I_3 = \emptyset \Rightarrow w = 1$
- $\qquad\qquad \neq \emptyset \Rightarrow x[I_2] \cap I_2 = \emptyset$

$\qquad \Rightarrow w = 1$

ALGEBRAIC CLOSURE

$F \subseteq X$ FINITE

$\mathrm{acl}(F) = \{x \in X \mid O(x)$ FINITE UNDER $G_F\}$

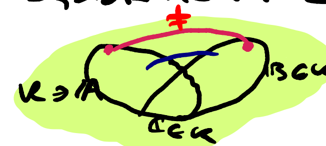
NO ALGEBRAICITY: $\forall F\ \mathrm{acl}(F) = F$

EXAMPLE

- $\mathrm{Sym}(X)$ ✓
- $\mathrm{Aut}(\mathbb{Q},<)$ ✓
- $\mathrm{Aut}(\text{random graph})$ ✓
- $GL(V)$ ✗

FRAÏSSÉ-LIMIT $\mathcal{K}$

NO ALGEBRAICITY $\Leftrightarrow$ SAP



THEOREM (MARIMON, P. '26)

CONJECTURE TRUE FOR ALL G WITHOUT ALGEBRAICITY!

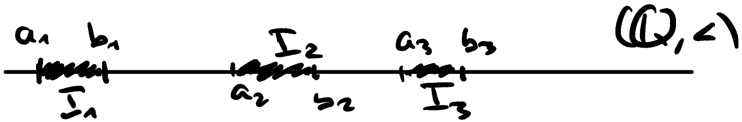
CONJECTURE (Bodirsky, Thom, Schneider '24)

G oligomorphic

$\Rightarrow$ no regular mixed identities

$\mathrm{Aut}(\mathbb{Q},<)$ OPEN!

EXAMPLE



$\mathrm{Supp}\, g_i \subseteq I_i$

$$w = [[g_1^x, g_3], [g_2^x, g_2]]$$

- $x(I_1) \cap I_3 = \emptyset \Rightarrow w = 1$
- $\qquad\qquad \neq \emptyset \Rightarrow x[I_2] \cap I_2 = \emptyset$
  $\Rightarrow w = 1$

ALGEBRAIC CLOSURE

$F \subseteq X$ finite

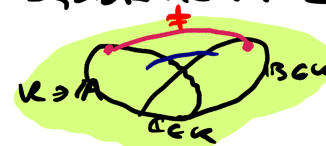
$\mathrm{acl}(F) = \{x \in X \mid O(x) \text{ finite under } G_F\}$

NO ALGEBRAICITY: $\forall F$ $\mathrm{acl}(F) = F$

EXAMPLE

- $\mathrm{Sym}(X)$ ✓
- $\mathrm{Aut}(\mathbb{Q},<)$ ✓
- $\mathrm{Aut}(\text{random graph})$ ✓
- $GL(V)$ ✗

FRAÏSSÉ-LIMIT $\mathcal{K}$:
NO ALGEBRAICITY $\Leftrightarrow$ SAP



THEOREM (Marimon, P. '26)

Conjecture true for all G without algebraicity!

INCLUDES $(\mathbb{Q},<)$!

TRUE SCOPE OF THE THEOREM

TRUE SCOPE OF THE THEOREM

- $\forall F$ $\mathrm{acl}(F)$ **FINITE**

TRUE SCOPE OF THE THEOREM

- $\forall F$ $acl(F)$ **FINITE**

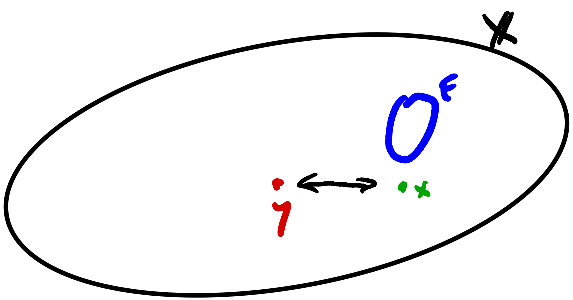
- acl SATISFIES **EXCHANGE**:

  $\forall x\ \forall y\ \forall F$

  $x \in acl(F \cup \{y\}) \setminus acl(F)$

  $\Rightarrow y \in acl(F \cup \{x\})$

  "PREGEOMETRY"



