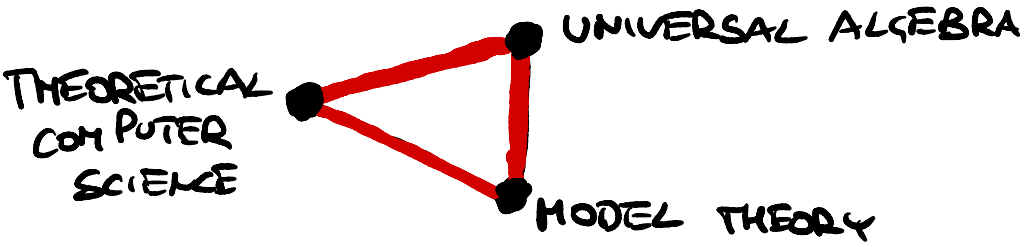
THE POWER OF POLYMORPHISMS



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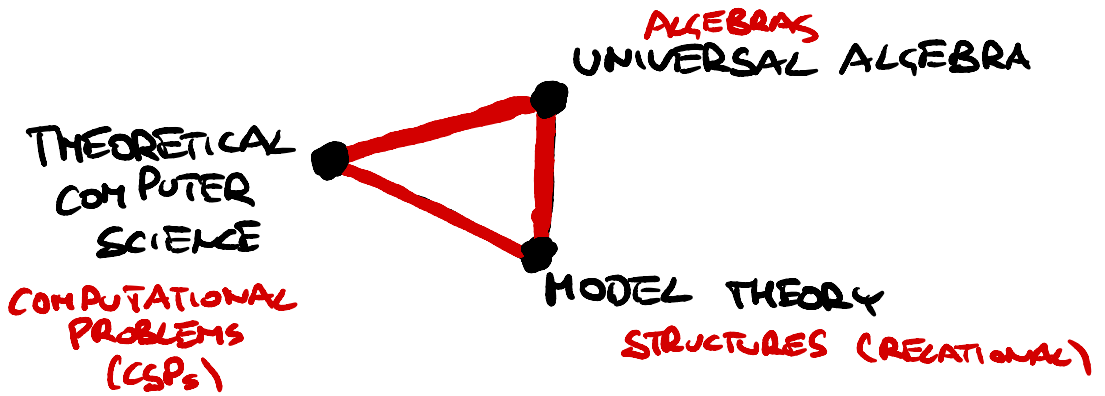

PART I THE MATHEMATICS OF FINITE-DOMAIN CSPs
ALGEBRAS
RELATIONAL STRUCTURES

PART II MODELLING PROBLEMS AS INFINITE-DOMAIN CSPs

PART III THE MATHEMATICS OF INFINITE-DOMAIN CSPs.
ALGEBRAS
RELATIONAL STRUCTURES

PART I : THE MATHEMATICS OF FINITE-DOMAIN CSPs

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES $x_1, \ldots, x_n$
- CONSTRAINTS ON THEM

QUESTION

- $\exists$ VALUES FOR $x_1 \ldots x_n$ SATISFYING ALL CONSTRAINTS ?

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- SUDOKU
- SCHEDULING
- SOLVING EQUATIONS

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EXAMPLE
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MODEL:

- SET V OF POSSIBLE VALUES (FIXED)
  E.G. $\{0,1\}$, $\{0,1,2\}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, ...
- ALLOWED CONSTRAINTS (FIXED)
  $C_1, \dots, C_m$ RELATIONS ON V
  $C_i \subseteq V^{d_i}$

SO $\mathbb{A} := (V, C_1, \dots, C_m)$ RELATIONAL STRUCTURE

CSP($\mathbb{A}$) = "TEMPLATE"

GIVEN
- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $C_{i_1}(\text{VARIABLES})$
  $\vdots$
  $C_{i_\ell}(\text{VARIABLES})$

QUESTION $\exists$ SOLUTION $x_i \mapsto a_i \in A$?

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ALTERNATIVE: GIVEN **PP-SENTENCE**
$\varphi = \exists x_1 \dots \exists x_n\, C_{i_1}(\text{VAR}) \wedge \dots \wedge C_{i_\ell}(\text{VAR})$
QUESTION: $\mathbb{A} \models \varphi$?

PP... PRIMITIVE POSITIVE

CONSTRAINT SATISFACTION PROBLEM CSP

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= "TEMPLATE"

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $C_{i_1}(\text{VARIABLES})$ ⋮ $C_{i_\ell}(\text{VARIABLES})$

QUESTION $\exists$ SOLUTION $x_i \mapsto a_i \in A$?

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QUESTION: $\mathbb{A} \models \varphi$?
PP... PRIMITIVE POSITIVE

META-QUESTION
FOR WHAT $\mathbb{A}$ IS CSP($\mathbb{A}$) EASY / HARD ?

P: $\exists$ ALGORITHM PROVIDING ANSWER IN $\wedge$(# VARIABLES) STEPS, $\wedge$ POLYNOMIAL
#
NP: NOT NECESSARILY IN P BUT VERIFYING "SOLUTION" IN P

EXAMPLE

TERNARY RELATIONS

- $\mathbb{A} = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1 \cdot x_1 = x_2$, $x_2 + x_1 = x_3$, $x_3 = 1$

QUESTION
- SOLUTION IN $\mathbb{Z}$?

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UNDECIDABLE (MATIYASEVIC '77)

HILBERT'S 10TH PROBLEM

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- SAME OVER $\mathbb{Z}_p$

NP-COMPLETE

↳ IF IN P $\Rightarrow$ P = NP

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IN P (GAUSS)

- $\mathbb{A} = (\mathbb{Q}, <)$

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $x_1 < x_2$, $x_2 < x_3$, $x_3 < x_1$, ⋮

QUESTION
- SOLUTION IN $\mathbb{Q}$?

IN P

EXAMPLE

TERNARY RELATIONS

- $\mathbb{A} = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

CSP($\mathbb{A}$)

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- VARIABLES $x_1, \dots, x_n$
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QUESTION
- SOLUTION IN $\mathbb{Z}$?

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IN P (GAUSS)

- $\mathbb{A} = (\mathbb{Q}, <)$

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $x_1 < x_2$, $x_2 < x_3$, $x_3 < x_1$, ⋮

QUESTION
- SOLUTION IN $\mathbb{Q}$?

IN P

- $\mathbb{A} = K_3 =$ $= (\{0,1,2\}, E)$

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $E(x_1, x_2)$, $E(x_2, x_3)$, ⋮

QUESTION
- SOLUTION IN $\{0,1,2\}$?

3-COLORING PROBLEM = NP-COMPLETE

ALTERNATIVE FORMULATION OF CSP(A):

GIVEN: FINITE $\mathbb{B}$

QUESTION: $\mathbb{B} \to \mathbb{A}$?, i.e.

$\exists h : \mathbb{B} \to \mathbb{A}$ HOMOMORPHISM?

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SAME PROBLEM:

- PP-SENTENCE $\varphi = \exists x_1 \cdots x_n \, C_{i_1}(\,) \wedge \cdots \wedge C_{i_\ell}(\,)$

  ⇝ STRUCTURE: DOMAIN $\{x_1 \cdots x_n\}$

  $\mathbb{B}_\varphi$ RELATIONS C_i GIVEN BY CONSTRAINTS

  "CANONICAL DATABASE"

  THEN $\mathbb{B}_\varphi \to \mathbb{A} \iff \mathbb{A} \models \varphi$

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- STRUCTURE $\mathbb{B} = (B, C_1, \ldots, C_n)$

  $B = \{b_1 \cdots b_\ell\}$

  ⇝ $\varphi_{\mathbb{B}} \equiv \exists b_1 \cdots b_\ell \bigwedge C_i(\,)$

  "CANONICAL QUERY OF $\mathbb{B}$"

ALTERNATIVE FORMULATION OF CSP(A):

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EXAMPLE $\mathbb{A} = K_2 =$

GIVEN DIGRAPH $\mathbb{B}$

QUESTION $\mathbb{B} \to K_2$?

2-COLORING PROBLEM

EXERCISE $CSP(\mathbb{Q}, <) = ?$

ALTERNATIVE FORMULATION OF CSP(𝔸):

GIVEN FINITE $\mathbb{B}$

QUESTION $\mathbb{B} \to \mathbb{A}$?, i.e.

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SAME PROBLEM:

- PP-SENTENCE $\varphi = \exists x_1 \cdots x_n \, C_{i_1}(\,) \wedge \cdots \wedge C_{i_\ell}(\,)$

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  $\{b_1 \cdots b_k\}$

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COMPUTATIONAL COMPLEXITY OF CSP(𝔸) CAN BE:

- UNDECIDABLE
- IN P
- NP-COMPLETE
- ANYTHING:

  EVERY COMPUTATIONAL PROBLEM IS POLYNOMIAL-TIME TURING-EQUIVALENT TO SOME CSP(𝔸)

$\mathbb{A}$ FINITE ⇒ CSP(𝔸) IN NP

THEOREM (BULATOV, ZHUK '17)
(CONJECTURE FEDER + VARDI '93)

$\mathbb{A}$ FINITE

$\Rightarrow$ $CSP(\mathbb{A})$ $\in$ P OR

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($P \neq NP$)

$CSP(\mathbb{A})$ $\in$ P $\Leftrightarrow$ $\mathbb{A}$ HAS ALGEBRAIC INVARIANT

$s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$

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ALGEBRAIC INVARIANTS = POLYMORPHISMS

$\mathbb{A} = (A; R_1, \dots, R_m)$ STRUCTURE

$f: A^k \to A$ **POLYMORPHISM** $:\Leftrightarrow$

f HOMOMORPHISM $\mathbb{A}^k \to \mathbb{A}$ $\Leftrightarrow$

$\forall i: \forall \bar{r}_1 \dots \bar{r}_k \in R_i$:

$f\left(\begin{pmatrix}\vdots\\ \\ \end{pmatrix} \dots \begin{pmatrix}\vdots\\ \\ \end{pmatrix}\right) \in R_i$

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$\mathbb{A}$ FINITE

$\Rightarrow$ $CSP(\mathbb{A}) \in P$ OR

NP-COMPLETE

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ALGEBRAIC INVARIANTS = POLYMORPHISMS

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f HOMOMORPHISM $\mathbb{A}^\ell \to \mathbb{A}$ $\Leftrightarrow$

$\forall i: \forall \bar{r}_1 \dots \bar{r}_\ell \in R_i:$

$f\begin{pmatrix} | & & | \\ \bar{r}_1 & \cdots & \bar{r}_\ell \\ | & & | \end{pmatrix} \in R_i$

$Pol(\mathbb{A}) := \{f \mid f \text{ POLYMORPHISM OF } \mathbb{A}\}$

- CONTAINS PROJECTIONS $(x_1, \dots x_\ell) \mapsto x_i$
- COMPOSITION-CLOSED

$\Rightarrow$ ESSENTIALLY TERM FUNCTIONS OF AN ALGEBRA ON A !

EXAMPLE

- $\min(x,y) : \mathbb{Q}^2 \to \mathbb{Q} \quad \in Pol(\mathbb{Q}, <)$

$\min\left(\begin{matrix} x_1 \\ \wedge \\ x_2 \end{matrix}, \begin{matrix} y_1 \\ \wedge \\ y_2 \end{matrix}\right) = \begin{matrix} \cdot \\ \wedge \\ \cdot \end{matrix}$

EXAMPLE

- $\min(x,y): \mathbb{Q}^2 \to \mathbb{Q} \quad \in Pol(\mathbb{Q}, <)$

  $\min\begin{pmatrix} x_1 \\ \wedge \\ x_2 \end{pmatrix}\begin{pmatrix} y_1 \\ \wedge \\ y_2 \end{pmatrix} = \begin{pmatrix} \wedge \end{pmatrix}$

- $(x,y,z) \mapsto x-y+z \in Pol(\mathbb{Z}_2, \{0\}, \{1\}, +)$
  - $0-0+0 = 0$
  - $1-1+1 = 1$
  - $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$

    $\in +\quad \in -\quad \in +\quad \in -$

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

EXAMPLE

- $\min(x,y): \mathbb{Q}^2 \to \mathbb{Q} \quad \in Pol(\mathbb{Q}, <)$

- $(x,y,z) \mapsto x - y + z \in Pol(\mathbb{Z}_p, \mathbb{Q}, \mathbb{R}, \dot{-})$
  - $0 - 0 + 0 = 0$
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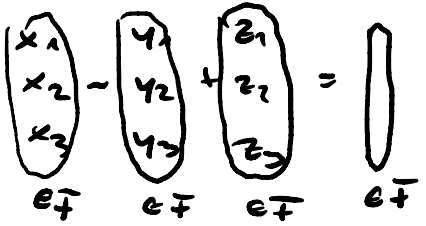
  - 

SOLUTIONS TO CSP-INSTANCE: AFFINE SPACE!

