

CONSTRAINT SATISFACTION PROBLEMS

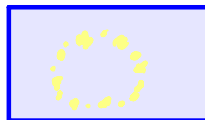
INTRODUCTION &

RECENT ALGEBRAIC AND MODEL-THEORETIC ADVANCES

MICHAEL PINSKER

TU WIEN

THE ROAMING LOGIC CONFERENCE WARSAW 2025



EUROPEAN RESEARCH COUNCIL

ERC SYNERGY GRANT

POCOCOP (CA 101071674)



FWF I5948

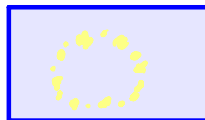
POLYMORPHISMS

MULTIVARIATE INVARIANTS FOR
LOGICIANS & COMPUTER SCIENTISTS

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/A ... FIRST-ORDER STRUCTURE

$\mathcal{A} \dots$ FIRST-ORDER STRUCTURE

$$\text{Aut}(\mathcal{A}) := \{ f: \mathcal{A} \rightarrow \mathcal{A} \mid f \text{ BIJECTIVE,} \\ f, f^{-1} \text{ HOMOMORPHISM} \}$$

$A \dots$ FIRST-ORDER STRUCTURE

$$\text{Aut}(A) := \{ f: A \rightarrow A \mid f \text{ BIJECTIVE,} \\ f, f^{-1} \text{ HOMOMORPHISM} \}$$

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$$\text{End}(\mathcal{A}) := \{ f: \mathcal{A} \rightarrow \mathcal{A} \mid f \text{ HOMOMORPHISM} \}$$

$$\text{Pol}(\mathcal{A}) := \bigcup_{n \geq 1} \{ f: \mathcal{A}^n \rightarrow \mathcal{A} \mid f \text{ HOMOMORPHISM} \}$$

OUTLINE

- 1 CONSTRAINT SATISFACTION PROBLEMS
⇒ RELATIONAL STRUCTURES
- 2 ALGEBRAIC INVARIANTS
(FOR FINITE STRUCTURES)
- 3 GOING INFINITE
(ω -CATEGORICAL)
- 4 POLYMORPHISMS FOR EVERYONE

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS ON THEM

QUESTION

- $\exists$ VALUES FOR $x_1 \dots x_n$
SATISFYING ALL CONSTRAINTS?

EXAMPLE

- SUDOKU
- SCHEDULING
- SOLVING EQUATIONS

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EXAMPLE

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MODEL :

- SET V OF POSSIBLE VALUES (FIXED)
E.G. $\{0,1\}$, $\{0,1,2\}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, ...
- ALLOWED CONSTRAINTS (FIXED)
 $C_1, \dots, C_m$ RELATIONS ON V
 $C_i \in V^{d_i}$

SO $A := (V, C_1, \dots, C_m)$ RELATIONAL STRUCTURE
= "TEMPLATE"

CSP(A)

GIVEN

- VARIABLES $x_1 \dots x_n$
- PP-SENTENCE

$$\varphi \equiv \exists x_1 \dots \exists x_n C_{i_1}(\text{VARIABLES}) \wedge C_{i_2}(\text{VARIABLES}) \wedge \dots C_{i_k}(\text{VARIABLES})$$

QUESTION

$$A \models \varphi ?$$

CONSTRAINT SATISFACTION PROBLEM CSP

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 $C_i \subseteq V^{a_i}$

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GIVEN

- VARIABLES $x_1 \dots x_n$
- PP-SENTENCE

$$\varphi \equiv \exists x_1 \dots \exists x_n C_1(\text{VARIABLES}) \wedge C_2(\text{VARIABLES}) \wedge \dots C_{i_2}(\text{VARIABLES})$$

QUESTION

$$\langle A \rangle \models \varphi ?$$

META-QUESTION

FOR WHAT $\langle A \rangle$ IS CSP(A) EASY/HARD?

P:

$\exists$ ALGORITHM PROVIDING ANSWER IN $p(\# \text{VARIABLES})$ STEPS, p POLYNOMIAL

NP:

NOT NECESSARILY IN P BUT VERIFYING "SOLUTION" IN P

EXAMPLES

• $A = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

TERNARY RELATIONS

CSP(A)

GIVEN

• VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS $x_1^2 + x_2^3 = x_3^4$

$\vdots$

QUESTION

• SOLUTION IN $\mathbb{Z}^?$

UNDECIDABLE (MATYASEWIC '77)

HILBERT 10

EXAMPLES

• $A = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

TERNARY RELATIONS

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• SAME OVER $\mathbb{Z}_p$:

