Reconstructing structures from their abstract clones

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Reconstructing structures from their automorphism groups and polymorphism clones

The topology of algebras

Reconstruction notions, results, problems

Michael Pinsker
Reconstructing structures from their automorphism groups and polymorphism clones
Outline

- Reconstructing structures from their automorphism groups and polymorphism clones
- The topology of algebras
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- Reconstructing structures from their automorphism groups and polymorphism clones
- The topology of algebras
- Reconstruction notions, results, problems
Part I

Reconstructing structures from their automorphism groups and polymorphism clones
Reconstructing structures up to first-order . . .

Theorem (Ryll-Nardzewski)

Let $\Delta$, $\Gamma$ be $\omega$-categorical structures on the same domain. Then $\text{Aut}(\Delta) = \text{Aut}(\Gamma) \iff \Delta, \Gamma$ are first-order interdefinable.

Aut as a topological group

Theorem (Ahlbrandt + Ziegler '86)

Let $\Delta$, $\Gamma$ be $\omega$-categorical structures. Then $\text{Aut}(\Delta) \cong \text{T} \text{Aut}(\Gamma) \iff \Delta, \Gamma$ are first-order bi-interpretable.

Reconstructing sheep from clones

Michael Pinsker
Reconstructing structures up to first-order . . .

\[ \text{countable} \]
Reconstructing structures up to first-order . . .

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countable, $\omega$-categorical
Reconstructing structures up to first-order . . .
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\[ \text{Aut}(\bullet) \]
Reconstructing structures up to first-order . . .

\[ \text{Aut}(\text{house}) \rightarrow \]
Reconstructing structures up to first-order . . .

\[ \text{Aut(\[\text{house}\])} \rightarrow \text{\[\text{house}\]} \text{ first-order interdefinable with } \text{\[\text{house}\]} \]
Reconstructing structures up to first-order . . .

\[ \text{Aut}(\Delta) \rightarrow \text{first-order interdefinable with } \text{Aut}(\Gamma) \]

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Michael Pinsker
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\[ \text{Aut}(\Delta) \text{ as a topological group } \rightarrow \text{first-order bi-interpretable with } \Gamma. \]
Reconstructing structures up to first-order . . .

\[
\text{Aut}(\mathcal{A}) \rightarrow \mathcal{A} \quad \text{first-order interdefinable with} \quad \mathcal{B}
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Theorem (Ahlbrandt + Ziegler ’86)

Let \( \Delta, \Gamma \) be \( \omega \)-categorical structures. Then \( \text{Aut}(\Delta) \cong^T \text{Aut}(\Gamma) \iff \Delta, \Gamma \) are first-order bi-interpretable.
Reconstruction from the abstract group

$\text{Aut}(\mathbb{H})$ as an abstract group $\rightarrow \ ?$
Can we reconstruct a \(\omega\)-categorical structure \(\Delta\) from the algebraic group structure of \(\text{Aut}(\Delta)\)?
Reconstruction from the abstract group

\[ \text{Aut}(\Delta) \text{ as an abstract group } \rightarrow ? \]

- Can we reconstruct an \( \omega \)-categorical structure \( \Delta \) from the algebraic group structure of \( \text{Aut}(\Delta) \)?

- Can we reconstruct the topological structure of \( \text{Aut}(\Delta) \) from its algebraic structure?
Let $\Delta$ be a structure. $\text{Aut}(\Delta)$ is the automorphism group of $\Delta$, $\text{End}(\Delta)$ is the endomorphism monoid of $\Delta$, and $\text{Pol}(\Delta)$ is the polymorphism clone of $\Delta$. All homomorphisms $f: \Delta \to \Delta$ belong to $\text{Pol}(\Delta)$, for all $n \geq 1$.

$\text{Pol}(\Delta)$ is a function clone: it is closed under composition and contains projections.

Observe: $\text{Pol}(\Delta) \supseteq \text{End}(\Delta) \supseteq \text{Aut}(\Delta)$.
Better reconstruction plans

Let $\Delta$ be a structure.