- PROJECTIONS, AUTOMORPHISMS $\in Pol(K_3)$

↑ NOT GREAT ☹

EXAMPLE

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- $(x,y,z) \mapsto x-y+z \in Pol(\mathbb{Z}_p, \{0\}, \{1\}, -+)$
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(BABY) THEOREM

(BODNARČUK + KALUŽNIN + KOTOV + ROMOV '69, GEIGER '68)

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$Pol(\mathbb{A}) \subseteq Pol(\mathbb{B}) \overset{(a)}{\Rightarrow} \mathbb{A}$ PP-DEFINES $\mathbb{B}$

$\overset{(a)}{\Rightarrow}$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

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- $\min(x,y): \mathbb{Q}^2 \to \mathbb{Q} \quad \in Pol(\mathbb{Q}, <)$

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$\in +$ $\in -$ $\in +$ $\in +$

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$\overset{②}{\Rightarrow} CSP(\mathbb{B})$ REDUCES TO $CSP(\mathbb{A})$

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③ **PROOF**:

LET $\varphi \equiv \exists x_1 \dots x_n\, C_{i_1}(\,) \wedge \dots \wedge C_{i_\ell}(\,)$ BE AN INSTANCE OF $CSP(\mathbb{B})$

REPLACE EACH C_{i_j} BY ITS PP-DEFINITION IN $\mathbb{A}$ $\rightsquigarrow \tilde{\varphi}$!

$$\mathbb{A} \models \tilde{\varphi} \iff \mathbb{B} \models \varphi$$

EXAMPLE

- $\min(x,y): \mathbb{Q}^2 \to \mathbb{Q} \quad \in Pol(\mathbb{Q}, <)$

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---

① **PROOF**:

LET $\varphi \equiv \exists x_1 \dots x_n\, C_{i_1}(\,) \wedge \dots \wedge C_{i_\ell}(\,)$ BE AN INSTANCE OF $CSP(\mathbb{B})$

REPLACE EACH C_{i_j} BY ITS PP-DEFINITION IN $\mathbb{A}$ $\rightsquigarrow \tilde{\varphi}$!

$\mathbb{A} \models \tilde{\varphi} \iff \mathbb{B} \models \varphi$

② **PROOF**:

LET R BE RELATION

SHOW: R INVARIANT UNDER $Pol(\mathbb{A})$

$\iff$

R PP-DEFINABLE IN $\mathbb{A}$

R INVARIANT UNDER $\mathrm{Pol}(\mathbb{A})$

$\Leftrightarrow$

R PP-DEFINABLE IN $\mathbb{A}$

R INVARIANT UNDER $\mathrm{Pol}(\mathbb{A})$

$\Leftrightarrow$

R PP-DEFINABLE IN $\mathbb{A}$

$\Leftarrow$ " EASY : $S(x_1,\dots x_\ell)$, $T(y_1\dots,y_n)$ INVARIANT

$\Rightarrow S \wedge T$ INVARIANT,

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R INVARIANT UNDER $Pol(\mathbb{A})$

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R PP-DEFINABLE IN $\mathbb{A}$

$\Leftarrow$ "EASY": $S(x_1,\dots x_k)$, $T(y_1\dots y_n)$ INVARIANT
$\Rightarrow$ $S \wedge T$ INVARIANT,
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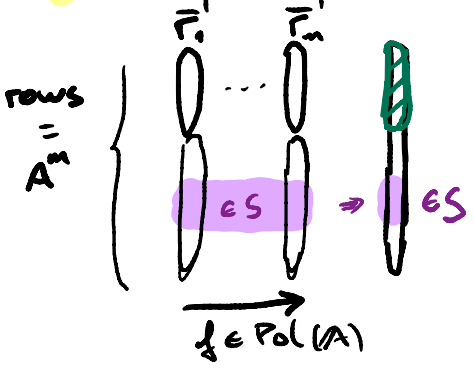
$\Rightarrow$ LET $R = \{\bar{r}_1 \dots \bar{r}_m\}$ ($\mathbb{A}$ FINITE!)



$R' := \{f(\bar{r}_1' \dots \bar{r}_m') \mid f \in Pol(\mathbb{A})\}$

R' IS $\{\wedge, =\}$-DEFINABLE!

R = PROJECTION OF R' ONTO FIRST COORDINATES!

□

R INVARIANT UNDER Pol(A)

$\Leftrightarrow$

R PP-DEFINABLE IN A

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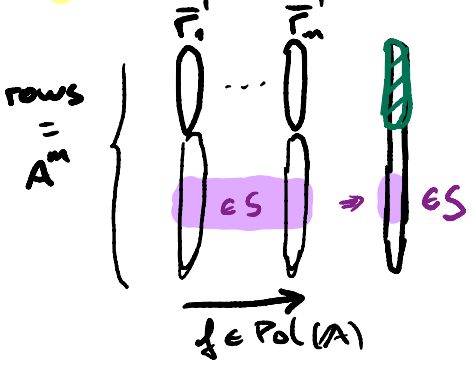

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(BABY) THEOREM (BODNARČUK + KALUŽNIN + KOTOV ROMOV '66, GEIGER '68)

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$Pol(\mathbb{A}) \subseteq Pol(\mathbb{B}) \overset{①}{\Rightarrow}$ $\mathbb{A}$ PP-DEFINES $\mathbb{B}$

$\overset{②}{\Rightarrow}$ CSP($\mathbb{B}$) REDUCES TO CSP($\mathbb{A}$)

R INVARIANT UNDER Pol(𝔸)

⇔

R PP-DEFINABLE IN 𝔸

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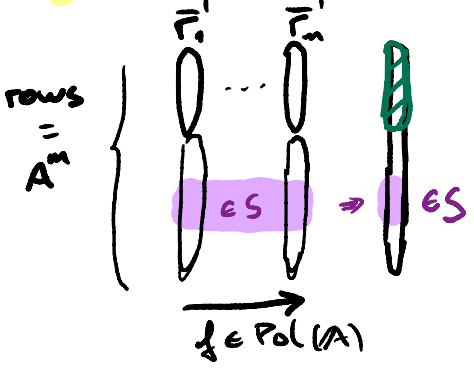

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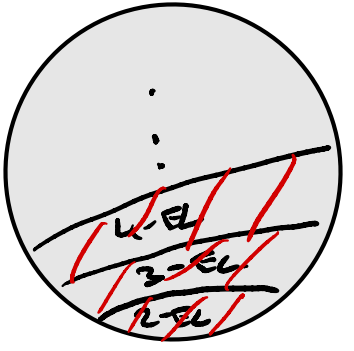
POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-DEFINABILITY



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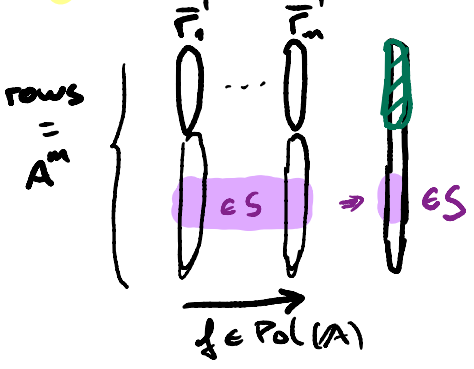

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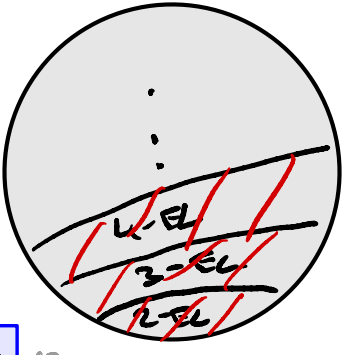

THEOREM (POST '41)

$\mathbb{A}$ BOOLEAN $\Rightarrow Pol(\mathbb{A})$ CONTAINS:

- CONSTANT
- MAX
- MIN
- MAJ
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OR $Pol(\mathbb{A}) = \{$"UNARY" INJECTIONS$\}$

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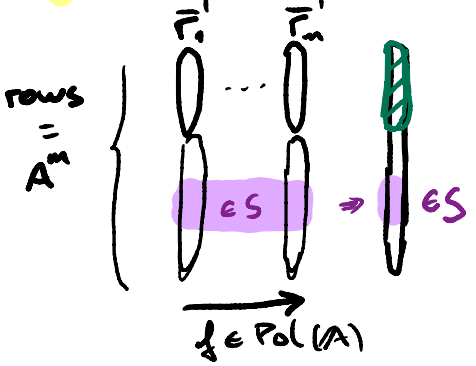

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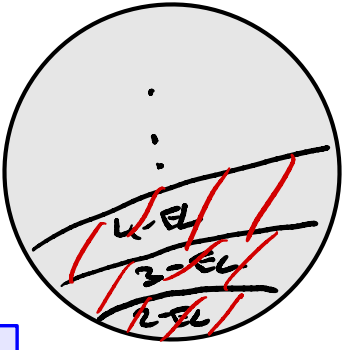

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THEOREM (SCHAEFER '78) COMPLEXITY DICHOTOMY

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ALGORITHMS

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  $x - y + z$!

  INVARIANT RELATIONS AFFINE OVER $\mathbb{Z}_2$

  GAUSS ALGORITHM

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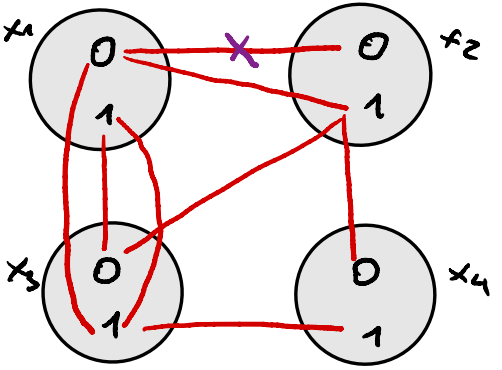
LOCAL CONSISTENCY ALGORITHM
(2,3)- MINIMALITY "

- KEEP LISTS OF POSSIBLE VALUES FOR PAIRS OF VARIABLES (EDGES)
- REMOVE EDGES LOOKING AT CONSISTENCY OF THE LISTS FOR 3 VARIABLES AT A TIME



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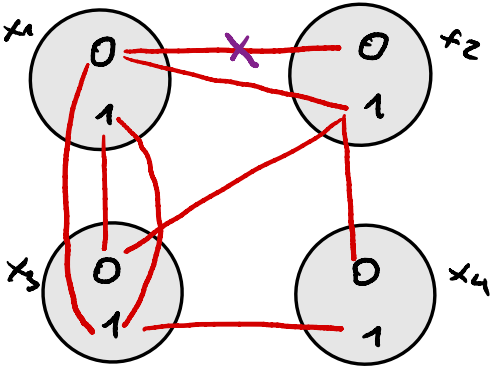


EXERCISE

SHOW LOCAL CONSISTENCY SOLVES CSP CORRECTLY IF SOLUTIONS INVARIANT UNDER MAJORITY

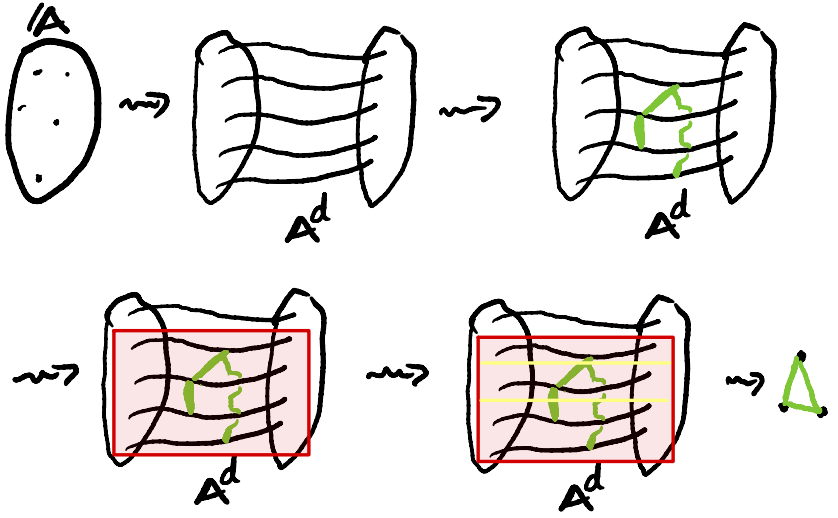
PP-DEFINABILITY REQUIRES EQUAL DOMAINS

PP-DEFINABILITY REQUIRES EQUAL DOMAIN!