TRUE SCOPE OF THE THEOREM

- $\forall F \quad \mathrm{acl}(F)$ **FINITE**
- acl SATISFIES **EXCHANGE**:

  $\forall x \, \forall y \, \forall F$

  $x \in \mathrm{acl}(F \cup \{y\}) \setminus \mathrm{acl}(F)$

  $\Longrightarrow y \in \mathrm{acl}(F \cup \{x\})$

  "PREGEOMETRY"

- acl **MODULAR**:

  $\dim(A \cup B) = \dim(A) + \dim(B)$

  $\qquad - \dim(A \cap B)$

TRUE SCOPE OF THE THEOREM

- $\forall F$ $\mathrm{acl}(F)$ **FINITE**

- acl SATISFIES **EXCHANGE**:

  $\forall x \; \forall y \; \forall F$

  $x \in \mathrm{acl}(F \cup \{y\}) \setminus \mathrm{acl}(F)$

  $\Longrightarrow y \in \mathrm{acl}(F \cup \{x\})$

  "PREGEOMETRY"

- acl **MODULAR**:

  $$\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B)$$

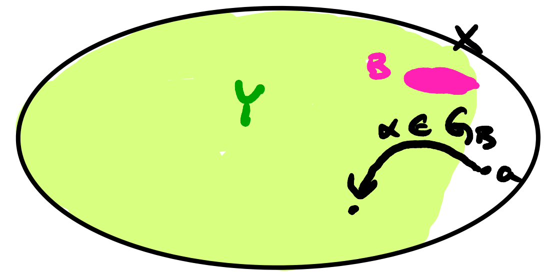
- acl **SUPERSPACIOUS**:

  $Y \subseteq X$ CO-FINITE DIMENSION

  $B \subseteq X$ FINITE DIMENSION

  $a \notin \mathrm{acl}(B)$

  $\Longrightarrow G_B(a) \cap Y \neq \emptyset$



TRUE SCOPE OF THE THEOREM

- $\forall F \quad \mathrm{acl}(F)$ **FINITE**

- acl SATISFIES **EXCHANGE**:

  $\forall x \ \forall y \ \forall F$

  $x \in \mathrm{acl}(F \cup \{y\}) \setminus \mathrm{acl}(F)$

  $\Longrightarrow y \in \mathrm{acl}(F \cup \{x\})$

  "PREGEOMETRY"

- acl **MODULAR**:

  $$\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B)$$

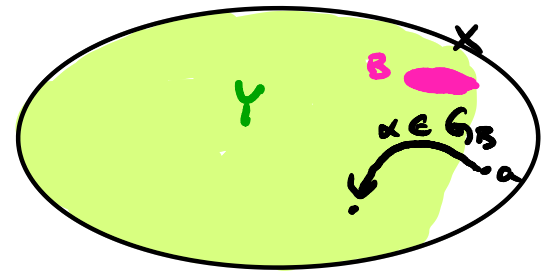
- acl **SUPERSPACIOUS**:

  $Y \subseteq X$ CO-FINITE DIMENSION

  $B \subseteq X$ FINITE DIMENSION

  $a \notin \mathrm{acl}(B)$

  $\Longrightarrow G_B(a) \cap Y \neq \emptyset$



INCLUDES VECTOR SPACES OVER FINITE FIELDS!

ZARISKI TOPOLOGY

ZARISKI TOPOLOGY

$S \subseteq X^X$ TRANSFORMATION MONOID

ZARISKI TOPOLOGY

$S \subseteq X^X$ TRANSFORMATION MONOID

- TOPOLOGY OF POINTWISE CONVERGENCE

  PW :

ZARISKI TOPOLOGY

$S \subseteq X^X$ TRANSFORMATION MONOID

- TOPOLOGY OF POINTWISE CONVERGENCE
  PW :

  SUBBASIS

$$O_{a,b} = \{ f \in S \mid f(a) = b \}$$

FOR $a, b \in X$.

ZARISKI TOPOLOGY

$S \subseteq X^X$ TRANSFORMATION MONOID

- TOPOLOGY OF POINTWISE CONVERGENCE PW :

  SUBBASIS

  $$O_{a,b} = \{ f \in S \mid f(a) = b \}$$

  FOR $a, b \in X$.

