CURRENT CHALLENGES IN INFINITE-DOMAIN CONSTRAINT SATISFACTION

DILEMMAS OF THE INFINITE SHEEP

MICHAEL PINSKER
TU WIEN
FWF P32337

ISMVL 2022 "DALLAS, TEXAS"

19.5.2022

CONSTRAINT SATISFACTION PROBLEM (CSP):

GIVEN
- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $C_1, \dots, C_m$ ON THEM

QUESTION ARE THERE VALUES FOR THE VARIABLES SATISFYING ALL CONSTRAINTS?

- THINK: SUDOKU
- WHAT ARE VALUES?
- WHAT ARE CONSTRAINTS?

FIXED-TEMPLATE CSP

- SET V OF POSSIBLE VALUES FIXED BEFOREHAND
  E.G. $\mathbb{Q}, \mathbb{R}, \mathbb{Z}, \mathbb{N}, \{0,1\}, \{0,1,2\}$
- ALLOWED CONSTRAINTS FIXED BEFOREHAND:
  $C_1^{\mathbb{A}}, \dots, C_m^{\mathbb{A}}$ ... RELATIONS ON V
  $C_i^{\mathbb{A}} \subseteq V^{n_i}$ n_i ... ARITY

$\Rightarrow (V, C_1^{\mathbb{A}}, \dots, C_m^{\mathbb{A}}) =: \mathbb{A}$ ... RELATIONAL STRUCTURE

$CSP(\mathbb{A})$ "TEMPLATE"

GIVEN
- VARIABLES $x_1, \dots, x_n$
- LIST OF CONSTRAINTS:

$$C_{i_1}(x_{j_1^1}, \dots, x_{j_{n_{i_1}}^1})$$
$$\vdots$$
$$C_{i_k}(x_{j_1^k}, \dots, x_{j_{n_{i_k}}^k})$$

QUESTION DOES THERE EXIST A SOLUTION,

I.E. $s: \{x_1, \dots, x_n\} \to V$

SUCH THAT FOR EVERY CONSTRAINT $C_i(x_{i_1}, \dots, x_{i_{n_i}})$

$(s(x_{i_1}), \dots, s(x_{i_{n_i}})) \in C_i^{\mathbb{A}}$?

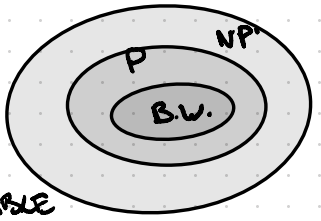
META-QUESTION

FOR WHICH $\mathbb{A}$ CAN $CSP(\mathbb{A})$ BE SOLVED EFFICIENTLY?

"EFFICIENTLY": MEASURES OF COMPLEXITY

- UNDECIDABLE: NO ALGORITHM SOLVES IT
- P: SOME ALGORITHM SOLVES IT IN # OF STEPS A POLYNOMIAL $p(n)$ IN # OF VARIABLES n
- NP: NOT NECESSARILY IN P BUT CHECKING A PROPOSED SOLUTION $S: \{x_1 \dots x_n\} \to V$ CAN BE DONE IN POLYNOMIAL TIME
- BOUNDED WIDTH: CSP($\mathbb{A}$) CAN BE SOLVED BY CHECKING CONSISTENCY OF THE CONSTRAINTS LOCALLY

$\Rightarrow$ CSP($\mathbb{A}$) $\in$ P



P ≠ NP MILLENIUM PROBLEM

EXAMPLES

- $\mathbb{A} = (\mathbb{Z}, \{0\}, \{1\}, \{(x,y,z) : x+y=z\}, \{(x,y,z) : x \cdot y = z\})$

CSP($\mathbb{A}$):

GIVEN:

- VARIABLES x_1, x_2, x_3
- CONSTRAINTS $x_1^3 + x_2^3 = x_3^3$

QUESTION: SOLUTION IN $\mathbb{Z}^3$.