$\text{Aut}(\Delta)$ . . . automorphism group of $\Delta$

$\text{End}(\Delta)$ . . . endomorphism monoid of $\Delta$

$\text{Pol}(\Delta)$ . . . polymorphism clone of $\Delta$

$\text{End}(\Delta)$ . . . all homomorphisms $f: \Delta \rightarrow \Delta$

$\text{Pol}(\Delta)$ . . . all homomorphisms $f: \Delta^n \rightarrow \Delta$, where $1 \leq n < \omega$

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\[ \text{End}(\Delta) \ldots \text{all homomorphisms } f : \Delta \rightarrow \Delta. \]

\[ \text{Pol}(\Delta) \ldots \text{all homomorphisms } f : \Delta^n \rightarrow \Delta, \text{ where } 1 \leq n < \omega. \]

\( \text{Pol}(\Delta) \) is a function clone:
- closed under composition
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**Observe:** \( \text{Pol}(\Delta) \supseteq \text{End}(\Delta) \supseteq \text{Aut}(\Delta) \).
Reconstruction up to primitive positive definitions

Theorem (Bodirsky + Nešetřil '03)
Let $\Delta, \Gamma$ be $\omega$-categorical structures on the same domain. Then $\text{Pol}(\Delta) = \text{Pol}(\Gamma)$ $\iff$ $\Delta, \Gamma$ are primitive positive interdefinable.

Why primitive positive definitions?
For $\Delta$ a structure with a finite relational signature $\tau$:

Definition (Constraint Satisfaction Problem) $\text{CSP}(\Delta)$ is the computational problem to decide whether a given primitive positive $\tau$-sentence holds in $\Delta$. 

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Reconstruction up to primitive positive definitions

Pol(🏠) → ?
Reconstruction up to primitive positive definitions

\[ \text{Pol}(\vdash) \rightarrow ? \]

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Reconstructing sheep from clones
Reconstruction up to primitive positive definitions

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Topological clones

Function clones carry:
- algebraic structure (composition / equations)
- topological structure (pointwise convergence)

Let $C, D$ be function clones.

$\xi: C \to D$ is a (clone) homomorphism iff it preserves arities; sends every projection in $C$ to the corresponding projection in $D$;

$\xi(f(g_1, \ldots, g_n)) = \xi(f)(\xi(g_1), \ldots, \xi(g_n))$ for all $f, g_1, \ldots, g_n \in C$.

\[ \Rightarrow \]

Topological clones

Theorem (Bodirsky + MP '12)

Let $\Delta, \Gamma$ be $\omega$-categorical structures. Then:

$Pol(\Delta) \cong TPol(\Gamma) \iff \Delta, \Gamma$ are primitive positive bi-interpretable.

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Michael Pinsker
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Let $\Delta, \Gamma$ be $\omega$-categorical structures. Then: $\text{Pol}(\Delta) \sim \text{T} \text{Pol}(\Gamma) \iff \Delta, \Gamma$ are primitive positive bi-interpretable.

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Michael Pinsker
Topological clones

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Reconstructing sheep from clones
Michael Pinsker
Topological clones

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$\Rightarrow$ Topological clones
Topological clones

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\[ \implies \text{Topological clones} \]

Theorem (Bodirsky + MP ’12)

Let $\Delta, \Gamma$ be $\omega$-categorical structures. Then:
$\text{Pol}(\Delta) \cong^T \text{Pol}(\Gamma) \iff \Delta, \Gamma \text{ are primitive positive bi-interpretable.}$
Reconstruction from the abstract clone

Can we reconstruct an \( \omega \)-categorical structure \( \Delta \) from the algebraic clone structure of \( \text{Pol}(\Delta) \)? Can we reconstruct the topological structure of \( \text{Pol}(\Delta) \) from its algebraic structure?
Reconstruction from the abstract clone

\( \text{Pol}(\text{\includegraphics[width=5em]{house.png}}) \) as an abstract clone \( \rightarrow \) ?
Can we reconstruct an $\omega$-categorical structure $\Delta$ from the algebraic clone structure of $\text{Pol}(\Delta)$?
Reconstruction from the abstract clone

Pol(-house) as an abstract clone $\rightarrow \ ?$

- Can we reconstruct an $\omega$-categorical structure $\Delta$ from the algebraic clone structure of Pol($\Delta$)?
- Can we reconstruct the topological structure of Pol($\Delta$) from its algebraic structure?
Part II

The topology of algebras
Clones from algebras

Let $A$ be an algebra. Term functions of $A$ (obtained by composition): function clone $\text{Clo}(A)$. $\text{Clo}(A)$ encodes the equations (=identities) which hold in $A$.