- ZARISKI TOPOLOGY Z :

ZARISKI TOPOLOGY

$S \subseteq X^X$ TRANSFORMATION MONOID

- TOPOLOGY OF POINTWISE CONVERGENCE PW:

  SUBBASIS

  $$O_{a,b} = \{ f \in S \mid f(a) = b \}$$

  FOR $a, b \in X$.

- ZARISKI TOPOLOGY Z:

  SUBBASIS

  $$O_{u,v} = \{ f \in S \mid u(f) \neq v(f) \}$$

  FOR $u(x), v(x)$ POLYNOMIALS

ZARISKI TOPOLOGY

$S \subseteq X^X$ TRANSFORMATION MONOID

- TOPOLOGY OF POINTWISE CONVERGENCE PW :

  SUBBASIS

  $$O_{a,b} = \{ f \in S \mid f(a) = b \}$$

  FOR $a, b \in X$.

- ZARISKI TOPOLOGY Z :

  SUBBASIS

  $$O_{u,v} = \{ f \in S \mid u(f) \neq v(f) \}$$

  FOR $u(x), v(x)$ POLYNOMIALS

(S, PW)

ZARISKI TOPOLOGY

$S \leq X^X$ TRANSFORMATION MONOID

- TOPOLOGY OF POINTWISE CONVERGENCE PW :

  SUBBASIS

  $$O_{a,b} = \{ f \in S \mid f(a) = b \}$$

  FOR $a, b \in X$.

- ZARISKI TOPOLOGY Z :

  SUBBASIS

  $$O_{u,v} = \{ f \in S \mid u(f) \neq v(f) \}$$

  FOR $u(x), v(x)$ POLYNOMIALS

(S, PW)

- TOPOLOGICAL SEMIGROUP

ZARISKI TOPOLOGY

$S \subseteq X^X$ TRANSFORMATION MONOID

- TOPOLOGY OF POINTWISE CONVERGENCE
  PW :

  SUBBASIS

  $$O_{a,b} = \{ f \in S \mid f(a) = b \}$$

  FOR $a, b \in X$.

- ZARISKI TOPOLOGY Z :

  SUBBASIS

  $$O_{u,v} = \{ f \in S \mid u(f) \neq v(f) \}$$

  FOR $u(x), v(x)$ POLYNOMIALS

(S, PW)

- TOPOLOGICAL SEMIGROUP
- METRIZABLE, SEPARABLE

ZARISKI TOPOLOGY

$S \subseteq X^X$ TRANSFORMATION MONOID

- TOPOLOGY OF POINTWISE CONVERGENCE PW:

  SUBBASIS

  $$O_{a,b} = \{ f \in S \mid f(a) = b \}$$

  FOR $a, b \in X$.

- ZARISKI TOPOLOGY Z:

  SUBBASIS

  $$O_{u,v} = \{ f \in S \mid u(f) \neq v(f) \}$$

  FOR $u(x), v(x)$ POLYNOMIALS

(S, PW)

- TOPOLOGICAL SEMIGROUP
- METRIZABLE, SEPARABLE
- $\mathbb{X}$ ω-CATEGORICAL

  $\Rightarrow$ (Aut $\mathbb{X}$, PW)

  "KNOWS" $\mathbb{X}$ UP TO FIRST-ORDER BI-INTERPRETATIONS

ZARISKI TOPOLOGY

$S \subseteq X^X$ TRANSFORMATION MONOID

- TOPOLOGY OF POINTWISE CONVERGENCE PW:

  SUBBASIS

  $$O_{a,b} = \{ f \in S \mid f(a) = b \}$$

  FOR $a, b \in X$.

- ZARISKI TOPOLOGY Z:

  SUBBASIS

  $$O_{u,v} = \{ f \in S \mid u(f) \neq v(f) \}$$

  FOR $u(x), v(x)$ POLYNOMIALS

(S, PW)

- TOPOLOGICAL SEMIGROUP
- METRIZABLE, SEPARABLE
- $\mathbb{X}$ ω-CATEGORICAL

  $\Rightarrow$ (Aut $\mathbb{X}$, PW)

  "KNOWS" $\mathbb{X}$ UP TO FIRST-ORDER BI-INTERPRETATIONS
- (End $\mathbb{X}$, PW) EXISTENTIAL POSITIVE
- (Pol $\mathbb{X}$, PW) PRIMITIVE POSITIVE

ZARISKI TOPOLOGY

$S \subseteq X^X$ TRANSFORMATION MONOID

- TOPOLOGY OF POINTWISE CONVERGENCE

  PW :

  SUBBASIS

  $$O_{a,b} = \{f \in S \mid f(a) = b\}$$

  FOR $a, b \in X$.