DEFINITION

$\mathbb{A}$ **PP-INTERPRETS** $\mathbb{B}$ IF

$\mathbb{B}$ IS A PP-DEFINABLE FACTOR
OF PP-DEFINABLE SUBSTRUCTURE
OF PP-DEFINABLE STRUCTURE
ON FINITE POWER OF $\mathbb{A}$



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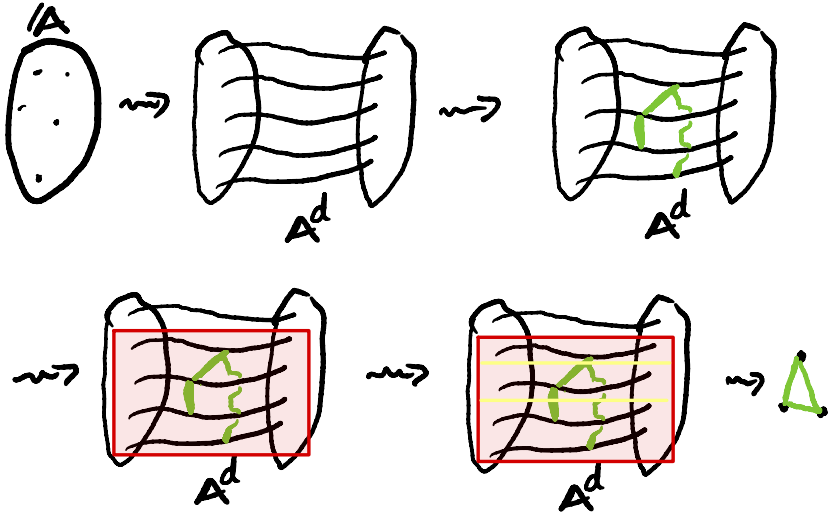

EXAMPLE

$(\mathbb{Z},+,\cdot,0,1) \rightsquigarrow (\mathbb{Z}^2,\tilde{+},\cdot,(0,0),(1,1)) \rightsquigarrow$

$\rightsquigarrow (\mathbb{Z}\times(\mathbb{Z}\setminus\{0\}),\tilde{+},\cdot,(0,1),(1,1))$

$\rightsquigarrow (\mathbb{Q},+,\cdot,0,1)$

PP-DEFINABILITY REQUIRES EQUAL DOMAINS

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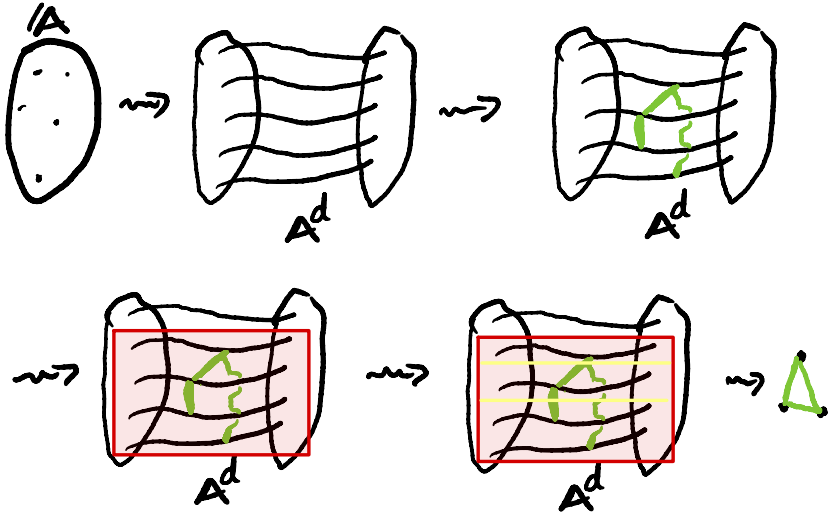

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EXERCISE

- SHOW THIS IS A PP-INTERPRETATION
- PP-INTERPRET $(\mathbb{Q},<)$

PP-DEFINABILITY REQUIRES EQUAL DOMAIN!

DEFINITION

$\mathbb{A}$ **PP-INTERPRETS** $\mathbb{B}$ $\Leftrightarrow$

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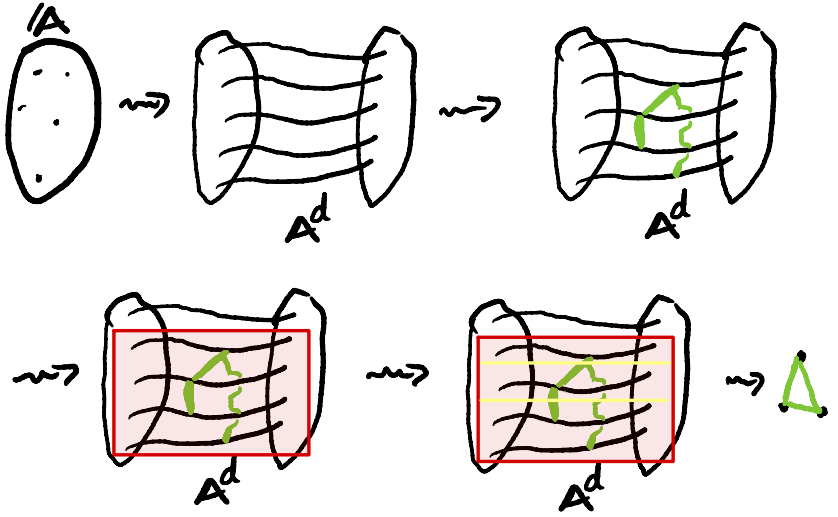

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POLYMORPHISMS & PP-DEFINITIONS ✓

PP-INTERPRETATIONS ?

PP-DEFINABILITY REQUIRES EQUAL DOMAIN!

DEFINITION

$\mathbb{A}$ **PP-INTERPRETS** $\mathbb{B}$ ⇔
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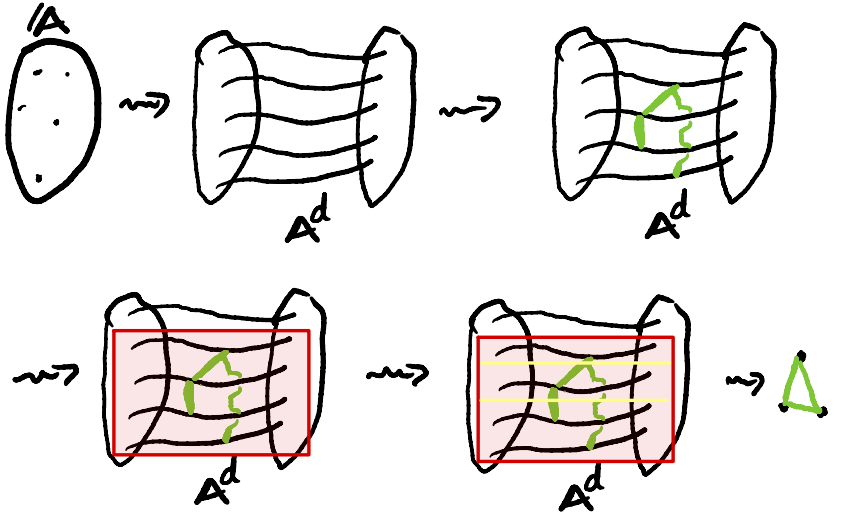

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POLYMORPHISMS & PP-DEFINITIONS ✓
PP-INTERPRETATIONS ?

EQUATIONAL CONDITIONS:

- $\exists f \in Pol(\mathbb{Q},<)\ \forall x,y \quad f(x,y)=f(y,x)$
- $\exists f \in Pol(\mathbb{Z},\{0\},\{1\},+)$
  $\forall x,y,z \quad f(x,x,y)=f(y,x,x)=y$
- $\exists f \in Pol(\mathbb{Z}_3) : \forall x \quad fff(x)=x$

DEFINITION

- EQUATIONAL ("STRONG MALCEV") CONDITION:

  SENTENCE

$$\Phi \equiv \exists f_1 \exists f_2 \cdots \forall x_1 \forall x_2 \cdots$$

$$\begin{aligned} & s_1(\text{variables}) = t_1(\text{variables}) \\ \wedge & \\ & \vdots \\ \wedge & \; s_n(\text{variables}) = t_n(\text{variables}) \end{aligned}$$

WHERE s_i, t_i ... TERMS OVER $f_1, f_2, \ldots$

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- Ψ TRIVIAL $:\Leftrightarrow$ $\mathrm{Pol}(\text{ANY STRUCTURE}) \models \Psi$

  $\Leftrightarrow$ Ψ SATISFIABLE BY PROJECTIONS

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(ADOLESCENT) THEOREM

(BULATOV + JEAVONS + KROKHIN '00)

$$\mathrm{EQ}(\mathrm{Pol}(\mathbb{A})) \subseteq \mathrm{EQ}(\mathrm{Pol}(\mathbb{B}))$$

$\Rightarrow$ $\mathbb{A}$ PP-INTERPRETS $\mathbb{B}$

$\Rightarrow$ $\mathrm{CSP}(\mathbb{B})$ REDUCES TO $\mathrm{CSP}(\mathbb{A})$

DEFINITION

- EQUATIONAL ("STRONG MALCEV") CONDITION:

  SENTENCE

  $$\Psi = \exists f_1 \exists f_2 \cdots \forall x_1 \forall x_2 \cdots$$
  $$s_1(\text{variables}) = t_1(\text{variables})$$
  $$\wedge \quad \vdots$$
  $$\wedge \; s_n(\text{variables}) = t_n(\text{variables})$$

  WHERE s_i, t_i ... TERMS OVER $f_1, f_2, \ldots$

- Ψ TRIVIAL $:\Leftrightarrow$ $\text{Pol}(\text{ANY STRUCTURE}) \models \Psi$

  $\Leftrightarrow$ Ψ SATISFIABLE BY PROJECTIONS

- $\text{EQ}(\text{Pol}(\mathbb{A})) := \{\Psi \mid \text{Pol}(\mathbb{A}) \models \Psi\}$

(ADOLESCENT) THEOREM (BULATOV + JEAVONS + KROKHIN '00)

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PROOF: AS WITH PP-DEFINITIONS

GIVEN INSTANCE OF $\text{CSP}(\mathbb{B})$:

$$\varphi = \exists x_1 \ldots x_n \, C_{i_1}(\;) \wedge \ldots \wedge C_{i_m}(\;)$$

- FOR EACH OCCURENCE OF x_i CREATE NEW VARIABLES $x_i^1 \ldots x_i^d$ (d ... POWER)
- ADD CONSTRAINTS THAT ALL TUPLES $(x_i^1 \ldots x_i^d)$ BE EQUIVALENT
- ADD CONSTRAINTS RESTRICTING $(x_i^1 \ldots x_i^d)$ TO SUBSET
- TRANSLATE CONSTRAINTS ON x_i INTO CONSTRAINTS ON x_i^j

EXERCISE CHECK THIS WORKS

- PROOF SKETCH OF:

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  preserving equations (compactness)

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$\mathbb{A}, \mathbb{B}$ RELATIONAL STRUCTURES

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CLONE HOMOMORPHISM :$\Leftrightarrow$

- φ PRESERVES ARITIES
- φ PROJECTIONS
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POLYMORPHISMS ALLOW FACTORING OF THE FINITE WORLD BY PP-~~DEFINABILITY~~ INTERPRETABILITY

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$CSP(\{0,1\}, R)$ IS 1-IN-3SAT (NP-COMPLETE)

$Pol(\{0,1\}, R) = \{f : f \text{ PROJECTION}\}$

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$Pol(\mathbb{A})$ FOR ANY $\mathbb{A}$!

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STRONGEST WRT PP-INTERPRETATIONS!

