CONSTRAINT SATISFACTION PROBLEMS

ALGEBRAIC & MODEL-THEORETIC CHALLENGES

DISTINGUISHING THE EASY FROM THE HARD

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TU WIEN

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EUROPEAN RESEARCH COUNCIL

ERC SYNERGY GRANT POCOCOP (GA 101071674)

HO!

OUTLINE

1. CONSTRAINT SATISFACTION PROBLEMS
   $\Leftrightarrow$ RELATIONAL STRUCTURES — 10 MINUTES
2. ALGEBRAIC INVARIANTS
   (FOR FINITE STRUCTURES) — 10 MINUTES
3. GOING INFINITE
   (ω-CATEGORICAL) — 10 MINUTES
4. GOING FINITE — 10 MINUTES

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50 MINUTES

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   (FOR FINITE STRUCTURES) — 10 MINUTES

3. GOING INFINITE
   (ω-CATEGORICAL) — 10 MINUTES

4. GOING FINITE — 10 MINUTES

~~50~~ 80 MINUTES

CONSTRAINT SATISFACTION PROBLEM CSP

GIVEN

- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS ON THEM

QUESTION

- $\exists$ VALUES FOR $x_1 \dots x_n$ SATISFYING ALL CONSTRAINTS ?

EXAMPLE

- SUDOKU
- SCHEDULING
- SOLVING EQUATIONS

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MODEL:

- SET V OF POSSIBLE VALUES (FIXED)

  E.G. $\{0,1\}, \{0,1,2\}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}, \dots$

- ALLOWED CONSTRAINTS (FIXED)

  $C_1, \dots, C_m$ RELATIONS ON V

  $C_i \subseteq V^{a_i}$

SO $\mathbb{A} := (V, C_1, \dots, C_m)$ RELATIONAL STRUCTURE

= "TEMPLATE"

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1 \dots x_n$
- PP-SENTENCE

$$\varphi \equiv \exists x_1 \dots \exists x_n \; C_{i_1}(\text{VARIABLES}) \wedge C_{i_2}(\text{VARIABLES}) \wedge \dots \wedge C_{i_\ell}(\text{VARIABLES})$$

QUESTION

$\mathbb{A} \models \varphi$?

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QUESTION

$$\mathbb{A} \models \varphi \; ?$$

META-QUESTION

FOR WHAT $\mathbb{A}$ IS CSP($\mathbb{A}$) EASY / HARD ?

P: $\exists$ ALGORITHM PROVIDING ANSWER IN p(# VARIABLES) STEPS, p POLYNOMIAL

NP: NOT NECESSARILY IN P BUT VERIFYING "SOLUTION" IN P

EXAMPLES

TERNARY RELATIONS

- $\mathbb{A} = (\mathbb{Z}, \{0\}, \{1\}, +, \cdot)$

CSP($\mathbb{A}$)

GIVEN
- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $x_1^2 + x_2^3 = x_3^4$

$\vdots$

QUESTION
- SOLUTION IN $\mathbb{Z}$?

<u>UNDECIDABLE</u> (MATIYASEVIC '77)

HILBERT 10

EXAMPLES

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- SAME OVER $\mathbb{Z}_p$:

NP-<u>COMPLETE</u>

$\hookrightarrow$ IF IN P $\Rightarrow$ P = NP

EXAMPLES

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- SAME WITHOUT $\cdot$:

IN P (GAUSS)

EXAMPLES

TERNARY RELATIONS

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IN P (GAUSS)

- $\mathbb{A} = (\mathbb{Q}, <)$

CSP($\mathbb{A}$) **GIVEN**
- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $x_1 < x_2$
  $x_2 < x_3$
  $x_3 < x_1$

$\vdots$

QUESTION
- SOLUTION IN $\mathbb{Q}$?

IN P

EXAMPLES

TERNARY RELATIONS

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QUESTION
- SOLUTION IN $\mathbb{Q}$?

IN P

- $\mathbb{A} = K_3 =$

$= (\{0,1,2\}, E)$

CSP($\mathbb{A}$) **GIVEN**
- VARIABLES $x_1 \dots x_n$
- CONSTRAINTS $E(x_1, x_2)$
  $E(x_2, x_3)$

$\vdots$

QUESTION
- SOLUTION IN $\{0,1,2\}$?