- ZARISKI TOPOLOGY Z :

  SUBBASIS

  $$O_{u,v} = \{f \in S \mid u(f) \neq v(f)\}$$

  FOR $u(x), v(x)$ POLYNOMIALS

(S, PW)

- TOPOLOGICAL SEMIGROUP
- METRIZABLE, SEPARABLE
- $\mathbb{X}$ ω-CATEGORICAL

  $\Rightarrow$ (Aut $\mathbb{X}$, PW)

  "KNOWS" $\mathbb{X}$ UP TO
  FIRST-ORDER BI-INTERPRETATIONS

- (End $\mathbb{X}$, PW) EXISTENTIAL POSITIVE
- (Pol $\mathbb{X}$, PW) PRIMITIVE POSITIVE

(S, Z)

ZARISKI TOPOLOGY

$S \leq X^X$ TRANSFORMATION MONOID

- TOPOLOGY OF POINTWISE CONVERGENCE PW:

  SUBBASIS

  $$O_{a,b} = \{ f \in S \mid f(a) = b \}$$

  FOR $a, b \in X$.

- ZARISKI TOPOLOGY Z:

  SUBBASIS

  $$O_{u,v} = \{ f \in S \mid u(f) \neq v(f) \}$$

  FOR $u(x), v(x)$ POLYNOMIALS

(S, PW)

- TOPOLOGICAL SEMIGROUP
- METRIZABLE, SEPARABLE
- $\mathbb{X}$ ω-CATEGORICAL

  $\Rightarrow (\mathrm{Aut}\,\mathbb{X}, \mathrm{PW})$

  "KNOWS" $\mathbb{X}$ UP TO FIRST-ORDER BI-INTERPRETATIONS
- $(\mathrm{End}\,\mathbb{X}, \mathrm{PW})$ EXISTENTIAL POSITIVE
- $(\mathrm{Pol}\,\mathbb{X}, \mathrm{PW})$ PRIMITIVE POSITIVE

(S, Z)

- NO TOPOLOGICAL SEMIGROUP
- NOT T_2
- COARSE:

  $\subseteq$ ANY T_2 SEMIGROUP TOPOLOGY

RECONSTRUCTION THEORY

CAN PW BE "RECONSTRUCTED" FROM GROUP/SEMIGROUP STRUCTURE ?

RECONSTRUCTION THEORY

CAN $P\omega$ BE "RECONSTRUCTED" FROM GROUP / SEMIGROUP STRUCTURE ?

THEOREM (P., SCHINDLER '22)

$P\omega$ UNIQUE POLISH TOPOLOGY ON $\mathrm{End}(\mathbb{Q}, \leq)$

RECONSTRUCTION THEORY

CAN PW BE "RECONSTRUCTED" FROM GROUP / SEMIGROUP STRUCTURE ?

THEOREM (P., SCHINDLER '22)

PW UNIQUE POLISH TOPOLOGY ON $\mathrm{End}(\mathbb{Q}, \leq)$

FOR MANY ω-CAT. $\mathbb{X}$:

$\mathrm{PW} = \mathcal{Z}$ ON $\mathrm{End}\,\mathbb{X}$!

RECONSTRUCTION THEORY

CAN Pw BE "RECONSTRUCTED" FROM GROUP / SEMIGROUP STRUCTURE ?

THEOREM (P., SCHINDLER '22)

Pw UNIQUE POLISH TOPOLOGY ON

$$\mathrm{End}\,(\mathbb{Q}, \leq)$$

FOR MANY ω-CAT. $\mathbb{X}$:

$$\mathrm{Pw} = \mathrm{Z} \text{ ON } \mathrm{End}\,\mathbb{X} \text{ !}$$

QUESTION (ELLIOTT, JONUŠAS, MITCHELL, PÉRESSE, P. '20)

ALWAYS ???