NP-COMPLETE

$\hookrightarrow$ IF IN P $\Rightarrow$ P = NP

EXAMPLES

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TERNARY RELATIONS

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- SAME OVER $\mathbb{Z}_p$:

NP-COMPLETE

$\hookrightarrow$ IF IN P $\Rightarrow$ P = NP

-
- SAME WITHOUT • :

IN P (GAUSS)

EXAMPLES

- $A = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

TERNARY RELATIONS

CSP(A)

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- VARIABLES $x_1, \dots, x_n$
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- SAME OVER $\mathbb{Z}_p$:

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$\hookrightarrow$ IF IN P $\Rightarrow$ P = NP

- SAME WITHOUT

IN P (GAUSS)

- $A = (\mathbb{Q}, <)$

CSP(A)

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1 < x_2$
 $x_2 < x_3$
 $x_3 < x_1$
 $\vdots$

QUESTION

- SOLUTION IN $\mathbb{Q}$?



IN P

EXAMPLES

- $A = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

TERNARY RELATIONS

CSP(A)

GIVEN

- VARIABLES $x_1, \dots, x_n$
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QUESTION

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 $x_2 < x_3$
 $x_3 < x_1$
 $\vdots$

QUESTION

- SOLUTION IN $\mathbb{Q}$?



IN P

- $A = \mathbb{K}_3 = \{0, 1, 2\} = (1, 0, 1, 2, 3, \mathbb{E})$

CSP(A)

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $\mathbb{E}(x_1, x_2)$
 $\mathbb{E}(x_2, x_3)$
 $\vdots$

QUESTION

- SOLUTION IN $\{0, 1, 2\}$?

3-COLORING PROBLEM = NP-COMPLETE

THEOREM

(BOGATOV, ZHUK '17)

(CONJECTURE FEDER+ VARDI '93)

$\mathbb{A}$ FINITE

$\Rightarrow \text{CSP}(\mathbb{A}) \in \text{P}$ OR

NP-COMPLETE

THEOREM

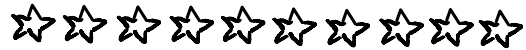
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$(\text{P} \neq \text{NP})$

$\text{CSP}(\mathcal{A}) \in \text{P} \Leftrightarrow \mathcal{A}$ HAS ALGEBRAIC
INVARIANT

$$S(x, y, x, z, y, z) =$$

$$S(y, x, z, x, z, y)$$

THEOREM

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A FINITE

$\Rightarrow \text{CSP}(A) \in P \text{ OR}$
 NP-COMplete



$(P \neq NP)$

$\text{CSP}(A) \in P \Leftrightarrow A$ HAS ALGEBRAIC
INVARIANT
 $S(x, y, x, z, y, z) =$
 $S(y, x, z, x, z, y)$



ALGEBRAIC INVARIANTS = POLYMORPHISMS

$A = (A; R_1, \dots, R_m)$ STRUCTURE

$f: A^r \rightarrow A$ POLYMORPHISM: $\Leftrightarrow$

f HOMOMORPHISM $A^r \rightarrow A \Leftrightarrow$

$\forall: \forall \bar{r}_1, \dots, \bar{r}_e \in R_i$

$$f\left(\begin{array}{c} \vdots \\ \bar{r}_1 \end{array} \dots \begin{array}{c} \vdots \\ \bar{r}_e \end{array}\right) \in R_i$$

$\text{POL}(A) := \{f \mid f \text{ POLYMORPHISM OF } A\}$

- CONTAINS PROJECTIONS $(x_1, \dots, x_e) \mapsto x_i$
- COMPOSITION-CLOSED

$\Rightarrow$ ESSENTIALLY TERM FUNCTIONS
OF AN ALGEBRA ON A !

EXAMPLES

- $\min(x, y): \mathbb{Q}^2 \rightarrow \mathbb{Q} \in \text{Pol}(\mathbb{Q}, <)$
- $(x, y, z) \mapsto x - y + z \in \text{Pol}(\mathbb{Z}_p, \{0\}, \{1\}, +)$
- $? \in \text{Pol}(K_3)$

EXAMPLES

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- PROJECTIONS, AUTOMORPHISMS $\in \text{Pol}(K_3)$

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(BABY) THEOREM

(BODNARČUK + KALUŽNIN + KOSTOV + ROMOV '09)
GEISER '08

A, B FINITE, SAME DOMAIN

$\text{Pol}(A) \subseteq \text{Pol}(B) \Rightarrow A$ PP-DEFINES B
 $\Rightarrow \text{CSP}(A)$ HARDER THAN $\text{CSP}(B)$

EXAMPLES

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- GOOD FOR CLASSIFYING CSPs ON $\{0, 1\}$
(SCHAEFER '78)