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$CSP(K_3)$ IS 3-COLORING

CLAIM $\forall f(x_1 \dots x_n) \in Pol(K_3)$

$\exists i \; \exists g \in Aut(K_3)$

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COROLLARY

$\varphi : Pol(K_3) \to Pol(\{0,1\}, R)$

$f \mapsto i\text{-th PROJ}$

CLONE HOMOMORPHISM

$\Rightarrow$ K_3 PP-INTERPRETS $(\{0,1\}, R)$ AND VICE-VERSA

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PROOF

CONSIDER $h(x) := f(x, \dots, x) \in Aut(\mathbb{K}_3)$

f, $h \circ f$, OR $h^2 \circ f$ IS IDEMPOTENT

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SHOW $\exists i \ g(x_1, \dots x_n) = x_i$

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SIMILARLY $\{1,3\}, \{1,2\}$

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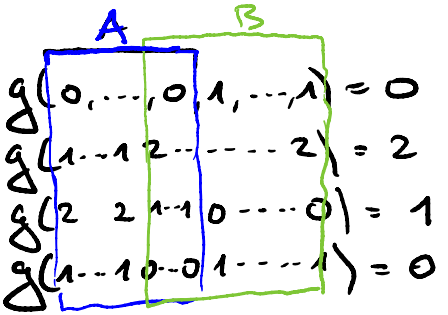
- $\forall A$ (A STRONG OR $\{1,\dots,n\}\setminus A$ STRONG)
- $\forall A,B$ A,B STRONG $\Rightarrow A\cap B$ STRONG



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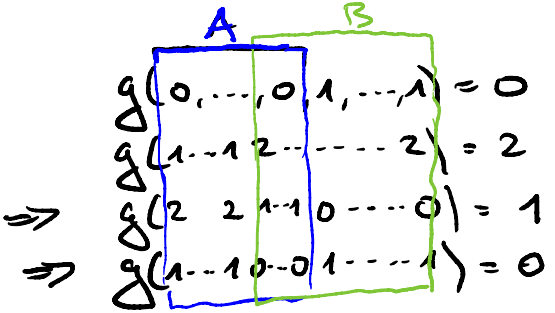

STRONG SUBSETS ULTRAFILTER

$\Rightarrow \exists i$ $\{i\}$ STRONG WLOG $i=1$

$g(0\cdots 0\ 1\cdots 1\ 2\cdots 2) = 0$

$g(1\cdots 1\ 0\cdots 0) = 1$

$g(2\cdots 2\ 0\cdots 0) = 2$

□

PP-INTERPRETATIONS INSUFFICIENT

... TO EXPLAIN ALL NP-HARDNESS

BY 1-IN-3SAT

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EXAMPLE

$\mathbb{A} = K_3 \dot{\cup} K_3$

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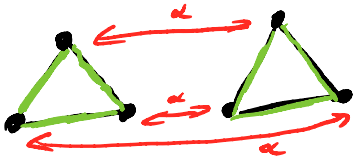

LET $S(x,y,z) = \begin{cases} z, & x,y \text{ IN SAME } K_3 \\ x, & \text{OTHERWISE} \end{cases}$

LET $\alpha(x)$ FLIP THE TWO COPIES

THEN $\alpha, S \in \mathrm{Pol}(\mathbb{A})$

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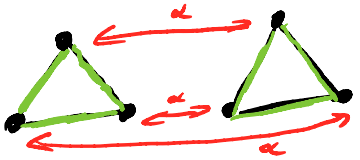

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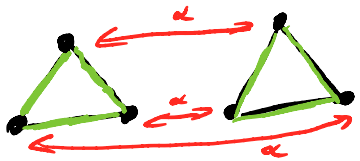

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HOMOMORPHICALLY EQUIVALENT :$\Leftrightarrow$
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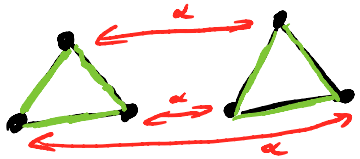

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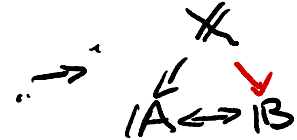

$\Leftarrow$ $\mathbb{A} \in \mathrm{CSP}(\mathbb{A}) \Rightarrow \mathbb{A} \to \mathbb{B}$

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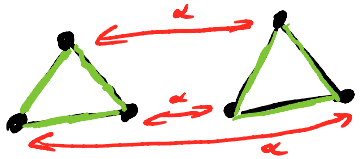

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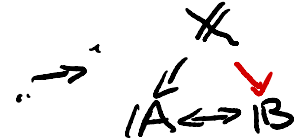

"$\Leftarrow$" $\mathbb{A} \in \mathrm{CSP}(\mathbb{A}) \Rightarrow \mathbb{A} \to \mathbb{B}$

- H.E. ALLOWS A DIFFERENT FACTORING OF THE WORLD

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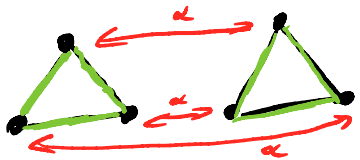
$\mathbb{A} = K_3 \dot{\cup} K_3$



LET $s(x,y,z) = \begin{cases} z, & x,y \text{ IN SAME } K_3 \\ x, & \text{OTHERWISE} \end{cases}$

LET $\alpha(x)$ FLIP THE TWO COPIES

THEN $\alpha, s \in \mathrm{Pol}(\mathbb{A})$

AND $s(x,x,y) = y = s(y,\alpha y,x)$

$\Rightarrow \mathrm{Pol}(\mathbb{A}) \not\to \mathrm{Pol}(K_3)$

BUT $\mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(K_3)$!

SINCE $\mathbb{A} \overset{\text{hom}}{\Rightarrow} K_3$, $K_3 \overset{\text{hom}}{\Rightarrow} \mathbb{A}$

DEFINITION

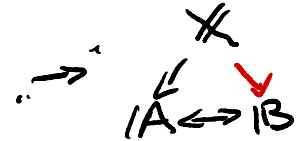
$\mathbb{A}, \mathbb{B}$ RELATIONAL STRUCTURES

HOMOMORPHICALLY EQUIVALENT :$\Leftrightarrow$

$\exists h: \mathbb{A} \to \mathbb{B}$ HOMOMORPHISM & VICE-VERSA

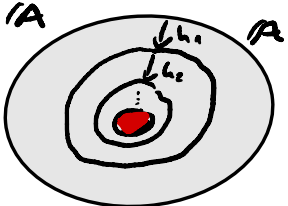
$\mathbb{A}, \mathbb{B}$ H.E. $\Rightarrow \mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(\mathbb{B})$

PROOF



"$\Leftarrow$" $\mathbb{A} \in \mathrm{CSP}(\mathbb{A}) \Rightarrow \mathbb{A} \to \mathbb{B}$

- H.E. ALLOWS A DIFFERENT FACTORING OF THE WORLD
- EACH EQUIVALENCE CLASS HAS UNIQUE SMALLEST REPRESENTATIVE: THE CORE OF $\mathbb{A}$



DEFINITION & THEOREM

$\mathbb{A}$ **PP-CONSTRUCTS** $\mathbb{B} :\Leftrightarrow$

$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathbb{C}_2 \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_3 \rightsquigarrow \ldots \rightsquigarrow \mathbb{B}$

DEFINITION & THEOREM

$\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B} :\Leftrightarrow$

$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathbb{C}_2 \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_3 \rightsquigarrow \dots \rightsquigarrow \mathbb{B}$

$\Leftrightarrow$

$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C} \overset{\text{H.E.}}{\rightsquigarrow} \mathbb{B}$

DEFINITION & THEOREM

$\mathbb{A}$ **PP-CONSTRUCTS** $\mathbb{B} :\Leftrightarrow$

$$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathfrak{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathfrak{C}_2 \overset{\text{PP-INT}}{\rightsquigarrow} \mathfrak{C}_3 \rightsquigarrow \dots \rightsquigarrow \mathbb{B}$$

$$\Leftrightarrow$$

$$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathfrak{C} \overset{\text{H.E.}}{\rightsquigarrow} \mathbb{B}$$

$$\Leftrightarrow$$

$$\mathbb{A} \overset{\text{PP-DEF.}}{\rightsquigarrow} \mathfrak{C} = (A^d, R_1 \dots R_n) \overset{\text{H.E.}}{\rightsquigarrow} \mathbb{B}$$

DEFINITION & THEOREM

$\mathbb{A}$ **PP-CONSTRUCTS** $\mathbb{B}$:$\Leftrightarrow$

$$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathbb{C}_2 \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_3 \rightsquigarrow \cdots \rightsquigarrow \mathbb{B}$$

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$$\Leftrightarrow$$

$\exists \varphi : \mathrm{Pol}(A) \to \mathrm{Pol}(B)$ **MINION HOM:**

- PRESERVES ARITIES
- PRESERVES MINORS

$$\varphi(f(x,\cdots,x,y\cdots y)) = \varphi(f)(x\cdots x\, y\cdots y)$$

DEFINITION & THEOREM

$\mathbb{A}$ **PP-CONSTRUCTS** $\mathbb{B}$:$\Leftrightarrow$

$$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathbb{C}_2 \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_3 \rightsquigarrow \dots \rightsquigarrow \mathbb{B}$$

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$\Leftrightarrow$

ALL HEIGHT 1 EQUATIONS SATISFIED IN $\mathrm{Pol}(\mathbb{A})$ ARE SATISFIED IN $\mathrm{Pol}(\mathbb{B})$

DEFINITION & THEOREM

$\mathbb{A}$ **PP-CONSTRUCTS** $\mathbb{B}$:$\Leftrightarrow$

$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathbb{C}_2 \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_3 \rightsquigarrow \dots \rightsquigarrow \mathbb{B}$

$\Leftrightarrow$

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$\Leftrightarrow$

ALL HEIGHT 1 EQUATIONS SATISFIED IN $\text{Pol}(\mathbb{A})$ ARE SATISFIED IN $\text{Pol}(\mathbb{B})$

EXAMPLE

HEIGHT 1 :

- $m(x,x,y) = m(x,y,x) = m(y,x,x)$
- $c(x_1 \dots x_n) = c(x_2 \dots x_n, x_1)$

NOT HEIGHT 1 :

- $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$
- $s(x,x,y) = y = s(y,x,y)$

DEFINITION & THEOREM

$\mathbb{A}$ **PP-CONSTRUCTS** $\mathbb{B}$:$\Leftrightarrow$

$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_1 \overset{\text{H.E}}{\rightsquigarrow} \mathbb{C}_2 \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C}_3 \rightsquigarrow \dots \rightsquigarrow \mathbb{B}$

$\Leftrightarrow$

$\mathbb{A} \overset{\text{PP-INT}}{\rightsquigarrow} \mathbb{C} \overset{\text{H.E.}}{\rightsquigarrow} \mathbb{B}$

$\Leftrightarrow$

$\mathbb{A} \overset{\text{PP-DEF.}}{\rightsquigarrow} \mathbb{C} = (A^d, R_1 \dots R_n) \overset{\text{H.E.}}{\rightsquigarrow} \mathbb{B}$

$\Leftrightarrow$

$\exists \varphi : \mathrm{Pol}(A) \to \mathrm{Pol}(B)$ **MINION HOM:**

- PRESERVES ARITIES
- PRESERVES MINORS

$$\varphi(f(x,\dots,x,y\dots y)) = \varphi(f)(x\dots x\, y\dots y)$$

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ALL HEIGHT 1 EQUATIONS SATISFIED IN $\mathrm{Pol}(\mathbb{A})$ ARE SATISFIED IN $\mathrm{Pol}(\mathbb{B})$

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- $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$
- $s(x,x,y) = y = s(y,x,y)$

EXAMPLE

$\mathbb{A} = K_3 \dot\cup K_3$ PP-CONSTRUCTS K_3

PP-~~INTERPRETS~~

$f \in \mathrm{Pol}(\mathbb{A}) \Rightarrow f|_{K_3}$ IS ESSENTIALLY UNARY

$\Rightarrow \exists$ MINION HOMOMORPHISM $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(K_3)$

(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

EQ^1 ... HEIGHT 1 - CONDITIONS

$f_i(\text{variables}) = g_j(\text{variables})$

~~$f_i \neq g_j$~~ ~~$f_i = g_j$~~

$EQ^1(Pol(\mathbb{A})) \subseteq EQ^1(Pol(\mathbb{B}))$

$\Rightarrow$ $\mathbb{A}$ pp-constructs $\mathbb{B}$

$\Rightarrow$ $CSP(\mathbb{A})$ harder than $CSP(\mathbb{B})$

(ADULT) THEOREM [BARTO + OPRŠAL + P. 16]