RECONSTRUCTION THEORY

CAN PW BE "RECONSTRUCTED" FROM GROUP / SEMIGROUP STRUCTURE ?

THEOREM (P., SCHINDLER '22)

PW UNIQUE POLISH TOPOLOGY ON $\mathrm{End}\,(\mathbb{Q}, \leq)$

FOR MANY ω-CAT. $\mathbb{X}$:

$\mathrm{PW} = \mathrm{Z}$ ON $\mathrm{End}\,\mathbb{X}$!

QUESTION (ELLIOTT, JONUŠAS, MITCHELL, PÉRESSE, P. '20)

ALWAYS ???

THEOREM (P., SCHINDLER '23)

$\mathbb{X}$ ω-CAT., NO ALGEBRAICITY

- $\overline{\mathrm{Aut}\,\mathbb{X}} = \mathrm{End}\,\mathbb{X}$ OR
- $\mathrm{End}\,\mathbb{X}$ CONTAINS FINITE RANGE FUNCTION

$\Longrightarrow$ $\mathrm{PW} = \mathrm{Z}$ ON $\mathrm{End}\,\mathbb{X}$!

RECONSTRUCTION THEORY

CAN Pw BE "RECONSTRUCTED" FROM GROUP / SEMIGROUP STRUCTURE ?

THEOREM (P., SCHINDLER '22)

Pw UNIQUE POLISH TOPOLOGY ON $\mathrm{End}(\mathbb{Q}, \leq)$

FOR MANY ω-CAT. $\mathbb{X}$:

$\mathrm{Pw} = \mathcal{Z}$ ON $\mathrm{End}\,\mathbb{X}$!

QUESTION (ELLIOTT, JONUŠAS, MITCHELL, PÉRESSE, P. '20)

ALWAYS ???

THEOREM (P., SCHINDLER '23)

$\mathbb{X}$ ω-CAT., NO ALGEBRAICITY

- $\overline{\mathrm{Aut}\,\mathbb{X}} = \mathrm{End}\,\mathbb{X}$ OR
- $\mathrm{End}\,\mathbb{X}$ CONTAINS FINITE RANGE FUNCTION

$\Rightarrow$ $\mathrm{Pw} = \mathcal{Z}$ ON $\mathrm{End}\,\mathbb{X}$!

THEOREM (DE LA NUEZ GHADERNEZHAD, MARIMON, P. '24)

$\mathbb{X}$ ω-CAT., $\mathrm{Aut}\,\mathbb{X}$ NON-TRIVIAL CENTER,

- $\overline{\mathrm{Aut}\,\mathbb{X}} = \mathrm{End}\,\mathbb{X}$

$\Rightarrow$ $\mathcal{Z}$ NOT T_2 !

THEOREM (DE LA NUEZ, GHADERNEZHAD, MARIMON, P. '24)

$\mathcal{X}$ ω-CAT., $\mathrm{Aut}\,\mathcal{X}$ NON-TRIVIAL CENTER,

- $\overline{\mathrm{Aut}\,\mathcal{X}} = \mathrm{End}\,\mathcal{X}$

$\Rightarrow$ Z NOT T_2 !

THEOREM (DE LA NUEZ, GHADERNEZHAD, MARIMON, P. '24)

$\mathcal{X}$ ω-CAT., $\mathrm{Aut}\,\mathcal{X}$ NON-TRIVIAL CENTER,

- $\overline{\mathrm{Aut}\,\mathcal{X}} = \mathrm{End}\,\mathcal{X}$

$\Rightarrow$ Z NOT T_2 !

PROOF SKETCH:

THEOREM (DE LA NUEZ, GHADERNEZHAD, MARIMON, P. '24)

$\mathcal{X}$ ω-CAT., $\mathrm{Aut}\,\mathcal{X}$ NON-TRIVIAL CENTER,

- $\overline{\mathrm{Aut}\,\mathcal{X}} = \mathrm{End}\,\mathcal{X}$

$\Rightarrow$ Z NOT T_2 !