UNDECIDABLE (MATYASEVIČ '77) HILBERT'S 10TH PROBLEM

- $\mathbb{A} = (\mathbb{Z}_p, \{0\}, \{1\}, \{(x,y,z) : x+y=z\})$

CSP($\mathbb{A}$):

GIVEN: • VARIABLES $x_1, \dots, x_n$

• CONSTRAINTS

$x_5 = 1$ $\quad x_1 + x_2 = x_3$

$x_3 = x_5 + x_5$ $\quad x_2 = x_1 \cdot x_4$

QUESTION: SOLUTION IN $\mathbb{Z}_p$?

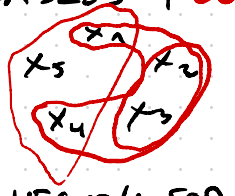
IN P (GAUSSIAN ELIMINATION)

BUT NOT BOUNDED WIDTH

- $\mathbb{A} = (\{0,1\}, \{(1,0,0), (0,1,0), (0,0,1)\})$

CSP($\mathbb{A}$):

GIVEN : VARIABLES , CONSTRAINTS



QUESTION: VALUES 0/1 FOR VARIABLES, EACH  PRECISELY ONE 1

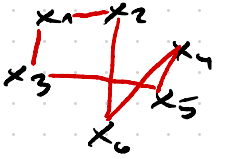
CSP($\mathbb{A}$) NP-COMPLETE, I.E.

IF IN P $\Rightarrow$ P=NP

- $\mathbb{A} = (\{0,1\}, \{(1,0), (0,1)\})$

CSP($\mathbb{A}$)

GIVEN :

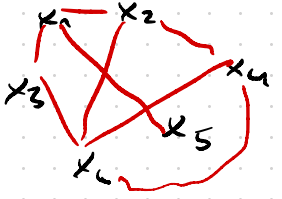


Q: SOLUTION ?

2-COLORING, HAS BOUNDED WIDTH

- $\mathbb{A} = (\{0,1,2\}, \{(a,b) : a \neq b\})$

CSP($\mathbb{A}$)



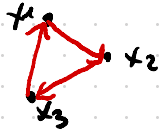
3-COLORING, NP-COMPLETE

- $\mathbb{A} = (\mathbb{Q}, <)$

GIVEN: VARIABLES $x_1, \ldots, x_n$
CONSTRAINTS $x_1 < x_2$
$x_2 < x_3$
$x_3 < x_1$

Q: SOLUTION IN $\mathbb{Q}$?

(YES IFF CONSTRAINTS DO NOT IMPLY CYCLE)



$\mathbb{A}$ FINITE STRUCTURE
$\Rightarrow$ CSP($\mathbb{A}$) $\in$ NP

- CHECKING PROPOSED SOLUTION CAN BE DONE LOCALLY
- FINDING ONE MIGHT BE HARD

$\mathbb{A}$ INFINITE
$\Rightarrow$ CSP($\mathbb{A}$) CAN BE UNDECIDABLE
E.G. $(\mathbb{Z}, 0, 1, +, \cdot)$
OR ANY COMPLEXITY
(BODIRSKY + GROHE '08)
(GILLIBERT + JONUŠAS + KOMPATSCHER + MOTTET + P. '20)
BUT SOMETIMES IN P,
EVEN BOUNDED WIDTH
E.G. $(\mathbb{Q}, <)$

THEOREM (BULATOV + ZHUK '17)
(CONJECTURE OF FEDER + VARDI '95)
$\mathbb{A}$ FINITE $\Rightarrow$ CSP($\mathbb{A}$) $\in$ P OR NP-COMPLETE

IN FACT

- EITHER $\mathbb{A}$ CAN "SIMULATE" ANY OTHER FINITE-DOMAIN STRUCTURE ("PP-CONSTRUCTION"), AND CSP($\mathbb{A}$) IS NP-COMPLETE
- OR OTHERWISE $\mathbb{A}$ HAS CERTAIN "SYMMETRY" (WEAK NU POLYMORPHISM) (MARÓTI + MCKENZIE '08)
AND CSP($\mathbb{A}$) $\in$ P (RECURSIVE APPLICATIONS OF {GAUSS / LOCAL CHECKING})

INFINITE A?

- $CSP(\mathbb{Q},<) \in P$
- $CSP(\mathbb{Z},0,1,+,\cdot)$ UNDECIDABLE

WHAT IS THE DIFFERENCE?