EQ^1 ... HEIGHT 1 - CONDITIONS

$f_i(\text{variables}) = f_j(\text{variables})$

~~$f_i \neq f_j$~~ ~~$f_i = f_j$~~

$EQ^1(Pol(\mathbb{A})) \subseteq EQ^1(Pol(\mathbb{B}))$

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$

$\Rightarrow$ $CSP(\mathbb{A})$ HARDER THAN $CSP(\mathbb{B})$

COROLLARY

$EQ^1(Pol(\mathbb{A}))$ TRIVIAL

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERYTHING

$\Rightarrow$ $CSP(\mathbb{A})$ NP-COMPLETE

(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

EQ^1 ... HEIGHT 1 - CONDITIONS

$f_i(\text{variables}) = f_j(\text{variables})$

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COROLLARY

$EQ^1(Pol(\mathbb{A}))$ TRIVIAL

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERYTHING

$\Rightarrow$ $CSP(\mathbb{A})$ NP-COMPLETE

THEOREM $EQ^1(Pol(\mathbb{A}))$ NON-TRIVIAL $\Rightarrow$

(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

EQ^1 ... HEIGHT 1 - CONDITIONS

$f_i(\text{variables}) = f_j(\text{variables})$

~~$i = j$~~ ~~$f_i \neq f_j$~~

$EQ^1(Pol(\mathbb{A})) \leq EQ^1(Pol(\mathbb{B}))$

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$

$\Rightarrow$ $CSP(\mathbb{A})$ HARDER THAN $CSP(\mathbb{B})$

COROLLARY

$EQ^1(Pol(\mathbb{A}))$ TRIVIAL

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERYTHING

$\Rightarrow$ $CSP(\mathbb{A})$ NP-COMPLETE

THEOREM $EQ^1(Pol(\mathbb{A}))$ NON-TRIVIAL $\Rightarrow$

$Pol(\mathbb{A}) \models \exists f \forall x,y,z$ $f(x,y,x,z,y,z) = f(y,x,z,x,z,y)$

6-ARY SIGGERS (SIGGERS '10)

(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

EQ^1 ... HEIGHT 1 - CONDITIONS

$f_i(\text{variables}) = f_j(\text{variables})$

~~$i = j$~~ ~~$f_i = f_j$~~

$EQ^1(Pol(\mathbb{A})) \leq EQ^1(Pol(\mathbb{B}))$

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$

$\Rightarrow$ $CSP(\mathbb{A})$ HARDER THAN $CSP(\mathbb{B})$

COROLLARY

$EQ^1(Pol(\mathbb{A}))$ TRIVIAL

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERYTHING

$\Rightarrow$ $CSP(\mathbb{A})$ NP-COMPLETE

THEOREM $EQ^1(Pol(\mathbb{A}))$ NON-TRIVIAL $\Rightarrow$

$Pol(\mathbb{A}) \models \exists f\, \forall x,y,z$ $f(x,y,x,z,y,z) = f(y,x,z,x,z,y)$

6-ARY SIGGERS

(SIGGERS '80)

$\models \exists f\, \forall a,r,e$ $f(a,r,e,a) = f(r,a,r,e)$

4-ARY SIGGERS

(KEARNES + MARKOVIĆ + MCKENZIE '16)

(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

EQ^1 ... HEIGHT 1 - CONDITIONS

$f_i(\text{variables}) = f_j(\text{variables})$

~~$f_i = f_j$~~ ~~$f_i = x$~~

$EQ^1(Pol(\mathbb{A})) \leq EQ^1(Pol(\mathbb{B}))$

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$

$\Rightarrow$ $CSP(\mathbb{A})$ HARDER THAN $CSP(\mathbb{B})$

COROLLARY

$EQ^1(Pol(\mathbb{A}))$ TRIVIAL

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERYTHING

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THEOREM $EQ^1(Pol(\mathbb{A}))$ NON-TRIVIAL $\Rightarrow$

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6-ARY SIGGERS (SIGGERS '80)

$\models \exists f\, \forall a,r,e$ $f(a,r,e,a) = f(r,a,r,e)$

4-ARY SIGGERS (KEARNES + MARKOVIĆ + MCKENZIE '16)

$\models \exists f\, \forall x_1 \dots x_n$

$f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$

$\forall\, n \geq |A|$ PRIME

CYCLIC (BARTO + KOZIK '11)

(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

EQ^1 ... HEIGHT 1 - CONDITIONS

$f_i(\text{variables}) = f_j(\text{variables})$

~~$f_i = f_j$~~ ~~$f_i \neq f_j$~~

$EQ^1(Pol(\mathbb{A})) \subseteq EQ^1(Pol(\mathbb{B}))$

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$

$\Rightarrow$ $CSP(\mathbb{A})$ HARDER THAN $CSP(\mathbb{B})$

COROLLARY

$EQ^1(Pol(\mathbb{A}))$ TRIVIAL

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERYTHING

$\Rightarrow$ $CSP(\mathbb{A})$ NP-COMPLETE

THEOREM

$EQ^1(Pol(\mathbb{A}))$ NON-TRIVIAL $\Rightarrow$

$Pol(\mathbb{A}) \models \exists f\, \forall x,y,z$ $f(x,y,x,z,y,z) = f(y,x,z,x,z,y)$

6-ARY SIGGERS (SIGGERS '80)

$\models \exists f\, \forall a,r,e$ $f(a,r,e,a) = f(r,a,r,e)$

4-ARY SIGGERS (KEARNES + MARKOVIĆ + MCKENZIE '16)

$\models \exists f\, \forall x_1 \dots x_n$

$f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$

$\forall n \geq |A|$ PRIME

CYCLIC (BARTO + KOZIK '11)

$\models \exists f\, \forall x,y$ $f(x \dots x_y) = \dots = f(yx \dots x)$

$\forall n \geq |A|$ PRIME

WNU (MARÓTI + MCKENZIE '08)

(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

EQ⁺ ... HEIGHT 1 - CONDITIONS

$f_i(\text{variables}) = f_j(\text{variables})$

~~$f_i = f_j$~~ ~~$f_i = x$~~

$EQ^+(Pol(\mathbb{A})) \subseteq EQ^+(Pol(\mathbb{B}))$

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS $\mathbb{B}$

$\Rightarrow$ $CSP(\mathbb{A})$ HARDER THAN $CSP(\mathbb{B})$

COROLLARY

$EQ^+(Pol(\mathbb{A}))$ TRIVIAL

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERYTHING

$\Rightarrow$ $CSP(\mathbb{A})$ NP-COMPLETE

THEOREM $EQ^+(Pol(\mathbb{A}))$ NON-TRIVIAL $\Rightarrow$

$Pol(\mathbb{A}) \models \exists f\, \forall x,y,z$ $f(x,y,x,z,y,z) = f(y,x,z,x,z,y)$

6-ARY SIGGERS (SIGGERS '10)

$\models \exists f\, \forall a,r,e$ $f(a,r,e,a) = f(r,a,r,e)$

4-ARY SIGGERS (KEARNES + MARKOVIĆ + MCKENZIE '16)

$\models \exists f\, \forall x_1 \dots x_n$

$f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$

$\forall n \geq |A|$ PRIME

CYCLIC (BARTO + KOZIK '11)

$\models \exists f\, \forall x,y$ $f(x \dots x y) = \dots = f(y x \dots x)$

$\forall n \geq |A|$ PRIME

WNU (MARÓTI + MCKENZIE '08)

$\Rightarrow CSP(\mathbb{A}) \in P$ (BULATOV, ZHUK '17)

THEOREM $\mathrm{EQ}^+(\mathrm{Pol}(\mathbb{A}))$ NON-TRIVIAL

$\Rightarrow \mathrm{CSP}(\mathbb{A}) \in \mathrm{P}$ (BULATOV, ZHUK '17)

THEOREM $EQ^+(Pol(\mathbb{A}))$ NON-TRIVIAL

$\Rightarrow CSP(\mathbb{A}) \in P$ (BULATOV, ZHUK '17)

ALGORITHM

REDUCES TO SMALLER STRUCTURE
BY SOLVING LINEAR EQUATIONS
OR LOCAL CONSISTENCY CHECKING

THEOREM $EQ^+(Pol(\mathbb{A}))$ NON-TRIVIAL

$\Rightarrow CSP(\mathbb{A}) \in P$ (BULATOV, ZHUK '17)

ALGORITHM

REDUCES TO SMALLER STRUCTURE
BY SOLVING LINEAR EQUATIONS
OR LOCAL CONSISTENCY CHECKING

THEOREM (BARTO + BODOR + LICHER + MOTTET + P. '23)

$EQ(Pol(\mathbb{A}))$ NON-TRIVIAL,
$\mathbb{A}$ CORE

$\Rightarrow Pol(\mathbb{A}) \models \exists f \exists \alpha_1 \dots \alpha_m \forall x,y,z$

$f(x,y,x,z,y,z,x,\dots,x) =$

$f(y,x,z,x,z,y,\alpha_1 y,\dots \alpha_m y)$

THEOREM $EQ^+(Pol(\mathbb{A}))$ NON-TRIVIAL

$\Rightarrow CSP(\mathbb{A}) \in P$ (BULATOV, ZHUK '17)

ALGORITHM

REDUCES TO SMALLER STRUCTURE
BY SOLVING LINEAR EQUATIONS
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THEOREM (BARTO + BODOR + LICHER + MOTTET + P. '23)

$EQ(Pol(\mathbb{A}))$ NON-TRIVIAL,
$\mathbb{A}$ CORE

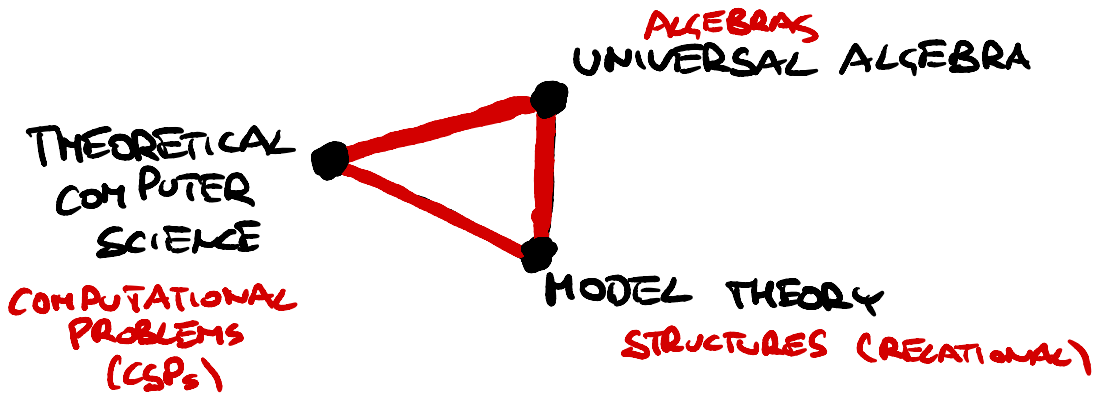
$\Rightarrow Pol(\mathbb{A}) \models \exists f \exists \alpha_1 \dots \alpha_m \forall x,y,z$

$f(x,y,x,z,y,z,x,\dots,x) =$
$f(y,x,z,x,z,y,\alpha_1 y,\dots \alpha_m y)$

OPEN PROBLEM

$EQ(Pol(\mathbb{A}))$ NON-TRIVIAL
$\mathbb{A}$ NOT A CORE

$\rightarrow$?

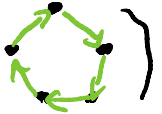


PART I THE MATHEMATICS OF FINITE-DOMAIN CSPs
ALGEBRAS
RELATIONAL STRUCTURES

PART II MODELLING PROBLEMS AS INFINITE-DOMAIN CSPs

PART III THE MATHEMATICS OF INFINITE-DOMAIN CSPs.
ALGEBRAS
RELATIONAL STRUCTURES

EXERCISES

- CSP  ALGORITHM? NON-TRIVIAL H1 EQUATION?
- CSP ——·· ——
- CSP $\left(\{0,1\};\ \left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right\},\ \left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}\right)$
- CSP

PART II : MODELLING PROBLEMS AS INFINITE-DOMAIN CSPs

FINITE-DOMAIN CSPs : RATHER LIMITED

FINITE-DOMAIN CSPs : RATHER LIMITED

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD

$\in NP$

FINITE-DOMAIN CSPs : RATHER LIMITED

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs : RESTRICTIONS OF SAT

$(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$

SATISFYING ASSIGNMENT?