PROOF SKETCH:

$$O_{u,v} = \{ f \in \mathrm{End}\,\mathcal{X} \mid u(f) \neq v(f) \}$$

$$u = \gamma_n \times \gamma_{n-1} \cdots \times \gamma_0$$
$$v = \delta_\ell \times \delta_{\ell-1} \cdots \times \delta_0$$

THEOREM (DE LA NUEZ, GHADERNEZHAD, MARIMON, P. '24)

$\mathcal{X}$ ω-CAT., $\mathrm{Aut}\,\mathcal{X}$ NON-TRIVIAL CENTER,

- $\overline{\mathrm{Aut}\,\mathcal{X}} = \mathrm{End}\,\mathcal{X}$

$\Rightarrow$ Z NOT T_2 !

PROOF SKETCH:

$$O_{u,v} = \{ f \in \mathrm{End}\,\mathcal{X} \mid u(f) \neq v(f) \}$$

$$u = \gamma_n \times \gamma_{n-1} \cdots \times \gamma_0$$

$$v = \delta_\ell \times \delta_{\ell-1} \cdots \times \delta_0$$

TYPE I : $n = m$ "SINGULAR"

TYPE II : $n \neq m$ "REGULAR"

THEOREM (DE LA NUEZ, GHADERNEZHAD, MARIMON, P. '24)

$\mathcal{X}$ ω-CAT., $\mathrm{Aut}\,\mathcal{X}$ NON-TRIVIAL CENTER,

- $\overline{\mathrm{Aut}\,\mathcal{X}} = \mathrm{End}\,\mathcal{X}$

$\Rightarrow$ Z NOT T_2 !

PROOF SKETCH:

$$O_{u,v} = \{ f \in \mathrm{End}\,\mathcal{X} \mid u(f) \neq v(f) \}$$

$$u = \gamma_n \times \gamma_{n-1} \cdots \times \gamma_0$$
$$v = \delta_\ell \times \delta_{\ell-1} \cdots \times \delta_0$$

TYPE I : $n = m$ "SINGULAR"

TYPE II : $n \neq m$ "REGULAR"

- $\alpha \in \mathrm{Aut}\,\mathcal{X}$, IN CENTER

$\Rightarrow$ α, id SATISFY SAME SINGULAR

THEOREM (DE LA NUEZ, GHADERNEZHAD, MARIMON, P. '24)

$\mathcal{X}$ ω-CAT., $\mathrm{Aut}\,\mathcal{X}$ NON-TRIVIAL CENTER,

- $\overline{\mathrm{Aut}\,\mathcal{X}} = \mathrm{End}\,\mathcal{X}$

$\Rightarrow$ Z NOT T_2 !

PROOF SKETCH:

$$O_{u,v} = \{ f \in \mathrm{End}\,\mathcal{X} \mid u(f) \neq v(f) \}$$

$$u = \gamma_n \times \gamma_{n-1} \cdots \times \gamma_0$$
$$v = \delta_\ell \times \delta_{\ell-1} \cdots \times \delta_0$$

TYPE I : $n = m$ "SINGULAR"

TYPE II : $n \neq m$ "REGULAR"

- $\alpha \in \mathrm{Aut}\,\mathcal{X}$, IN CENTER

$\Rightarrow$ α, id SATISFY SAME SINGULAR

- TYPE II SETS ARE COMEAGER IN PW

"NO REGULAR IDENTITIES" !!

THEOREM (DE LA NUEZ, GHADERNEZHAD, MARIMON, P. '24)

$\mathbb{X}$ ω-CAT., $\mathrm{Aut}\,\mathbb{X}$ NON-TRIVIAL CENTER,

- $\overline{\mathrm{Aut}\,\mathbb{X}} = \mathrm{End}\,\mathbb{X}$

$\Rightarrow$ Z NOT T_2 !

PROOF SKETCH:

$$O_{u,v} = \{ f \in \mathrm{End}\,\mathbb{X} \mid u(f) \neq v(f) \}$$

$$u = \gamma_n \times \gamma_{n-1} \cdots \times \gamma_0$$

$$v = \delta_\ell \times \delta_{\ell-1} \cdots \times \delta_0$$

TYPE I : $n = m$ "SINGULAR"

TYPE II : $n \neq m$ "REGULAR"

- $\alpha \in \mathrm{Aut}\,\mathbb{X}$, IN CENTER

$\Rightarrow$ α, id SATISFY SAME SINGULAR

- TYPE II SETS ARE COMEAGER IN PW

"NO REGULAR IDENTITIES" !!