3-COLORING PROBLEM = NP-COMPLETE

THEOREM (BULATOV, ZHUK '17)

(CONJECTURE FEDER + VARDI '93)

$\mathbb{A}$ FINITE

$\Rightarrow$ CSP($\mathbb{A}$) $\in$ P OR

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(P $\neq$ NP)

CSP($\mathbb{A}$) $\in$ P $\Leftrightarrow$ $\mathbb{A}$ HAS ALGEBRAIC INVARIANT

$s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$

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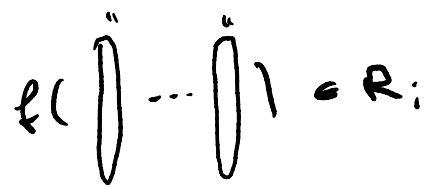
ALGEBRAIC INVARIANTS = POLYMORPHISMS

$\mathbb{A} = (A; R_1, \dots, R_m)$ STRUCTURE

$f: A^\ell \to A$ POLYMORPHISM $:\Leftrightarrow$
f HOMOMORPHISM $\mathbb{A}^\ell \to \mathbb{A}$ $\Leftrightarrow$
$\forall i$: $\forall \bar{r}_1 \dots \bar{r}_\ell \in R_i$



$\mathrm{Pol}(\mathbb{A}) := \{ f \mid f \text{ POLYMORPHISM OF } \mathbb{A} \}$

- CONTAINS PROJECTIONS $(x_1 \dots x_\ell) \mapsto x_i$
- COMPOSITION-CLOSED

$\Rightarrow$ ESSENTIALLY TERM FUNCTIONS OF AN ALGEBRA ON A !

EXAMPLES

- $\min(x,y) : \mathbb{Q}^2 \to \mathbb{Q} \in \mathrm{Pol}(\mathbb{Q}, <)$
- $(x,y,z) \mapsto x-y+z \in \mathrm{Pol}(\mathbb{Z}_p, \{0\}, \{1\}, +)$
- ? $\in \mathrm{Pol}(K_3)$

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- PROJECTIONS, AUTOMORPHISMS $\in \mathrm{Pol}(K_3)$

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(BABY) THEOREM

(BODNARČUK + KALUŽNIN + KOTOV + ROMOV '66)
GEIGER '68

$\mathbb{A}, \mathbb{B}$ FINITE, SAME DOMAIN

$\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B}) \Rightarrow \mathbb{A}$ PP-DEFINES $\mathbb{B}$

$\Rightarrow \mathrm{CSP}(\mathbb{A})$ HARDER THAN $\mathrm{CSP}(\mathbb{B})$

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- <u>GOOD</u> FOR CLASSIFYING CSPs ON $\{0,1\}$ (SCHAEFER '78)
- <u>BETTER</u>: EQUATIONAL CONDITIONS:
  - $\exists f \in \mathrm{Pol}(\mathbb{Q}, <)\ \forall x,y \quad f(x,y) = f(y,x)$
  - $\exists f \in \mathrm{Pol}(\mathbb{Z}_p, \{0\}, \{1\}, +)$
    $\forall x,y,z \quad f(x,x,y) = f(y,x,x) = y$
  - $\exists f \in \mathrm{Pol}(K_3) : \forall x \quad fff(x) = x$ ☹

EXAMPLES

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DEFINITION

- EQUATIONAL (STRONG MALCEV) CONDITION:

  SENTENCE

  $$\Psi \equiv \exists f_1 \exists f_2 \cdots \forall x_1 \forall x_2 \cdots$$
  $$s_1(\text{variables}) = t_1(\text{variables})$$
  $$\wedge$$
  $$\vdots$$
  $$\wedge \ s_k(\text{variables}) = t_k(\text{variables})$$

  WHERE s_i, t_i ... TERMS OVER $f_1, f_2, \ldots$

- Ψ TRIVIAL $:\Leftrightarrow \mathrm{Pol}(K_3) \models \Psi$

  $\Leftrightarrow \Psi$ SATISFIABLE BY PROJECTIONS

- $\mathrm{EQ}(\mathrm{Pol}(\mathbb{A})) := \{\Psi \mid \mathrm{Pol}(\mathbb{A}) \models \Psi\}$

(ADOLESCENT) THEOREM

(BULATOV + JEAVONS + KROKHIN '00)

$$\mathrm{EQ}(\mathrm{Pol}(\mathbb{A})) \subseteq \mathrm{EQ}(\mathrm{Pol}(\mathbb{B}))$$