$\in$ NP

FINITE-DOMAIN CSPs : RATHER LIMITED

$\in NP$

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs : RESTRICTIONS OF SAT

  $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$

  SATISFYING ASSIGNMENT?
- n-COLORING PROBLEM

  GENERALIZATION : H-COLORING PROBLEM

  H GRAPH, GIVEN GRAPH G, DECIDE IF $G \to H$ (HELL + NEŠETŘIL '91)

  FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

FINITE-DOMAIN CSPs : RATHER LIMITED

$\in NP$

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs : RESTRICTIONS OF SAT

  $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$

  SATISFYING ASSIGNMENT?
- n-COLORING PROBLEM

  GENERALIZATION : H-COLORING PROBLEM

  H GRAPH, GIVEN GRAPH G, DECIDE IF $G \to H$ (HELL + NEŠETŘIL '90)

  FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

INFINITE-DOMAIN CSPs

FINITE-DOMAIN CSPs: RATHER LIMITED

$\in$ NP

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

  $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$

  SATISFYING ASSIGNMENT?
- n-COLORING PROBLEM

  GENERALIZATION: H-COLORING PROBLEM

  H GRAPH, GIVEN GRAPH G, DECIDE IF $G \to H$ (HELL + NEŠETŘIL '90)

  FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

INFINITE-DOMAIN CSPs

- (LINEAR) EQUATIONS OVER $\mathbb{Q}/\mathbb{Z}$

FINITE-DOMAIN CSPs: RATHER LIMITED

$\in$ NP

- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

  $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge \dots \wedge (x_{100} \vee \neg x_{105} \vee \neg x_{106})$

  SATISFYING ASSIGNMENT?
- n-COLORING PROBLEM

  GENERALIZATION: H-COLORING PROBLEM

  H GRAPH, GIVEN GRAPH G, DECIDE IF $G \to H$ (HELL + NEŠETŘIL '90)

  FOR DIGRAPHS AS GENERAL AS ALL FINITE CSPs

INFINITE-DOMAIN CSPs

- (LINEAR) EQUATIONS OVER $\mathbb{Q}/\mathbb{Z}$
- $CSP(\mathbb{Z}, <)$ AND FO-DEFINABLE STRUCTURES E.G. $CSP(\mathbb{Z}, \{(x,y) \mid x-y \le 1337\})$

FINITE-DOMAIN CSPs: RATHER LIMITED

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- SOLVING (LINEAR) EQUATIONS / FINITE FIELD
- BOOLEAN-DOMAIN CSPs: RESTRICTIONS OF SAT

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EXERCISE

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WHY ALL THOSE FORBIDDEN PATTERN PROBLEMS?

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PATTERNS CONNECTED, MONOTONE $\Rightarrow$ K IS A CSP

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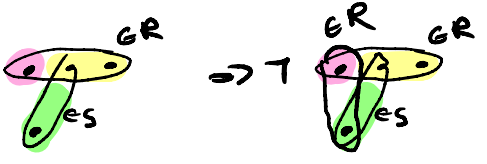

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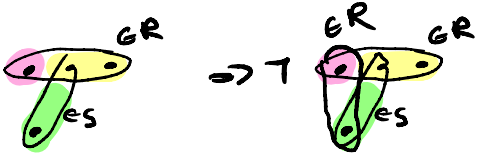

EXERCISE

WRITE DOWN NO MONOCHROMATIC TRIANGLE

WHEN DO POLYMORPHISMS WORK?

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DEFINITION

$\mathbb{A}$ COUNTABLE

$\mathbb{A}$ **ω-CATEGORICAL** :$\Leftrightarrow$

$\mathrm{Aut}(\mathbb{A}) \curvearrowright A^n$ FINITELY MANY ORBITS FOR ALL $n \geq 1$

"$\mathbb{A}$ INFINITE BUT $\mathbb{A}^n / \mathrm{Aut}\,\mathbb{A}$ FINITE"

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EXAMPLES

- $(\mathbb{N}, =)$
- $(\mathbb{Q}, <)$
- RANDOM GRAPH (V, E)
- COUNTABLE VECTOR SPACE OVER FINITE FIELD

× $(\mathbb{Z}, <)$, $(\mathbb{N}, <)$

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SPOILER

$\mathbb{A}$ ω-CATEGORICAL ⇒

- **PP-DEFINABILITY** DEPENDS ONLY ON $\mathrm{Pol}(\mathbb{A})$
- **PP-INTERPRETABILITY OF FINITE STRUCTURES** DEPENDS ON LOCAL EQUATIONS OF $\mathrm{Pol}(\mathbb{A})$
- **PP-CONSTRUCTABILITY** OF FINITE STRUCTURE : LOCAL HA-EQUATIONS OF $\mathrm{Pol}(\mathbb{A})$
- $\mathbb{A}$ HAS A **"CORE"**
- ∃ $\mathbb{A}$ ω-CATEGORICAL, **CSP($\mathbb{A}$) UNDECIDABLE**, $\mathbb{A}$ DOES NOT PP-CONSTRUCT $\mathbb{K}_3$
- ∃ **NICE SUBCLASS** OF ω-CAT. WITH P/NP-COMPLETE CONJECTURE ("FORBIDDEN PATTERN PROBLEMS FOR FINITE SETS OF PATTERNS")

CONSTRUCTING ω-CATEGORICAL STRUCTURES

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HP: $\forall F \leq G \in \mathcal{K} : F \in \mathcal{K}$

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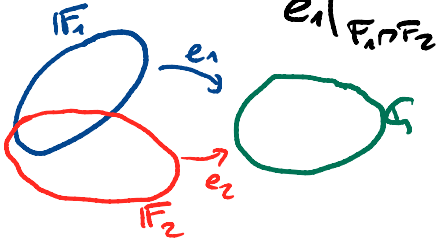
$\exists G \in \mathcal{K}$ $\exists e_i : F_i \to G$ EMBEDDINGS

$e_1|_{F_1 \cap F_2} = e_2|_{F_1 \cap F_2}$



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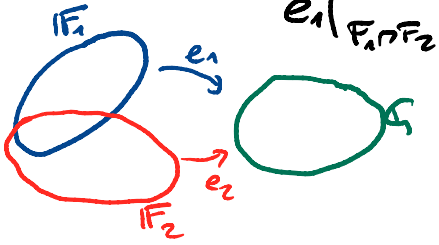

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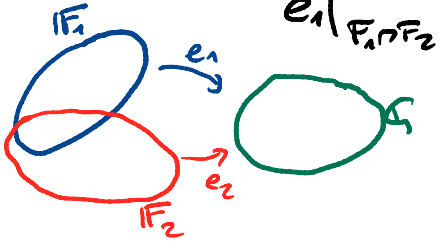

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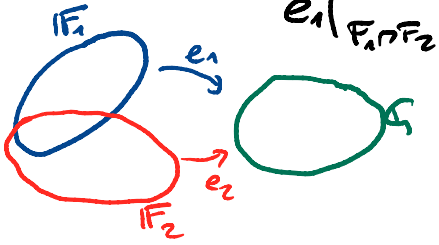

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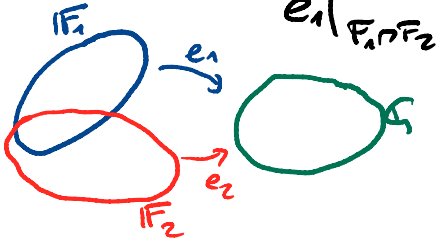

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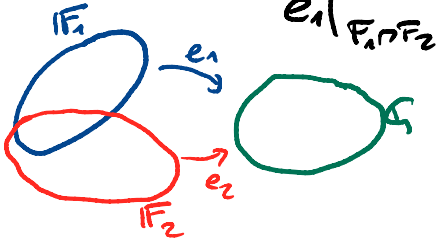

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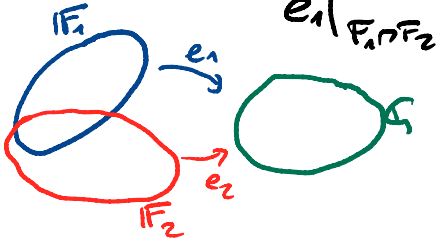

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- GRAPHS, DIGRAPHS, TOURNAMENTS, POSETS, ...

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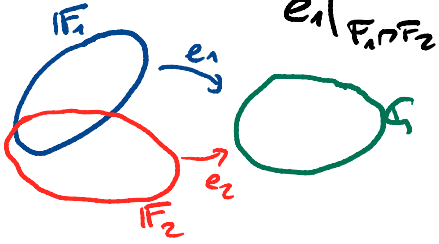

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- GRAPHS, DIGRAPHS, TOURNAMENTS, POSETS, ...

EXERCISE: NON-TRIVIAL CLASS OF GRAPHS W/ HP, AP

FIRST-ORDER REDUCTS OF $(\mathbb{Q}, <)$

FIRST-ORDER REDUCTS OF $(\mathbb{Q}, <)$

- $\mathrm{CSP}(\mathbb{Q}, <)$

  GIVEN FINITE DIGRAPH $\mathbb{D}$, $\mathbb{D} \stackrel{?}{\to} (\mathbb{Q}, <)$

  $\mathbb{D} \stackrel{?}{\to}$ FINITE LINEAR ORDER ?

FIRST-ORDER REDUCTS OF $(\mathbb{Q}, <)$

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  GIVEN FINITE DIGRAPH $\mathbb{D}$, $\mathbb{D} \xrightarrow{?} (\mathbb{Q}, <)$

  $\mathbb{D} \xrightarrow{?}$ FINITE LINEAR ORDER ?

- $\mathrm{CSP}(\mathbb{Q}, B(x,y,z))$

  $B(x,y,z) :\Leftrightarrow (x<y \wedge y<z) \vee (z<y \wedge y<x)$

FIRST-ORDER REDUCTS OF $(\mathbb{Q}, <)$

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- $\mathrm{CSP}(\mathbb{Q}, R)$

  $R(x,y,z) \Leftrightarrow (x=y<z) \vee (x=z<y) \vee (y=z<x)$

  EXERCISE: COMPLEXITY ?

FIRST-ORDER REDUCTS OF $(\mathbb{Q}, <)$

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CSPs OF FO-REDUCTS OF $(\mathbb{Q}, <)$

"LINEAR-ORDER-SAT", "TEMPORAL CSPs"

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"LINEAR-ORDER-SAT", "TEMPORAL CSPs"

THEOREM

- LINEAR-ORDER-SAT $(\mathbb{A})$ IN P UNLESS $\mathbb{A}$ pp-CONSTRUCTS K_3 (BODIRSKY + KÁRA '07)
- GRAPH-SAT (BODIRSKY + P. '11)
- POSET-SAT (VAN PHAM + KOMPATSCHER '16)
- TOURNAMENT-SAT (MOTTET + P. '21)
- HYPERGRAPH-SAT (MOTTET + NAGY + P. 23)

⋮

FIRST-ORDER REDUCTS OF $(\mathbb{Q}, <)$

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CONJECTURE (BODIRSKY + P. '11)

TRUE FOR ALL FIRST-ORDER REDUCTS OF HOMOGENEOUS STRUCTURES GIVEN BY FINITELY MANY FORBIDDEN SUBSTRUCTURES

FIRST-ORDER REDUCTS OF $(\mathbb{Q}, <)$

- $\mathrm{CSP}(\mathbb{Q}, <)$

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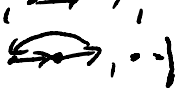
EXAMPLE

- LINEAR ORDERS = Forb()
- GRAPHS = Forb( )

THE GRAPH ORIENTATION PROBLEM AS CSP

$\mathcal{F}$... FINITE SET OF FINITE TOURNAMENTS

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$\mathbb{K} = \{ D \mid D$ $\mathcal{F}$-FREE ORIENTED FINITE GRAPH $\}$

$\mathbb{K}$ HAS HP, AP

$\mathbb{A} = (A, \to) := \mathrm{Flim}\, \mathbb{K}$

$G := (A, E)$ $\quad E(x,y) :\Leftrightarrow x \to y \vee y \to x$

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LET $\mathbb{H}$ FINITE GRAPH $\Rightarrow$

$\mathbb{H} \in \mathrm{CSP}(\mathcal{G}) \Leftrightarrow \mathbb{H} \xrightarrow{\text{hom}} \mathcal{G}$

$\overset{\text{Ex.}}{\Leftrightarrow} \mathbb{H} \xrightarrow{\text{emb}} \mathcal{G}$

$\Leftrightarrow$ $\mathbb{H}$ HAS $\mathcal{F}$-FREE ORIENTATION

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THEOREM (BODIRSKY + GUZMÁN - PRO '23) [FELLER + P. '24]

$\mathrm{CSP}(\mathcal{G}) \in \mathrm{P}$ UNLESS $\mathcal{G}$ PP-CONSTRUCTS K_3

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$\mathrm{CSP}(\mathcal{G}) \in P$ UNLESS $\mathcal{G}$ PP-CONSTRUCTS K_3

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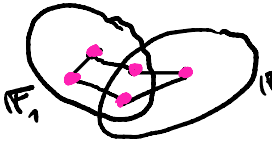
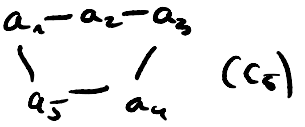
EXAMPLE

GRAPHS WITHOUT $a_1 \cdots a_5$:





(C_5)

OFTEN WE CAN EXTEND SIGNATURE TO OBTAIN AP, THEN TAKE REDUCT

THE GRAPH ORIENTATION PROBLEM AS CSP

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$\mathcal{K} = \{ \mathbb{D} \mid \mathbb{D}$ $\mathcal{F}$-FREE ORIENTED FINITE GRAPH $\}$