CONSIDER AN INSTANCE OF $CSP(\mathbb{Q},<)$

- VARIABLES $x_1,\dots,x_n$
- CONSTRAINTS $x_1<x_2,\ x_2<x_3,\ x_3<x_1$

$s: \{x_1,\dots,x_n\} \to \mathbb{Q}$ POTENTIAL SOLUTION

WHETHER OR NOT s IS CORRECT ONLY DEPENDS ON ORBIT OF $(s(x_1),\dots,s(x_n))$

ORBIT: $\{\alpha\cdot(s(x_1),\dots,s(x_n)) : \alpha \in Aut(\mathbb{Q},<)\}$

EQUIVALENTLY: ORBIT = ALL n-TUPLES ORDERED THE SAME WAY

ONLY FINITELY MANY OPTIONS!
= ω-CATEGORICITY

IN $(\mathbb{Z},0,1,+,\cdot)$: INFINITELY MANY ORBITS!

VALUES OF SOLUTION MIGHT BE HUGE!

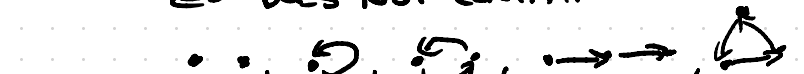
MOREOVER:

FINITE FORMULATION OF $CSP(\mathbb{Q},<)$:

GIVEN: . VARIABLES $x_1,\dots,x_n$
. CONSTRAINTS $x_1<x_2,\ x_2<x_3,\ x_3<x_1$

Q: IS THERE A FINITE TOTAL ORDER IN WHICH THE CONSTRAINTS ARE SATISFIED?

NOTE: A STRUCTURE WITH BINARY RELATION $<$ IS A TOTAL ORDER $\Leftrightarrow$ DOES NOT CONTAIN



AS SUBSTRUCTURES

SO: HAVE TO SATISFY CONSTRAINTS WHILE AVOIDING THESE PATTERNS

VERY DIFFERENT FROM $(\mathbb{Z}, 0, 1, +, \cdot)$:
"$(\mathbb{Q}, <)$ FINITELY BOUNDED"

GENERAL

$\tau = \{R_1, \dots, R_s\}$ RELATIONAL LANGUAGE

$\mathcal{F} = \{\mathbb{F}_1, \dots, \mathbb{F}_t\}$ FORBIDDEN PATTERNS

$\mathcal{C} = \{C_1, \dots, C_u\}$ CONSTRAINT LANGUAGE: EACH C_i BOOLEAN COMBINATION OF $R_1 \dots R_s$, =

E.G. $\tau = \{<\}$ $\mathcal{F}$ = AS BEFORE

$C_1(x,y,z) = \neg(x=y) \wedge \neg(x=z) \wedge \neg(y=z) \wedge (x<y \wedge y<z) \vee (z<y \wedge y<x)$

"y IS BETWEEN x, z"

Forb CSP $(\mathcal{F}, \mathcal{C})$:

GIVEN
- VARIABLES $x_1, \dots, x_n$
- CONSTRAINTS $C_i(x_{i_1}, \dots x_{i_e})$

Q
- CAN CONSTRAINTS BE SATISFIED AVOIDING $\mathcal{F}$?

EXAMPLE $\quad CSP(\mathbb{Q}, <) = \text{Forb CSP}(\mathcal{F}, \mathcal{C})$

$\mathcal{F} = \{\bullet\,\bullet, \circlearrowright, \bullet \rightarrow \bullet, \bullet \rightarrow\!\rightarrow \bullet, \circlearrowleft\}$

$\mathcal{C} = \{<\}$ $\quad$ IN P

EXAMPLE $\quad CSP(\mathbb{Q}, Betw(x,y,z)) = \text{Forb CSP}(\mathcal{F}, \mathcal{C})$

↑ BETWEENNESS RELATION

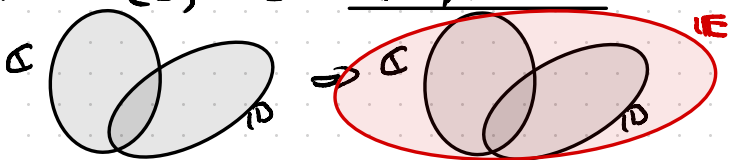
$\mathcal{F}$ ... AS BEFORE

$\mathcal{C} = \{\varphi\}$ WHERE $\varphi =$

IN GENERAL : Forb CSP $(\mathcal{F}, \mathcal{C})$ IN NP

FOR WHICH $\mathcal{F}, \mathcal{C}$ IS THERE $\mathbb{A}$ SUCH THAT Forb CSP $(\mathcal{F}, \mathcal{C}) = CSP(\mathbb{A})$?