$\Rightarrow$ $\mathbb{A}$ PP-INTERPRETS $\mathbb{B}$

$\Rightarrow$ CSP($\mathbb{A}$) HARDER THAN CSP($\mathbb{B}$)

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(ADULT) THEOREM (BARTO + OPRŠAL + P. '16)

EQ^1 ... HEIGHT 1 - CONDITIONS

$f_i(\text{variables}) = f_j(\text{variables})$

$\cancel{s}$ $\cancel{t}$

$EQ^1(Pol(\mathbb{A})) \subseteq EQ^1(Pol(\mathbb{B}))$

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COROLLARY

$EQ^1(Pol(\mathbb{A}))$ TRIVIAL

$\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERYTHING

$\Rightarrow$ CSP($\mathbb{A}$) NP-COMPLETE

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THEOREM $EQ^1(Pol(\mathbb{A}))$ NON-TRIVIAL

$\Rightarrow$

$Pol(\mathbb{A}) \models \exists f\, \forall x,y,z \quad f(x,y,x,z,y,z) = f(y,x,z,x,z,y)$

(SIGGERS '10)

$\models \exists f\, \forall a,r,e \quad f(a,r,e,a) = f(r,a,r,e)$

(KEARNES + MARKOVIĆ + MCKENZIE '16)

$\models \exists f\, \forall x_1 \dots x_n$

$f(x_1 \dots x_n) = f(x_2 \dots x_n, x_1)$

$\forall\, n \geq |A|$ PRIME

(BARTO + KOZIK '11)

$\models \exists f\, \forall x,y \quad f(x \dots xy) = \dots = f(yx \dots x)$

$\forall\, n \geq |A|$ PRIME

(MARÓTI + MCKENZIE '08)

$\Rightarrow$ CSP($\mathbb{A}$) $\in$ P (BULATOV, ZHUK '17)

GOING INFINITE

FINITE-DOMAIN CSPs ... COMBINATORIAL PROBLEMS

E.G. 3-COLORING

INFINITE-DOMAIN CSPs:

EXAMPLES

- $(\mathbb{Q}, <)$
- $(\mathbb{Q}, \text{between}(x,y,z))$
- RANDOM GRAPH
- RANDOM POSET
- ATOMLESS BOOLEAN ALGEBRA

LOGIC PROBLEMS:

- $(\mathbb{Z}, +, 1, <)$
- $(\mathbb{Z}, +, \cdot)$
- $(\mathbb{Q}, +, \cdot)$

NUMERIC PROBLEMS

TEMPLATE = (COUNTABLE) ω-CATEGORICAL STRUCTURE

OFTEN: DEFINABLE IN HOMOGENEOUS STRUCTURE

"ARE THE CONSTRAINTS SATISFIABLE IN A LINEAR ORDER?" ETC.

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(ADULT) THEOREM

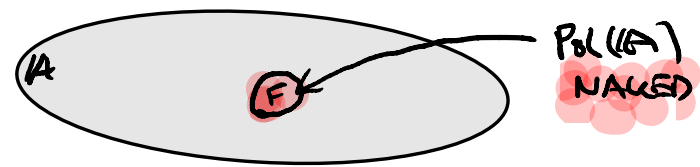
(BARTO + OPRŠAL + P. '16)

$\mathbb{A}$ ω-CATEGORICAL

$EQ^\wedge_{\text{LOCAL}}(\mathrm{Pol}(\mathbb{A}))$ TRIVIAL $\Rightarrow$ $\mathbb{A}$ PP-CONSTRUCTS EVERYTHING

$\Rightarrow$ $\mathrm{CSP}(\mathbb{A})$ NP-HARD

$\exists F \subseteq A$ FINITE: $EQ^\wedge(\mathrm{Pol}(\mathbb{A})|_F)$ TRIVIAL



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- $(\mathbb{Z}, +, 1, <)$
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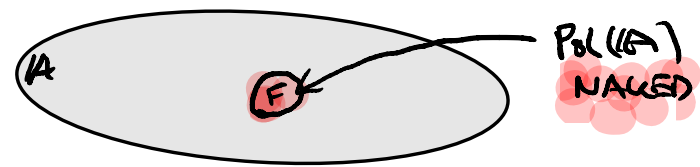