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$G := (A, E)$ $\quad E(x,y) :\Leftrightarrow x \to y \vee y \to x$

LET $\mathbb{H}$ FINITE GRAPH $\Rightarrow$

$\mathbb{H} \in \text{CSP}(G) \Leftrightarrow \mathbb{H} \xrightarrow{\text{HOM}} G$

$\overset{\text{EX.}}{\Leftrightarrow} \mathbb{H} \xrightarrow{\text{EMB}} G$

$\Leftrightarrow$ $\mathbb{H}$ HAS $\mathcal{F}$-FREE ORIENTATION

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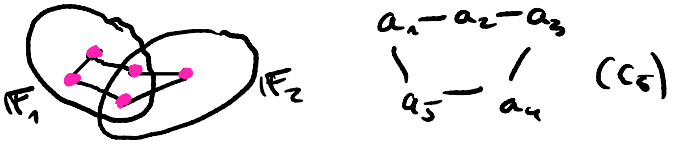


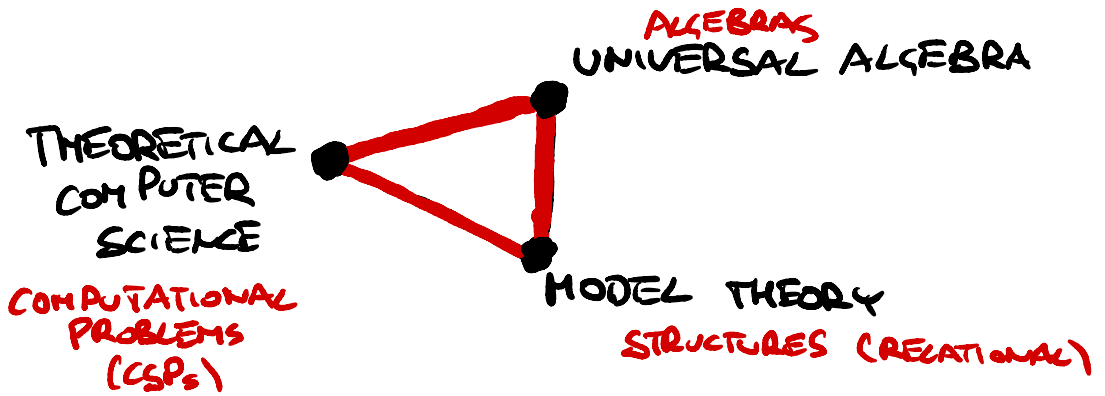
OFTEN WE CAN EXTEND SIGNATURE TO OBTAIN AP, THEN TAKE REDUCT

THEOREM [CHERLIN-SHELAH-SHI]

$\mathcal{F}$ ... FINITE SET OF STRUCTURES, CONNECTED, HOM-CLOSED

$\Rightarrow \exists \mathbb{A}$ ω-CATEGORICAL, FO-REDUCT OF HOMOGENEOUS STRUCTURE GIVEN BY FINITE SET OF FORBIDDEN STRUCTURES

$\text{CSP}(\mathbb{A}) = \text{Forb}(\mathcal{F})$



PART I THE MATHEMATICS OF FINITE-DOMAIN CSPs
ALGEBRAS
RELATIONAL STRUCTURES

PART II MODELLING PROBLEMS AS INFINITE-DOMAIN CSPs

PART III THE MATHEMATICS OF INFINITE-DOMAIN CSPs.
ALGEBRAS
RELATIONAL STRUCTURES

EXERCISES

- LET $G = (V, E)$ GRAPH $\quad P(x,y) :\Leftrightarrow \exists a\ E(x,a) \wedge E(a,y)$
  $Q(x,y) :\Leftrightarrow \exists a,b\ E(x,a) \wedge E(a,b) \wedge E(b,y)$

  $\mathbb{K} := \{ G \mid G \text{ FINITE}, \ C_5 \not\xrightarrow{\text{HOM}} G \}$

  SHOW: THE STRUCTURES OF $\mathbb{K}$, EXPANDED BY P AND Q, HAVE THE AP

- $\mathcal{F}$ ... (POSSIBLY INFINITE) SET OF FINITE TOURNAMENTS

  SHOW: $\mathbb{K} := \{ D \mid D \text{ DIGRAPH}, \ \mathcal{F}\text{-FREE} \}$ HAS AP

  CONCLUDE: THERE EXIST UNDECIDABLE ω-CATEGORICAL CSPs.

- WHEN IS A CLASS OF STRUCTURES A CSP?

PART III : THE MATHEMATICS OF ω-CATEGORICAL CSPs

SHORT VERSION:

BASIC THEORY ... AS FOR FINITE STRUCTURES

ADVANCED THEORY ... MUCH HARDER / FALSE

SHORT VERSION:

BASIC THEORY ... AS FOR FINITE STRUCTURES
ADVANCED THEORY ... MUCH HARDER / FALSE

(BABY) THEOREM (BODIRSKY + NEŠETŘIL '03)

$\mathbb{A}$ ω-CATEGORICAL, $R \subseteq A^m$ RELATION

$\Rightarrow$ R PP-DEFINABLE IN $\mathbb{A}$ $\Leftrightarrow$
R INVARIANT UNDER $Pol(\mathbb{A})$

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BASIC THEORY ... AS FOR FINITE STRUCTURES

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"$\rightarrow$" TRIVIAL

"$\leftarrow$"

R IS A UNION OF ORBITS OF
$Aut(\mathbb{A}) \curvearrowright A^m$

LET $\bar{r}_1, \ldots, \bar{r}_l \in R$ BE REPRESENTATIVES OF THESE ORBITS

LET $(\bar{a}_0, \bar{a}_1, \ldots)$ ENUMERATION OF A^k

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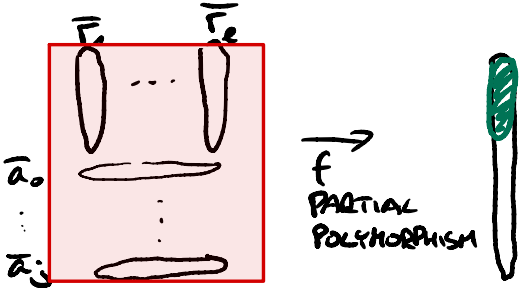

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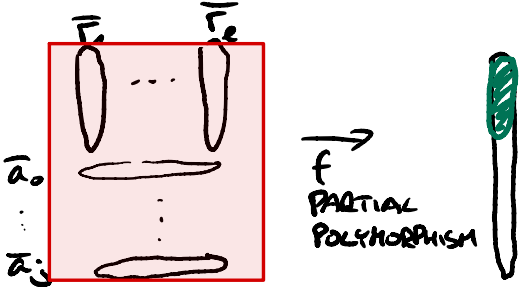

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BASIC THEORY ... AS FOR FINITE STRUCTURES

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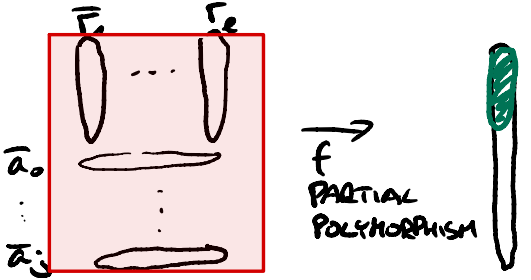

- $R_j \supseteq R$ (PROJECTIONS)
- $R_j \supseteq R_{j+1}$, EVENTUALLY CONSTANT (R_j UNION OF ORBITS)

$\Rightarrow$ $\bigcap_j R_j =: R_\infty$ PP-DEFINABLE

$R_\infty = R$

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BASIC THEORY ... AS FOR FINITE STRUCTURES

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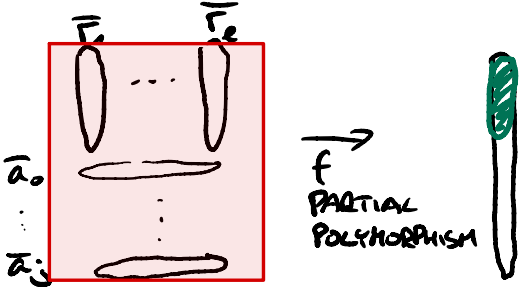

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- $R_j \supseteq R_{j+1}$, EVENTUALLY CONSTANT (R_j UNION OF ORBITS)

$\Rightarrow \bigcap_j R_j =: R_\infty$ PP-DEFINABLE

$R_\infty = R$

- $\bar{r} \in R_\infty$

$\Rightarrow \exists f_j$ PART. POLYMORPHISMS WITNESSES

.. $\exists$ SUBSEQUENCE SUCH THAT f_j EXTENDS f_i, $i<j$ MOD. ORBIT

.. COMPOSING WITH AUTOMORPHISMS, f_j EXTENDS f_i $f := \bigcup f_i$

$\Rightarrow \bar{r} = f(\bar{r}_1, \dots \bar{r}_l) \in R$

ADOLESCENT / ADULT THEOREMS:

- CHARACTERISE
  - PP-INTERPRETATIONS BY EQUATIONS / CLONE HOMOMORPHISMS
  - PP-CONSTRUCTIONS BY MI EQUATIONS / MINION HOMOMORPHISMS

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TOPOLOGY

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TOPOLOGY

$A^{A^n} = \{f : A^n \to A\}$ CARRIES

TOPOLOGY OF POINTWISE CONVERGENCE:

$(f_j)_{j \in \omega} \to f \iff$

$\forall F \subseteq A^n$ FINITE

$f_j|_F = f|_F$ EVENTUALLY

ADOLESCENT / ADULT THEOREMS:

- CHARACTERISE
  - PP-INTERPRETATIONS BY EQUATIONS / CLONE HOMOMORPHISMS
  - PP-CONSTRUCTIONS BY MI EQUATIONS / MINION HOMOMORPHISMS
- ONLY PP-INTERPRET FINITE IB CONSTRUCT IN ω-CATEGORICAL IA
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Pol(A) CLOSED SUBSPACE:

$f \notin Pol(A) \Rightarrow \exists F \subseteq A$ FINITE:

$O_{f|_F} = \{g \mid g|_F = f|_F\} \cap Pol(A) = \emptyset$

THEOREM (ADOLESCENT + ADULT)

BODIRSKY P. '15

BARTO + OPRŠAL + P. '17

$\mathbb{A}$ ω-CATEGORICAL, $\mathbb{B}$ FINITE $\Rightarrow$

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$\Leftarrow$ EASY

$\rightarrow$ $\mathrm{Pol}(\mathbb{A}) \rightarrow \mathrm{Pol}(\mathbb{B})$

$\xrightarrow{\text{u.c.}}$ $\mathrm{Pol}(\mathbb{B}) \supseteq \mathrm{Pol}(\mathbb{C}) \in \mathrm{HSP}^{\mathrm{fin}}(\mathrm{Pol}(\mathbb{A}))$

$\Rightarrow$ ... AS IN FINITE CASE

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WHAT IF $\mathrm{Pol}(\mathbb{A}) \xrightarrow[\text{MINION}]{\text{u.c.}} \mathrm{Pol}(\mathbb{K}_3)$?

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$\Rightarrow$ $Pol(\mathbb{A}) \rightarrow Pol(\mathbb{B})$

$\xrightarrow{u.c.}$ $Pol(\mathbb{B}) \supseteq Pol(\mathbb{C}) \in HSP^{FIN}(Pol(\mathbb{A}))$

$\Rightarrow$ ... AS IN FINITE CASE

WHAT IF $Pol(\mathbb{A}) \xrightarrow[\text{MINION}]{u.c.} Pol(\mathbb{K}_3)$?

RECALL $\mathbb{A}$ FINITE $\Rightarrow$

$Pol(\mathbb{A})$ HAS

- CYCLIC OPERATION $c(x_1 \cdots x_n) = c(x_2 \cdots x_n x_1)$
- WNU OPERATION $w(x \cdots xy) = \cdots = w(yx \cdots x)$
- 4-ARY SIGGERS OPERATION $s(a,r,e,a) = s(r,a,r,e)$
- 6-ARY SIGGERS OPERATION $s(xyxzyz) = s(yxzxzy)$

...

FALSE FOR ω-CATEGORICAL $\mathcal{A}$!

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EXAMPLE

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL

$CSP(A) \in P$

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

EXAMPLE

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL
$CSP(\mathbb{A}) \in P$

- DOES $Pol(\mathbb{A})$ HAVE CYCLIC $c(x_1 \dots x_n)$?