- IF Forb$(\mathcal{F})$ HAS AMALGAMATION :



IF Forb(F) HAS AMALGAMATION
⇒ ∃ UNIVERSAL OBJECT CONTAINING Forb(F)
"HOMOGENEOUS"
UNIQUE! (FRAÏSSÉ)

CONJECTURE (BODIRSKY & P. '11)

IF F HAS AMALGAMATION
⇒ ∀ℰ CONSTRAINT LANGUAGES
Forb CSP(F, ℰ) ∈ P OR NP-COMPLETE

⇔

CONJECTURE (BODIRSKY & P. 11)

Ⓤ HOMOGENEOUS, FINITELY BOUNDED
↕
LIMIT OF Forb(F)

𝔸 FIRST ORDER DEFINABLE
⇒ CSP(𝔸) ∈ P OR NP-COMPLETE

HOW TO SOLVE IT ?

OVERALL STRATEGY: REDUCE TO FINITE-DOMAIN CSP

- $\mathbb{A} \rightsquigarrow Pol(\mathbb{A}) = \{f : \mathbb{A}^n \to \mathbb{A} \mid f \text{ HOMOMORPHISM}\}$

"POLYMORPHISM CLONE"

EXAMPLE

$\mathbb{A} = (\mathbb{Q}, <)$

$\Rightarrow f : \mathbb{Q}^n \to \mathbb{Q} \in Pol(\mathbb{A})$

$\Leftrightarrow f$ MONOTONE

E.G. $\max(x,y) : \mathbb{Q}^2 \to \mathbb{Q}$

- COMPLEXITY OF CSP(𝔸)
  - ONLY DEPENDS ON Pol(𝔸) (BODIRSKY + NEŠETŘIL '05)
  - ONLY DEPENDS ON LOCAL IDENTITIES IN Pol(𝔸) $\max(x,y) = \max(y,x)$ (BODIRSKY & P. '11)

- IF THERE ARE NO $e, f, s \in \mathrm{Pol}(\mathbb{A})$ SUCH THAT $\forall x,y,z \; e \circ s(x,y,x,z,y,z) = f \circ s(y,x,z,x,z,y)$

$\Rightarrow$ $\mathrm{CSP}(\mathbb{A})$ NP-COMPLETE

(BARTO + P. '16, BARTO + KOMPATSCHER + OLŠÁK + VAN PHAM + P. '18)

REASON

NO SUCH e, f, s

$\Rightarrow \exists F \subseteq A$ FINITE, $\mathrm{Pol}(\mathbb{A})|_F$ "TRIVIAL"

$\Rightarrow \mathbb{A}|_F$ "SIMULATES" (PP-CONSTRUCTS) 3-COLORING $\square$

THEREFORE CAN ASSUME THERE ARE $e, f, s \in \mathrm{Pol}(\mathbb{A})$ WITH $e \circ s(x,y,x,z,y,z) = f \circ s(y,x,z,x,z,y)$

FIRST DILEMMA

$\mathbb{A}$ FINITE $\Rightarrow$ HAVE BETTER IDENTITIES

E.G. WNU

$w(x,\dots,x,y) = \dots = w(y,x,\dots,x)$

BECAUSE CAN ASSUME THAT $\mathbb{A}$ HAS NO AUTOMORPHISMS "IDEMPOTENCY"