CONJECTURE (BODIRSKY, THOM, SCHNEIDER '24)

G OLIGOMORPHIC

$\Rightarrow$ NO REGULAR MIXED IDENTITIES

THEOREM (DE LA NUEZ, GHADERNEZHAD, MARIMON, P. '24)

$\mathcal{X}$ ω-CAT., Aut $\mathcal{X}$ NON-TRIVIAL CENTER,

- $\overline{\mathrm{Aut}\,\mathcal{X}} = \mathrm{End}\,\mathcal{X}$

$\Rightarrow$ $\mathbb{Z}$ NOT T_2 !

PROOF SKETCH:

$O_{u,v} = \{ f \in \mathrm{End}\,\mathcal{X} \mid u(f) \neq v(f) \}$

$u = \gamma_n \times \gamma_{n-1} \cdots \times \gamma_0$

$v = \delta_\ell \times \delta_{\ell-1} \cdots \times \delta_0$

TYPE I : $n = m$ "SINGULAR"

TYPE II : $n \neq m$ "REGULAR"

- $\alpha \in \mathrm{Aut}\,\mathcal{X}$, IN CENTER

$\Rightarrow$ α, id SATISFY SAME SINGULAR

- TYPE II SETS ARE COMEAGER IN PW

"NO REGULAR IDENTITIES" !!

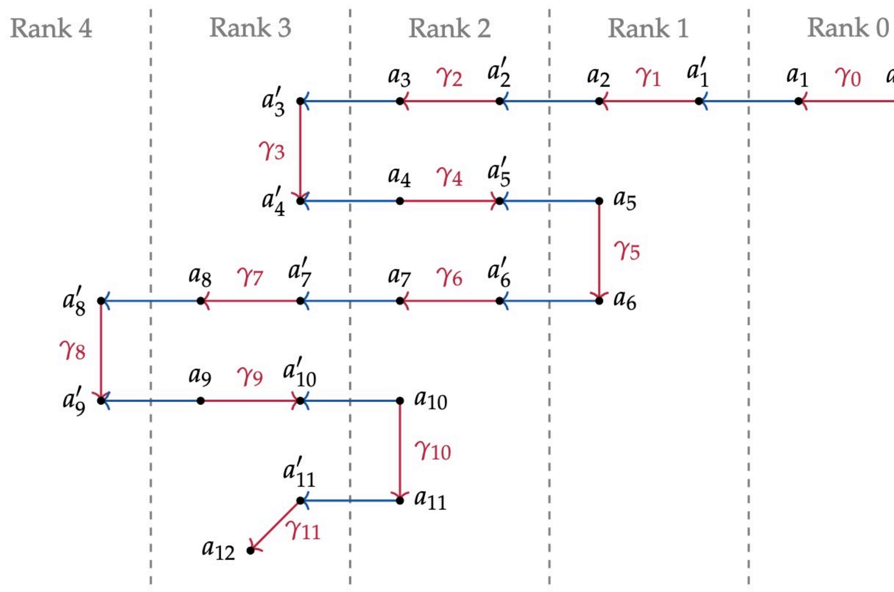
CONJECTURE (BODIRSKY, THOM, SCHNEIDER '24)

G OLIGOMORPHIC

$\Rightarrow$ NO REGULAR MIXED IDENTITIES

THEOREM (MARIMON, P. '26)

CONJECTURE TRUE FOR ALL G WITHOUT ALGEBRAICITY !



$$\gamma_{11} \times \gamma_{10} \times^{-1} \gamma_9 \times^{-1} \gamma_8 \times \gamma_7 \times \gamma_6 \times \gamma_5 \times^{-1} \gamma_4 \times^{-1} \gamma_3 \times \gamma_2 \times \gamma_1 \times \gamma_0 = 1$$

Thank you!

HO!

FUNDING STATEMENT : FUNDED BY THE EUROPEAN UNION (ERC, POCOCOP, 101071674).
VIEWS AND OPINIONS EXPRESSED ARE HOWEVER THOSE OF THE AUTHOR(S) ONLY AND DO NOT NECESSARILY REFLECT THOSE OF THE EUROPEAN UNION OR THE EUROPEAN RESEARCH COUNCIL GRANT AGENCY.
NEITHER THE EUROPEAN UNION NOR THE GRANTING AUTHORITY CAN BE HELD RESPONSIBLE FOR THEM.