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

EXAMPLE

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL
CSP($\mathbb{A}$) $\in$ P

- DOES Pol($\mathbb{A}$) HAVE CYCLIC $c(x_1 \dots x_n)$?
  PICK $a_1, \dots, a_n$ DISTINCT

$$c(a_1 \dots a_n) = a$$
$$\| \; \| \; \| \; \| \; \| \; \|$$
$$c(a_2 \dots a_{n-1} a_1) = a$$

↯

- $s(a,r,e,a) = s(r,a,r,e)$? ↯
- $s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$ ↯
- ALL SINGLE NONTRIVIAL HI-EQUATION

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

EXAMPLE

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL
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- ALL SINGLE NONTRIVIAL H1-EQUATION

EXERCISE

- HOW ABOUT WNU?
  $w(x \dots xy) = \dots = w(yx \dots x)$
- WHY DOES K_ω NOT PP-CONSTRUCT K_3?

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

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  $w(x \dots xy) = \dots = w(yx \dots x)$
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EXAMPLE

$NAE_n = \{(a_1 \dots a_n) \mid a_i$ NOT ALL EQUAL$\}$
$\mathbb{A} = (\mathbb{N}, NAE_n)$ ω-CATEGORICAL,
CSP($\mathbb{A}$) TRIVIAL

↯ $w(a,\dots,a,b) =$
$\vdots$
$w(b,a,\dots,a) =$ ↯

FALSE FOR ω-CATEGORICAL $\mathbb{A}$!

EXAMPLE

LET $\mathbb{A} = K_\omega = (\mathbb{N}, \neq)$ ω-CATEGORICAL, $CSP(\mathbb{A}) \in P$

- DOES $Pol(\mathbb{A})$ HAVE CYCLIC $c(x_1 \dots x_n)$?
  PICK $a_1, \dots, a_n$ DISTINCT

$$c(a_1 \cdots a_n) = a$$
$$\| \; \| \; \| \; \| \; \| \qquad \lightning$$
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- $s(a,r,e,a) = s(r,a,r,e)$? $\lightning$
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- ANY SINGLE NONTRIVIAL H1-EQUATION

EXERCISE

- HOW ABOUT WNU?
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$NAE_n = \{(a_1 \cdots a_n) \mid a_i$ NOT ALL EQUAL$\}$

$\mathbb{A} = (\mathbb{N}, NAE_n)$ ω-CATEGORICAL,
$CSP(\mathbb{A})$ TRIVIAL

$\lightning$ $w(a, \dots, a, b) = \begin{pmatrix} a \\ \vdots \\ b \end{pmatrix}$ $\lightning$

EXAMPLE

LET $\mathcal{C} := \{f : A^n \to A \mid n \geq 1,$ f INJECTIVE UP TO DUMMY VARIABLES$\}$

$=: \mathbb{A}$

- $\mathcal{C} = Pol(A; \neq, x=y \to u=v)$ **EXERCISE**
- DOES NOT SATISFY ANY H1-EQUATIONS!
- $CSP(\mathbb{A}) \in P$
- SATISFIES $u \circ f(x,y) = v \circ f(y,x)$
  "PSEUDO-COMMUTATIVE"

PSEUDO-EQUATIONS

EQUIVALENCE RELATION ON A^{A^n}:

$$f \sim g \iff g \in \overline{\{\alpha f \mid \alpha \in \mathrm{Aut}(A)\}}$$

$$\iff \forall F \subseteq A \text{ FINITE}$$

$$\exists \alpha \in \mathrm{Aut}\, A$$

$$g|_F = \alpha f|_F$$

"f, g ARE EQUAL MODULO ORBITS"

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COMPACT (POLISH)

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IDEA: TRY TO MIMIC PROOFS FOR FINITE STRUCTURES IN CASE OF $\mathrm{Pol}(\mathbb{A})/\mathrm{Aut}\,\mathbb{A}$

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FACT $\mathrm{Pol}(\mathbb{A})$ SATISFIES EQUATION MODULO $\mathrm{Aut}(\mathbb{A})$

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EXAMPLE

$$f(g(x,y),h(z)) \sim h(f(z,x),g(y,y))$$

$$\Rightarrow \exists u, v \in \overline{\mathrm{Aut}(\mathbb{A})}$$

$$u \circ f(g(x,y),h(z)) = v \circ h(f(z,x),g(y,y))$$

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THEOREM (BARTO+P. '17)

$\mathbb{A}$ ω-CATEGORICAL, $\mathrm{Pol}(\mathbb{A})$ NON-TRIVIAL:

$\mathrm{Pol}(\mathbb{A}) \not\xrightarrow{\text{u.c.}}$ (minion homomorphism) $\mathrm{Pol}(K_3)$

$\Rightarrow$ $\mathrm{Pol}(\mathbb{A})$ HAS PSEUDO-SIGGERS OPERATION.

$$u \circ s(x,y,x,z,y,z) = v \circ s(y,x,z,x,z,y)$$

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CONVERSE TRUE IF (BARTO+KOMPATSCHER+KOZIK+PHAM+P. '18)

- $\mathbb{A}$ "CORE"
- $\mathbb{A}$ "GIVEN BY FINITELY MANY FORBIDDEN STR.

EXAMPLE

$\mathbb{A} = (\mathbb{Q}, <, x=y \to u=v)$: w-CAT, CSP $\in$ P

NO PSEUDO-CYCLIC POLYMORPHISM :

EXAMPLE

$\mathbb{A} = (\mathbb{Q}, <, x=y \to u=v)$: ω-CAT, CSP $\in$ P

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- ALL POLYMORPHISMS INJECTIVE (UP TO DUMMY VARIABLES)
- PSEUDO-CYCLIC CANNOT HAVE DUMMY VARIABLES
- SUPPOSE c IS INJECTIVE PSEUDO-CYCLIC, $a_1 \cdots a_n$ NOT ALL EQUAL

WLOG $c(a_1 \cdots a_n) < c(a_2 \cdots a_n a_1)$

$$\begin{aligned} c(a_1 \cdots a_n) &= \alpha\, c(a_2 \cdots a_n a_1) \\ &= \alpha^2 c(a_3 \cdots a_n a_1 a_2) \\ &\vdots \\ &= \alpha^n (a_1 \cdots a_n) \end{aligned}$$

↯

EXAMPLE

$\mathbb{A} = (\mathbb{Q}, <, x=y \to u=v)$: ω-CAT, CSP ∈ P

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- SUPPOSE c IS INJECTIVE PSEUDO-CYCLIC, $a_1 \dots a_n$ NOT ALL EQUAL

WLOG $c(a_1 \dots a_n) < c(a_2 \dots a_n a_1)$

$$\begin{aligned} c(a_1 \dots a_n) &= \alpha\, c(a_2 \dots a_n a_1) \\ &= \alpha^2 c(a_3 \dots a_n a_1 a_2) \\ &\vdots \\ &= \alpha^n (a_1 \dots a_n) \end{aligned}$$

↯

THEOREM (BARTO + BODOR + KOZIK + MOTTET + P. '23)

$\exists \mathbb{A}$ ω-CATEGORICAL ("CORE")

- $\mathrm{Pol}(\mathbb{A}) \xrightarrow[\text{MINION HOM.}]{\cancel{\text{UNIF.}}} \mathrm{Pol}(\mathbb{K}_3)$
- $\mathbb{A}$ DOES NOT HAVE PSEUDO-WNU

$\mathbb{A}$ NOT GIVEN BY FINITELY MANY FORBIDDEN STRUCTURES

EXAMPLE

$\mathbb{A} = (Q, <, x=y \to u=v)$: ω-CAT, CSP ∈ P

NO PSEUDO-CYCLIC POLYMORPHISM :

- ALL POLYMORPHISMS INJECTIVE (UP TO DUMMY VARIABLES)
- PSEUDO-CYCLIC CANNOT HAVE DUMMY VARIABLES
- SUPPOSE c IS INJECTIVE PSEUDO-CYCLIC, $a_1 \dots a_n$ NOT ALL EQUAL

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$$= \alpha^2 c(a_3 \dots a_n a_1 a_2)$$
$$\vdots$$
$$= \alpha^n (a_1 \dots a_n) \quad \lightning$$

THEOREM (BARTO + BODOR + KOZIK + MOTTET + P. '23)

$\exists \mathbb{A}$ ω-CATEGORICAL ("CORE")

- $\mathrm{Pol}(\mathbb{A}) \xrightarrow[\text{MINION HOM.}]{\not\to} \mathrm{Pol}(\mathbb{K}_3)$
- $\mathbb{A}$ DOES NOT HAVE PSEUDO-WNU

$\mathbb{A}$ NOT GIVEN BY FINITELY MANY FORBIDDEN STRUCTURES

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$\mathbb{A} = (Q, <, x=y \to u=v)$: ω-CAT, CSP ∈ P

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THEOREM

(BARTO + BODOR + KOZIK + MOTTET + P. '23)

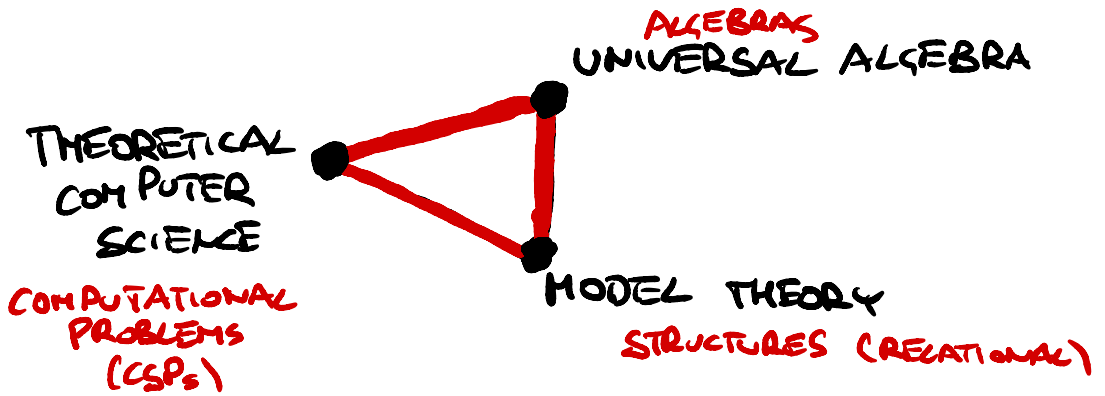
$\exists \mathbb{A}$ ω-CATEGORICAL ("CORE")

- $\mathrm{Pol}(\mathbb{A}) \not\to \mathrm{Pol}(\mathbb{K}_3)$ (NOT MINION HOM.)
- $\mathbb{A}$ DOES NOT HAVE PSEUDO-WNU

$\mathbb{A}$ NOT GIVEN BY FINITELY MANY FORBIDDEN STRUCTURES

OPEN PROBLEM

- PSEUDO-4-ARY SIGGERS
- MINION ⇒ CLONE HOMOMORPHISM
- PSEUDO-WNU FOR $\mathbb{A}$ GIVEN BY FINITELY MANY FORBIDDEN STRUCTURES



PART I THE MATHEMATICS OF FINITE-DOMAIN CSPs
ALGEBRAS
RELATIONAL STRUCTURES

PART II MODELLING PROBLEMS AS INFINITE-DOMAIN CSPs

PART III THE MATHEMATICS OF INFINITE-DOMAIN CSPs.
ALGEBRAS
RELATIONAL STRUCTURES

EXERCISES

- FIND A COMPLETE METRIC THAT INDUCES THE TOPOLOGY OF POINTWISE CONVERGENCE

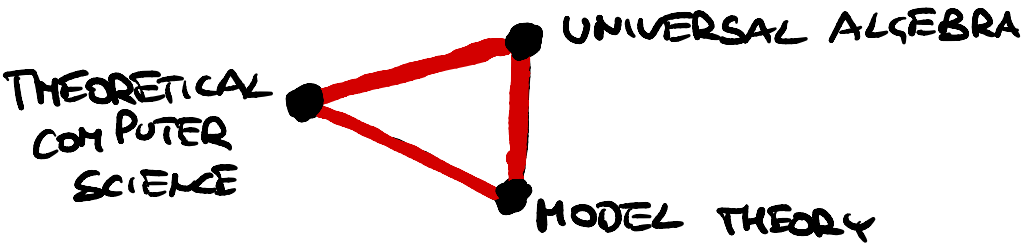
- LET $\mathbb{A}$ BE ω-CATEGORICAL

  $\forall$ FINITE $F \subseteq A$ $\exists$ $f_F, g_F, h_F \in \mathrm{Pol}(\mathbb{A})$:

  $$f_F(g_F(x,y), h_F(y,x)) = f_F(g_F(y,x), h_F(y,x))$$

  $\Rightarrow \exists$ $f, g, h \in \mathrm{Pol}(\mathbb{A})$

  $$f(g(x,y), h(y,x)) = f(g(y,x), h(y,x))$$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Thank you!