$\mathbb{A}$ INFINITE: LOTS OF AUTOMORPHISMS

E.G. $(\mathbb{Q}, <)$

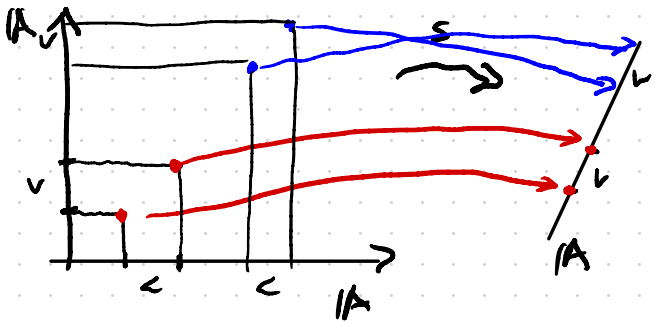
(COMES FROM HOMOGENEITY / ω-CATEGORICITY / FINITE # OF ORBITS)

IDEA MODEL-COMPLETE CORES

SECOND DILEMMA

HAVE $e \circ S(x,y,x,z,y,z) = f \circ S(y,x,z,x,z,y)$

BUT S IS AN INFINITE POSSIBLY CHAOTIC OBJECT



IDEA: USE RAMSEY'S THEOREM TO FIND REGULAR BEHAVIOUR WITH A FINITE DESCRIPTION

(BOORSUK + P. ŚWIĄTKOWSKI "CANONICAL POLYMORPHISMS" '11)

THIRD DILEMMA

TO APPLY RAMSEY'S THEOREM, THE STRUCTURE HAS TO BE ORDERED (LIKE $(\mathbb{Q}, <)$)

BUT REGULAR BEHAVIOUR WITH RESPECT TO AN ORDER IS INCOMPATIBLE WITH SATISFACTION OF

$$e \circ S(x,y,x,z,y,z) = f \circ S(y,z,z,x,z,y)$$

IDEA:

- ADD ORDER
- USE RAMSEY'S THEOREM TO FIND REGULAR BEHAVIOUR
- FORGET ORDER
- OBTAIN IDENTITY SATISFIED BY CANONICAL POLYMORPHISM

SYSTEMATIC APPROACH: SMOOTH APPROXIMATIONS

- IF $e \cdot s(x,y,x,z,y,z) = f \cdot s(y,x,z,x,z,y)$ IS SATISFIED BY s WITH REGULAR BEHAVIOUR $\Rightarrow$ $CSP(\mathbb{A}) \in P$

CONJECTURE (BODIRSKY + P. '11)

$\mathbb{U}$ HOMOGENEOUS, FINITELY BOUNDED $\leftrightarrow$ LIMIT OF $Forb_{fin}(\mathcal{F})$

$\mathbb{A}$ FIRST ORDER DEFINABLE

$\Rightarrow$ $CSP(\mathbb{A}) \in P$ OR NP-COMPLETE

CONFIRMED FOR:

UNORDERED STRUCTURES: (REDUCTION TO FINITE)

CSPs OVER
- GRAPHS
- K_n-FREE GRAPHS
- TOURNAMENTS
- TREES
- PARTIAL ORDERS
- UNARY STRUCTURES

ORDERED:
- $(\mathbb{Q}, <)$

THEOREM (MOTTET + NAGY + P. '23)

LET $\mathbb{H}$ BE HOMOGENEOUS UNIVERSAL ℓ-HYPERGRAPH

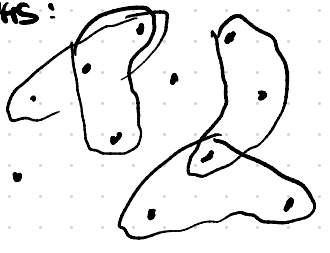
$\mathbb{A}$ FIRST-ORDER DEFINABLE

$\Rightarrow$ $CSP(\mathbb{A}) \in P$ OR NP-COMPLETE

"FORGETTING ORDER" DOES NOT WORK:
THERE ARE $\mathbb{A}$ WHERE NO ELEMENT OF $pol(\mathbb{A})$ SIMULTANEOUSLY:
- BEHAVES REGULARLY
- SATISFIES $e\,s(x_1y_1x_2y_2) = f\,s(y_1x_2x_2y_1)$

SOLUTION: DIRECT ALGORITHMS RATHER THAN REDUCTION TO FINITE

3-HYPERGRAPHS:



Prof.

MICHAEL.PINSKER@TUWIEN.AC.AT

Polymorphisms, and how to use them

THANK YOU!