Universal Algebra: Structure of $A$ $\iff$ equations in $\text{Clo}(A)$.

Reconstructing sheep from clones
Let $\mathcal{A}$ be an algebra.
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Universal Algebra: Structure of $A \iff$ equations in $\text{Clo}(A)$. 
Birkhoff’s theorem

For an algebra $A$, consider the algebras obtained by taking homomorphic images, subalgebras, and powers/finite powers.

Theorem (Birkhoff 1935)
Let $A, B$ be algebras. Then $\text{Clo}(B) = \text{Clo}(C)$ for some $C \in \text{HSP}(A) \iff \exists$ clone homomorphism from $\text{Clo}(A)$ onto $\text{Clo}(B)$.

Theorem (Bodirsky + MP ‘11)
Let $A, B$ be countable. Then $\text{Clo}(B) = \text{Clo}(C)$ for some $C \in \text{HSP}_{\text{fin}}(A) \iff \exists$ uniformly continuous clone homomorphism from $\text{Clo}(A)$ onto $\text{Clo}(B)$.
Birkhoff’s theorem

For an algebra $\mathfrak{A}$ consider the algebras obtained by taking

Theorem (Birkhoff 1935)

Let $\mathfrak{A}$, $\mathfrak{B}$ be algebras. Then $\text{Clo}(\mathfrak{B}) = \text{Clo}(\mathfrak{C})$ for some $\mathfrak{C} \in \text{HSP}(\mathfrak{A})$ if and only if there exists a clone homomorphism from $\text{Clo}(\mathfrak{A})$ onto $\text{Clo}(\mathfrak{B})$.

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Let $\mathfrak{A}$, $\mathfrak{B}$ be countable. Then $\text{Clo}(\mathfrak{B}) = \text{Clo}(\mathfrak{C})$ for some $\mathfrak{C} \in \text{HSP}_{\text{fin}}(\mathfrak{A})$ if and only if there exists a uniformly continuous clone homomorphism from $\text{Clo}(\mathfrak{A})$ onto $\text{Clo}(\mathfrak{B})$. 
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- Homomorphic images

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- Homomorphic images
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Reconstructing sheep from clones

Michael Pinsker
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HSP vs. HSP$^\text{fin}$

When do HSP and HSP$^\text{fin}$ coincide for an algebra?

When can HSP$^\text{fin}$ be described algebraically?

Can we reconstruct the topological structure of function clones from their algebraic structure?
HSP vs. HSP$^\text{fin}$

- When do HSP and HSP$^\text{fin}$ coincide for an algebra?
HSP vs. $\text{HSP}^{\text{fin}}$

- When do HSP and $\text{HSP}^{\text{fin}}$ coincide for an algebra?
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HSP vs. $\text{HSP}^\text{fin}$

- When do HSP and $\text{HSP}^\text{fin}$ coincide for an algebra?
- When can $\text{HSP}^\text{fin}$ be described algebraically?
- Can we reconstruct the topological structure of function clones from their algebraic structure?
Part III

Reconstruction notions & results
Reconstruction notions

Let $O$ be the largest function clone on $\omega$, and $C$ be a closed subclone.

Definition

$C$ has reconstruction $\iff C \cong D$ implies $C \cong T D$ for all closed subclones $D$ of $O$;

$C$ has automatic homeomorphicity $\iff$ every clone isomorphism between $C$ and a closed subclone of $O$ is a homeomorphism;

$C$ has automatic continuity $\iff$ every clone homomorphism from $C$ into $O$ is continuous.

Observation. (2) $\Rightarrow$ (1).

Fact. For groups (3) $\Rightarrow$ (2).

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Reconstruction notions

Let $\mathcal{O}$ be the largest function clone on $\omega$, and $\mathcal{C}$ be a closed subclone.

Definition $\mathcal{C}$ has reconstruction $\iff \mathcal{C} \sim = \mathcal{T} \mathcal{D}$ implies $\mathcal{C} \sim = \mathcal{T} \mathcal{D}$ for all closed subclones $\mathcal{D}$ of $\mathcal{O}$;

$\mathcal{C}$ has automatic homeomorphicity $\iff$ every clone isomorphism between $\mathcal{C}$ and a closed subclone of $\mathcal{O}$ is a homeomorphism;

$\mathcal{C}$ has automatic continuity $\iff$ every clone homomorphism from $\mathcal{C}$ into $\mathcal{O}$ is continuous.