THEOREM

$\mathbb{A}$ ω-CATEGORICAL

$EQ^{\wedge}_{\text{LOCAL}}(\text{Pol}(\mathbb{A}))$ NON-TRIVIAL

$\Rightarrow$

$\text{Pol}(\mathbb{A}) \models \exists a, b, f\ \forall x, y, z$

$a \circ f(x,y,x,z,y,z) = b \circ f(y,x,z,x,z,y)$

(BARTO + P. '17)

$\models$ ... MANY MORE

(BARTO + BODOR + KOZIK + MOTTET + P. '23)

GOING ~~IN~~FINITE

$\mathrm{Pol}(\mathbb{A}) \models \exists u, v, f\ \forall x, y, z$

$$u \circ f(x,y,x,z,y,z) = v \circ f(y,x,z,x,z,y)$$

... SO WHAT ?

<u>PROBLEM</u>: f MIGHT BE <u>UGLY</u> !

<u>UGLINESS</u>: IF $\mathbb{A}$ IS DEFINABLE IN $\mathbb{B}$
WHERE $\mathbb{B}$ IS HOMOGENEOUS,
d-ARY LANGUAGE

$\mathrm{CSP}(\mathbb{A})$: GIVEN $x_1 \dots x_n$
WANT TO ASSIGN $\mathrm{Aut}(\mathbb{B})$-ORBITS
TO d-TUPLES OF VARIABLES

... BUT f MIGHT NOT ACT ON ORBITS

EXAMPLE $\mathrm{CSP}(\mathbb{Q}, \mathrm{Betw}(x,y,z))$

GIVEN VARIABLES $x_1, \dots, x_n$
WANT TO ASSIGN "$<$", "$>$", "$=$"
TO EVERY PAIR (x_i, x_j)

GOING ~~I~~NFINITE

$\mathrm{Pol}(\mathbb{A}) \models \exists u, v, f\ \forall x, y, z$
$u \circ f(x,y,x,z,y,z) = v \circ f(y,x,z,x,z,y)$

... SO WHAT ?

PROBLEM: f MIGHT BE <u>UGLY</u> !

UGLINESS: IF $\mathbb{A}$ IS DEFINABLE IN $\mathbb{B}$
WHERE $\mathbb{B}$ IS HOMOGENEOUS,
d-ARY LANGUAGE

$\mathrm{CSP}(\mathbb{A})$: GIVEN $x_1 \dots x_n$
WANT TO ASSIGN $\mathrm{Aut}\,\mathbb{B}$-ORBITS
TO d-TUPLES OF VARIABLES

... BUT f MIGHT NOT ACT ON ORBITS

EXAMPLE $\mathrm{CSP}(\mathbb{Q}, \mathrm{Betw}(x,y,z))$

GIVEN VARIABLES $x_1, \dots, x_n$
WANT TO ASSIGN "<", ">", "="
TO EVERY PAIR (x_i, x_j)

DEFINITION $\mathbb{B}$ HOMOGENEOUS, d-ARY

$f : B^d \to B$ CANONICAL WRT $\mathbb{B}$ $:\Leftrightarrow$
f PRESERVES EQUIVALENCE RELATION
INDUCED BY $\mathrm{Aut}(\mathbb{B}) \curvearrowright B^d$

EXAMPLE

- $f : \mathbb{Q} \to \mathbb{Q}$, $x \mapsto -x$ CANONICAL WRT $(\mathbb{Q}, <)$
- $\min : \mathbb{Q}^2 \to \mathbb{Q}$ <u>NOT</u> CANONICAL

$\min(a, b) = u$
$\min(c, d) = v$?

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THEOREM (MOTTET + BODIRSKY '18)

$\mathbb{A}$ FO-DEFINABLE IN $\mathbb{B}$, $\mathbb{B}$ HOMOGENEOUS
FINITE BOUNDED

$f \in \mathrm{Pol}(\mathbb{A})$ CANONICAL WRT $\mathbb{B}$,
ACTION ON ORBITS NON-TRIVIAL

