SMOOTH APPROXIMATIONS & CSPs

OVER FINITELY BOUNDED HOMOGENEOUS STRUCTURES

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I THE TOPIC

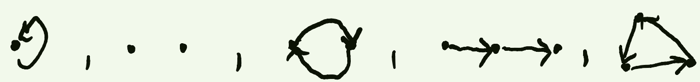
- τ ... RELATIONAL SIGNATURE
- $\mathcal{F}$ ... FINITE SET OF FORBIDDEN τ-STRUCTURES
- $\mathcal{C}$ ... CONSTRAINT LANGUAGE:
  FINITE SET OF QF FO-FORMULAS

$\mathrm{CSP}(\mathcal{F},\mathcal{C})$

GIVEN: VARIABLES $x_1, \dots, x_n$, CONSTRAINTS FROM $\mathcal{C}$

Q: CONSTRAINTS SATISFIABLE WITHIN $\mathrm{Forb}(\mathcal{F}) :=$ STRUCTURES AVOIDING $\mathcal{F}$?

EXAMPLE

- $\tau = \{<\}$ BINARY SYMBOL
- $\mathcal{F} = \{$  $\}$
- $\mathcal{C} = \{\underbrace{(x<y<z) \vee (z<y<x)}_{=:B(x,y,z)}\}$

$\mathrm{CSP}(\mathcal{F},\mathcal{C})$: GIVEN: $x_1, \dots, x_n$, CONSTRAINTS $B(x_1, x_3, x_2), B(x_4, x_9, x_1) \dots$

Q: SATISFIABLE IN TOTAL ORDER?

EXAMPLE

- $\tau = \{E\}$ BINARY
- $\mathcal{F} = \{ \bullet\!\circlearrowleft \; , \; \bullet\!\longrightarrow\!\bullet \}$
- $$\begin{aligned} e = \{ & (E(x,y) \wedge \neg E(x,z) \wedge \neg E(y,z)) \vee \\ & (\neg E \quad\quad E \quad\quad \neg E \quad) \vee \\ & (\neg E \quad\quad \neg E \quad\quad E \quad) \; \} \end{aligned}$$

$\underline{CSP(\mathcal{F}, e)}$ ... ESSENTIALY 1-IN-3SAT ON EDGES/NON-EDGES OF GRAPHS

FURTHER EXAMPLES

SATISFYING FO-CONSTRAINTS IN A
- TOURNAMENT
- K_n-FREE GRAPH
- EQUIVALENCE RELATION
- TREE
- PARTIAL ORDER
-
- ...
-

$\boxed{\mathrm{CSP}(\mathcal{F}, \mathcal{L}) \overset{?}{=} \mathrm{CSP}(\mathbb{A})}$

CLASSICAL FIXED-TEMPLATE CSP

VARIABLES $\rightsquigarrow$ VALUES IN STRUCTURE $\mathbb{A}$

IDEA

① $\mathrm{Forb}(\mathcal{F})$ = ALLOWED FINITE STRUCTURES

$\rightsquigarrow$ UNIVERSAL INFINITE OBJECT $\mathbb{U}$

EXAMPLE
- TOTAL ORDERS $\rightsquigarrow (\mathbb{Q}, <)$
- GRAPHS $\rightsquigarrow$ RANDOM GRAPH
- K_n-FREE GRAPHS $\rightsquigarrow$ " K_n-FREE GRAPH
- ⋮ ...

② CONSTRAINT LANGUAGE $\mathcal{L}$ (FO-FORMULAS) $\rightsquigarrow$ $\mathbb{A}$ FO-DEFINABLE IN $\mathbb{U}$

EXAMPLE: $\mathbb{A} = (\mathbb{Q}, \mathrm{Betw}(x,y,z))$

CONJECTURE (BODIRSKY + P. '11)

$\mathbb{U}$ ... FINITELY BOUNDED HOMOGENEOUS

$\mathbb{A}$ ... FO-DEFINABLE

$\Rightarrow$ $\mathrm{CSP}(\mathbb{A})$ IN P OR NP-C

ADVANTAGE: ALGEBRAIC APPROACH

$\mathrm{Pol}(\mathbb{A})$ =

POLYMORPHISMS OF $\mathbb{A}$ =

$\mathrm{HOM}(\mathbb{A}^n \to \mathbb{A})$, $n \geq 1$

INTERESTING ?