- BETTER : EQUATIONAL CONDITIONS:

- $\exists f \in \text{Pol}(\mathbb{Q}, <) \forall x, y \quad f(x, y) = f(y, x)$

- $\exists f \in \text{Pol}(\mathbb{Z}_p, \{0, 1\}, \{1, 1\}, +)$

$\forall x, y, z \quad f(x, x, y) = f(y, x, x) = y$

- $\exists f \in \text{Pol}(\mathbb{K}_3) : \forall x \quad f(f(x)) = x$ 😞

EXAMPLES

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DEFINITION

- EQUATIONAL (STRONG MAL'CEV) CONDITION: SENTENCE

$$\Psi = \exists f_1, \exists f_2 \dots \forall x_1, \forall x_2, \dots$$

$$\begin{aligned} & S_1(\text{variables}) = t_1(\text{variables}) \\ & \wedge \\ & \vdots \\ & \wedge S_k(\text{variables}) = t_k(\text{variables}) \end{aligned}$$

WHERE $S_i, t_i, \dots$ TERMS OVER $f_1, f_2, \dots$

- Ψ TRIVIAL: $\Leftrightarrow \text{Pol}(\text{ANY STRUCTURE}) \models \Psi$
 $\Leftrightarrow \Psi$ SATISFIABLE BY PROJECTIONS
- $\text{EQ}(\text{Pol}(A)) := \{\Psi \mid \text{Pol}(A) \models \Psi\}$

(ADOLESCENT) THEOREM (BULATOV + JEAVONS + KROKHIN 1994)

$$\text{EQ}(\text{POL}(A)) \subseteq \text{EQ}(\text{POL}(B))$$

$\Rightarrow$ A PP-INTERPRETS B

$\Rightarrow$ CSP(A) HARDER THAN CSP(B)

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(ADULT) THEOREM (BARTO + OPRŠAL + P. 16)

$EQ^1 \dots$ HEIGHT 1 - CONDITIONS

$$\underset{\times}{f_i}(\underset{\times}{\text{variables}}) = \underset{\times}{f_j}(\underset{\times}{\text{variables}})$$

$$EQ^1(POL(A)) \leq EQ^1(POL(B))$$

$\Rightarrow$ A PP-CONSTRUCTS B

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~~9~~ ~~+~~

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COROLLARY

$EQ^1(POL(A))$ TRIVIAL

$\Rightarrow$ A PP-CONSTRUCTS EVERYTHING

$\Rightarrow$ CSP(A) NP-COMPLETE

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$$f_i(\text{variables}) = f_j(\text{variables})$$

~~x~~

~~x~~

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THEOREM $EQ^1(Pol(A))$ NON-TRIVIAL

$\Rightarrow$

$$Pol(A) \models \exists f \forall x, y, z \quad f(x, y, x, z, y, z) = f(y, x, z, x, z, y) \quad (\text{SUGGERS '0})$$

$$\models \exists f \forall a, r, e, a \quad f(a, r, e, a) = f(r, a, r, e)$$

(KEARNES + MARCOWICZ + MCKENZIE '14)

$$\models \exists f \forall x_1 \dots x_n$$

$$f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$$

$\forall n \geq |A|$ PRIME

(BARTO + HODGSON '11)

$$\models \exists f \forall x, y \quad f(x \dots x, y) = \dots = f(y, x \dots x)$$

$\forall n \geq |A|$ PRIME

(MARSTI + MCKENZIE '08)

$\Rightarrow$ CSP(A) $\in$ P (BULATOV, ZHUK '17)

GOING INFINITE

FINITE-DOMAIN CSPs ... COMBINATORIAL PROBLEMS

E.G. 3-COLORING

INFINITE-DOMAIN CSPs:

EXAMPLES

- $(\mathbb{Q}, <)$
- $(\mathbb{Q}, \text{between}(x, y, z))$
- RANDOM GRAPH
- RANDOM POSET
- ATOMLESS BOOLEAN ALGEBRA

LOGIC PROBLEMS:

TEMPLATE = (COUNTABLE) ω -CATEGORICAL STRUCTURE

OFTEN: DEFINABLE IN HOMOGENEOUS STRUCTURE

"ARE THE CONSTRAINTS SATISFIABLE
IN A LINEAR ORDER?" ETC.