$\Rightarrow \mathrm{CSP}(\mathbb{A}) \in \mathrm{P}$

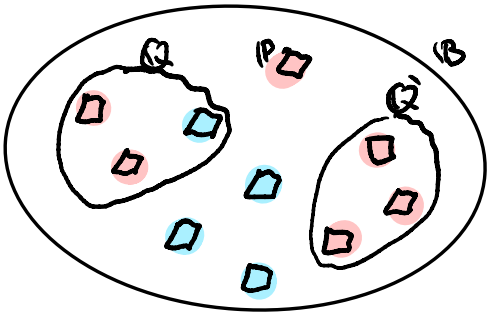
DEFINITION

$\mathbb{B}$ RAMSEY $:\Longleftrightarrow$

$\forall$ FINITE $\mathbb{P}, \mathbb{Q} \subseteq \mathbb{B}$

$\forall \chi : \binom{\mathbb{B}}{\mathbb{P}} \to 2$

$\exists \mathbb{Q}' \in \binom{\mathbb{B}}{\mathbb{Q}} : \chi \restriction \binom{\mathbb{Q}'}{\mathbb{P}}$ CONSTANT



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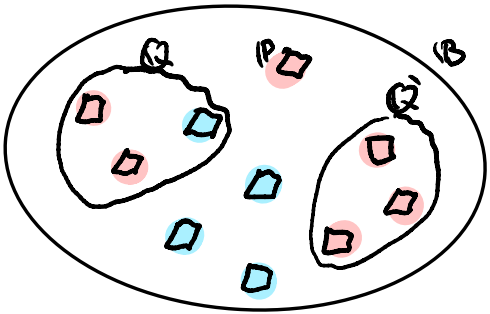

THEOREM (BODIRSKY + P. + TSANKOV '13)

$\mathbb{B}$ HOMOGENEOUS, d-ARY, RAMSEY

$\Rightarrow \forall f : B^d \to B$

$\exists\, \tilde{f} \in \{\alpha \circ f(\beta_1, \dots, \beta_n) \mid \alpha, \beta_i \in \mathrm{Aut}(\mathbb{B})\}$

$\tilde{f}$ CANONICAL

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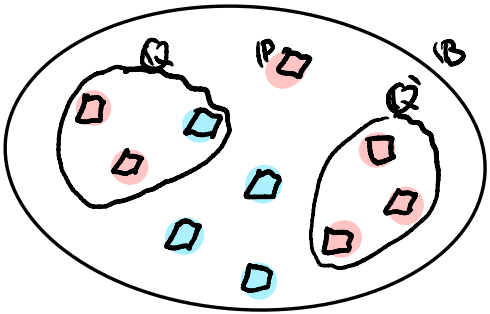

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BUT CANONICAL POLYMORPHISMS TRIVIAL

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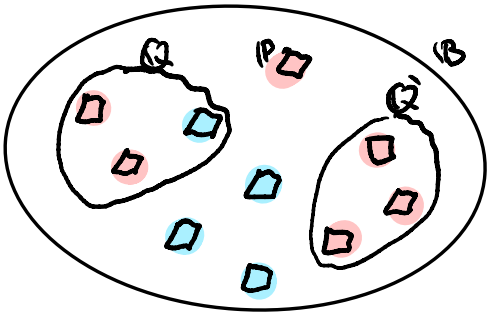

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BUT CANONICAL POLYMORPHISMS TRIVIAL

$\Rightarrow \exists s \in \mathrm{Pol}(\mathbb{A})$ ($\forall a, b \dots$

THEOREM

LET $\mathbb{A}$ BE FIRST-ORDER DEFINABLE IN:

- $\mathbb{Q}$ (BODIRSKY + KÁRA '07)
- THE RANDOM GRAPH (BODIRSKY + P. '11)
- ANY HOMOGENEOUS GRAPH (BODIRSKY + MARTIN + PONGRÁCZ + P. '16)
- THE UNIVERSAL HOMOGENEOUS TOURNAMENT (MOTTET + P. '20)
- PARTIAL ORDER (KOMPATSCHER + VAN PHAM '17)
- 2-BRANCHING C-RELATION (BODIRSKY + JONSSON + VAN PHAM '16)
- ANY FINITELY BOUNDED HOMOGENEOUS HYPERGRAPH WITH RAMSEY EXPANSION BY GENERIC TOTAL ORDER (MOTTET + NAGY + P. '23)

- IF $\mathrm{Pol}(\mathbb{A}) \models \exists u, v, f\ \forall x, y, z \quad u \circ f(x, y, x, z, y, z) = v \circ f(y, x, z, x, z, y)$ THEN $\mathrm{CSP}(\mathbb{A}) \in \mathrm{P}$
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CONJECTURE (BODIRSKY + P. '11, BARTO + P. '16)

TRUE $\forall \mathbb{A}$ FO-DEFINABLE IN FINITELY BOUNDED HOMOGENEOUS $\mathbb{B}$

THANK YOU !

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