- DICHOTOMY : NOT AT ALL !
- STRUCTURAL REASONS FOR (NON-) TRACTABILITY / LOGIC FOR P :
  VERY !
- METHODS : MODEL THEORY
  UNIVERSAL ALGEBRA
  RAMSEY THEORY
- SCOPE : HUGE

II RESULTS

A CONJECTURE (P/NPc DICHOTOMY) TRUE FOR:

Forb($\mathcal{F}$) =
- TOTAL ORDERS (JACM '09)
- GRAPHS (JACM '15)
- K_n-FREE GRAPHS (SICOMP '19)
- TOURNAMENTS NEW
- ANY OTHER PREVIOUSLY KNOWN CLASS E.G. MMSNP (SICOMP '21)

B CSP($\mathbb{A}$) $\in$ P $\underset{P \neq NP}{\Longleftrightarrow}$ Pol($\mathbb{A}$) CONTAINS e, f, s:

$$e\,s(x,y,x,z,y,z) = f\,s(y,x,z,x,z,y)$$

C FOR Forb($\mathcal{F}$) =
- GRAPHS
- K_n-FREE GRAPHS
- TOURNAMENTS

NEW

CSP($\mathbb{A}$) SOLVABLE BY DATALOG $\Longleftrightarrow$ Pol($\mathbb{A}$) CONTAINS $e_1, \dots, e_n, w$

$e_1 w(x \dots xy) = \dots = e_n w(yx \dots x)$

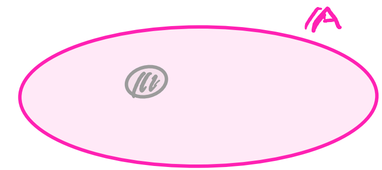
FOR ALL $n \geq 3$

D $\boxed{\text{TRUE RESULT = METHOD !}}$

III METHOD

$\mathrm{Pol}(\mathbb{A})$ … POLYMORPHISMS OF $\mathbb{A}$

$\mathrm{Col}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{A})$ … "CANONICAL POLYMORPHISMS"
BEHAVE REGULARLY



KNOWN:

① $\mathrm{Pol}(\mathbb{A})$ "NAKED SOMEWHERE"
$\Rightarrow$ $\mathrm{CSP}(\mathbb{A})$ NP-c (TAMS '14)

② $\mathrm{Col}(\mathbb{A})$ NOT NAKED $\Rightarrow$ $\mathrm{CSP}(\mathbb{A}) \in P$ (LICS '16)

⊕ $\neg$① $\Rightarrow$ $\mathrm{Pol}(\mathbb{A})$ SATISFIES
$$e\,s(xyxzyz) = f\,s(yxzxzy)$$
(SICOMP '21)

OLD METHOD

$\neg$① / $\mathrm{Pol}(\mathbb{A})$ NOT NAKED
USE RAMSEY THEORY
TO PROVE ②

- ⊕ NOT USED
- EXHAUSTIVE CASE DISTINCTION OVER CANONICAL FUNCTIONS

NEW METHOD

$\neg$② / $\mathrm{Col}(\mathbb{A})$ NAKED

SMOOTH APPROX
$\Rightarrow$ $\mathrm{Pol}(\mathbb{A})$ NAKED
①

$\mathrm{Pol}(\mathbb{A})$ HAS "COMMUTATIVE FCT"
⊕ $e\,s(\ ) = f\,s(\ )$
$\Rightarrow$ ② ↯

IV ACHIEVEMENTS & FUTURE

- NEW PROOFS < 10% OF OLD ONES
- PROOF FOR GRAPHS ⊆ PROOF FOR TOURNAMENT
- SCALABLE : USE OF EQUATIONS INSTEAD OF CONCRETE POLYMORPHISMS

  SIMILAR TO BOOLEAN CSP ⇝ FINITE DOMAIN CSP

- FUTURE :

  - DATALOG FOR E.G. MMSNP (ICALP '21)
  - DICHOTOMY FOR HYPERGRAPHS (MOTTET + NAGY + P. '22)

    HYPERGRAPHS NOT CLASSIFIABLE / ABSTRACT CONDITIONS !

  - GENERAL ARGUMENT USING esc( ) = fsc( )

    + "COMMUTATIVE" POLYMORPHISM

    ⟹ CONJECTURE ?

    LIFTING OF ZHUK'S ALGORITHM FROM FINITE CSP

Thank you:

- the audience
- all reviewers

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