• $(\mathbb{Z}, +, 1, <)$

• $(\mathbb{Z}, +, \cdot)$

• $(\mathbb{Q}, +, \cdot)$

NUMERIC PROBLEMS

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NUMERIC PROBLEMS

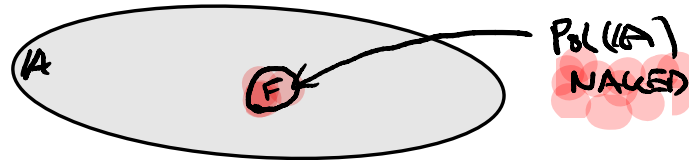
(ADULT) THEOREM (BARTO + OPREŠAL + P. '16)

$\mathcal{A}$ ω -CATEGORICAL

$\text{EQ}^{\text{LOCAL}}(\text{POL}(\mathcal{A}))$ TRIVIAL $\Rightarrow \mathcal{A}$ PP-CONSTRUCTS EVERYTHING

$\Rightarrow \text{CSP}(\mathcal{A})$ NP-HARD

$\exists$ FSA FINITE: $\text{EQ}^{\text{LOCAL}}(\text{POL}(\mathcal{A})|_F)$ TRIVIAL



GOING INFINITE

FINITE-DOMAIN CSPs ... COMBINATORIAL PROBLEMS

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NUMERIC PROBLEMS

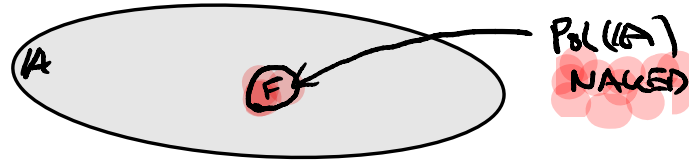
(ADULT) THEOREM (BARTO + OPREA + P. '16)

ω -CATEGORICAL

$\text{EQ}^{\text{LOCAL}}(\text{POL}(\omega))$ TRIVIAL $\Rightarrow \omega$ PP-CONSTRUCTS EVERYTHING

$\Rightarrow \text{CSP}(\omega)$ NP-HARD

$\exists F \subseteq \omega$ FINITE: $\text{EQ}^{\text{LOCAL}}(\text{POL}(\omega)|_F)$ TRIVIAL



THEOREM

ω -CATEGORICAL

$\text{EQ}^{\text{LOCAL}}(\text{POL}(\omega))$ NON-TRIVIAL $\Rightarrow$

$\text{POL}(\omega) \models \exists a, b, f \forall x, y, z$

$a \circ f(x, y, x, z, y, z) = b \circ f(y, x, z, x, z, y)$
(BARTO + P. '17)

$\models \dots$ MANY MORE

(BARTO + BODOR + LOZIK + MOTTEGGER, '23)

$\models \exists f$ BINARY, ESSENTIAL (MARINON + P. '25)

POLYMORPHISMS FOR EVERYONE

POLYMORPHISMS FOR EVERYONE

POLYMORPHISMS & AC

/A W-CATEGORICAL

$\text{EQ}_{\text{LOCAL}}(\text{POL}(A))$ NON-TRIVIAL $\stackrel{?}{\Rightarrow}$

$\text{EQ}(\text{POL}(A))$ NON-TRIVIAL

POLYMORPHISMS FOR EVERYONE

POLYMORPHISMS & AC

/A ω -CATEGORICAL

$\text{EQ}_{\text{LOCAL}}(\text{Pol}(A))$ NON-TRIVIAL $\stackrel{?}{\Rightarrow}$

$\text{EQ}(\text{Pol}(A))$ NON-TRIVIAL

STRENGTHENING:

$$\forall \varphi: \text{Pol}(A) \xrightarrow{\text{hom}} \text{Pol}(K_3)$$

φ CONTINUOUS

(NO: BODARSKY + P. + PONGRÁCZ '13)

AXIOM OF DETERMINACY ?

POLYMORPHISMS FOR EVERYONE

POLYMORPHISMS & AC

I/A ω -CATEGORICAL

$EQ_{\text{LOCAL}}(Pol(A))$ NON-TRIVIAL $\stackrel{?}{\Rightarrow}$

$EQ(Pol(A))$ NON-TRIVIAL

STRENGTHENING:

$\forall \varphi: Pol(A) \xrightarrow{\text{hom}} Pol(K_3)$

φ CONTINUOUS

(NO: BODIRSKY + P. + PONGRÁČ '13)

AXIOM OF DETERMINACY?

POLYMORPHISMS & TOPOLOGY

$I/A, I/B$ ω -CATEGORICAL

$Pol(A) \cong Pol(B) \stackrel{?}{\Rightarrow} Pol(A) \cong_{\text{top}} Pol(B)$

(NO: BODIRSKY + EVANS + KOMPATSKHER + P. '17)

POLYMORPHISMS FOR EVERYONE

POLYMORPHISMS & AC

$\mathcal{A}$ ω -CATEGORICAL

$\text{EQ}_{\text{LOCAL}}(\text{Pol}(\mathcal{A}))$ NON-TRIVIAL $\stackrel{?}{\Rightarrow}$

$\text{EQ}(\text{Pol}(\mathcal{A}))$ NON-TRIVIAL

STRENGTHENING:

$\forall \varphi: \text{Pol}(\mathcal{A}) \xrightarrow{\text{hom}} \text{Pol}(\mathcal{K}_3)$

φ CONTINUOUS

(NO: BODIRSKY + P. + PONGRÁČ '13)

AXIOM OF DETERMINACY?

POLYMORPHISMS & TOPOLOGY

$\mathcal{A}, \mathcal{B}$ ω -CATEGORICAL

$\text{Pol}(\mathcal{A}) \cong \text{Pol}(\mathcal{B}) \stackrel{?}{\Rightarrow} \text{Pol}(\mathcal{A}) \cong_{\text{TOP}} \text{Pol}(\mathcal{B})$

(NO: BODIRSKY + EVANS + KOMPATSKHER + P. '17)

POLYMORPHISMS & DESCRIPTIVE SET THEORY

$\{ \text{Pol}(\mathcal{A}) \mid \mathcal{A} \text{ } \omega\text{-CATEGORICAL} \}$

$\cong_{\text{TOP}}$ SMOOTH?

(PAOLINI, NIES + PAOLINI '24 AUT)

FUN FACT: DECIDABLE FOR FINITE $\mathcal{A}$

POLYMORPHISMS & SET THEORY

$\mathcal{A}$ FINITE

$$\varphi_{\mathcal{A}} \equiv \forall \mathcal{X} : \mathcal{X} \xrightarrow{\text{hom}} \mathcal{A} \Leftrightarrow \forall \mathcal{I} \subseteq^{\text{fin}} \mathcal{X} \left(\mathcal{I} \xrightarrow{\text{hom}} \mathcal{A} \right)$$



$\mathcal{A} = \mathcal{U}_3$: ERDŐS - DEBRAJN THEOREM

THEOREM (ZF) (KÁTAI, TÓTH, VIDNYÁNSZUK '24)

- $\text{EQ}^1(\text{Pol}(\mathcal{A}))$ TRIVIAL
 $\Rightarrow \varphi_{\mathcal{A}} \Leftrightarrow$ ULTRAFILTER LEMMA
- $\text{EQ}^1(\text{Pol}(\mathcal{A}))$ NON-TRIVIAL
 $\rightarrow \varphi_{\mathcal{A}}$ STRICTLY WEAKER

$\text{EQ}^1(\text{Pol}(\mathcal{A}))$ VERY NON-TRIVIAL $\Rightarrow ?$

$\mathcal{A}$ ω -CATEGORICAL $\Rightarrow ?$

POLYMORPHISMS & SET THEORY

$\mathcal{A}$ FINITE

$$\varphi_{\mathcal{A}} \equiv \forall \mathbb{X} : \mathbb{X} \xrightarrow{\text{hom}} \mathcal{A} \Leftrightarrow \forall IF \subseteq^{\text{fin}} \mathbb{X} \quad (IF \xrightarrow{\text{hom}} \mathcal{A})$$



$\mathcal{A} = \mathcal{U}_3$: ERDŐS - DEBRAJN THEOREM

THEOREM (ZF) (KÁTAI, TÓTH, VIDNYÁNSZUK '24)

- $\text{EQ}^1(\text{Pol}(\mathcal{A}))$ TRIVIAL
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- $\text{EQ}^1(\text{Pol}(\mathcal{A}))$ NON-TRIVIAL
 $\rightarrow \varphi_{\mathcal{A}}$ STRICTLY WEAKER

$\text{EQ}^1(\text{Pol}(\mathcal{A}))$ VERY NON-TRIVIAL $\Rightarrow ?$

$\mathcal{A}$ ω -CATEGORICAL $\Rightarrow ?$

POLYMORPHISMS & MODEL THEORY

$\mathcal{A}$ ω -CATEGORICAL

- $\text{Pol}(\mathcal{A}) \models \exists a, b, f \quad \forall x, y, z$
 $a \cdot f(x, y, x, z, y, z) = b \cdot f(y, x, z, x, z, x)$

AND

- $\text{EQ}^1(\text{Pol}(\mathcal{A}))$ TRIVIAL
LOCAL



$\text{Aut}(\mathcal{A}) \restriction \mathcal{A}^n$ HAS $\geq 2^{2^n}$ ORBITS
FOR ALL n

THANK YOU !



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