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- $\mathcal{C}$ has reconstruction $\iff \mathcal{C} \cong \mathcal{D}$ implies $\mathcal{C} \cong^T \mathcal{D}$
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Reconstruction notions

Let $\mathcal{O}$ be the largest function clone on $\omega$, and $\mathcal{C}$ be a closed subclone.

**Definition**

- $\mathcal{C}$ has **reconstruction** $\iff \mathcal{C} \cong D$ implies $\mathcal{C} \cong^T D$ for all closed subclones $D$ of $\mathcal{O}$;

- $\mathcal{C}$ has **automatic homeomorphicity** $\iff$ every clone isomorphism between $\mathcal{C}$ and a closed subclone of $\mathcal{O}$ is a homeomorphism;

**Observation.** $(2) \implies (1)$.

**Fact.** For groups $(3) \implies (2)$. 

Reconstructing sheep from clones

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Let $\mathcal{O}$ be the largest function clone on $\omega$, and $\mathcal{C}$ be a closed subclone.

**Definition**

- **$\mathcal{C}$ has reconstruction** $\iff$ $\mathcal{C} \cong D$ implies $\mathcal{C} \cong^T D$ for all closed subclones $D$ of $\mathcal{O}$;
- **$\mathcal{C}$ has automatic homeomorphicity** $\iff$ every clone isomorphism between $\mathcal{C}$ and a closed subclone of $\mathcal{O}$ is a homeomorphism;
- **$\mathcal{C}$ has automatic continuity** $\iff$ every clone homomorphism from $\mathcal{C}$ into $\mathcal{O}$ is continuous.

Observation. (2) $\Rightarrow$ (1).

Fact. For groups (3) $\Rightarrow$ (2).

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Reconstruction notions

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**Observation.** (2) $\implies$ (1).
Reconstruction notions

Let \( O \) be the largest function clone on \( \omega \), and \( C \) be a closed subclone.

**Definition**

- **C** has reconstruction \( \iff C \cong D \implies C \cong^T D \)
  for all closed subclones \( D \) of \( O \);

- **C** has automatic homeomorphicity \( \iff \) every clone isomorphism between \( C \) and a closed subclone of \( O \) is a homeomorphism;

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Groups: the small index property

Automorphism groups with automatic continuity: 

- $(G;\cong)$ (Dixon+Neumann+Thomas'86)
- $(G;\triangleleft)$ and the atomless Boolean algebra (Truss'89)
- the random graph (Hodges+Hodkinson+Lascar+Shelah'93)
- the random $K_n$-free graphs (Herwig'98)
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- \((\mathbb{N}; =)\) (Dixon+Neumann+Thomas’86)
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Groups: Rubin’s forall-exists interpretations

Automorphism groups with automatic homeomorphicity: the random graph \((\mathbb{Q};<)\) all homogeneous countable graphs various \(\omega\)-categorical semilinear orders the random partial order the random tournament (Rubin '94) the random \(k\)-hypergraphs the Henson digraphs (Barbina+MacPherson '07).

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Observation
If $\Delta$ is $\omega$-categorical, then $\text{Emb}(\Delta)$ does not have automatic continuity.

Theorem (Evans + Hewitt '90)
There exists an $\omega$-categorical $\Delta$ such that $\text{Aut}(\Delta)$ does not have reconstruction.

Theorem (Bodirsky + Evans + Kompatscher + MP '16)
$\text{Pol}(\Delta)$, $\text{End}(\Delta)$, $\text{Aut}(\Delta)$ do not have reconstruction.
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Method I: Automatic continuity via Birkhoff’s theorem

Let $C$ be a closed subclone of $O$, and $\xi: C \to O$ be a homomorphism.

Theorem (Birkhoff ’35)

The algebra $(\omega; \xi[C])$ is an HSP of the algebra $(\omega; C)$.

The only possibly discontinuous step is an infinite product.

Theorem (Bodirsky + MP + Pongrácz ’13)

Any closed subclone of $O$ containing $\omega \omega$ has automatic continuity and automatic homeomorphicity.
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Method II: Automatic homeomorphicity via groups

Let $C$ be a closed subclone of $O$ whose group $G_C$ of invertibles has automatic homeomorphicity. Show:
- the closure of $G_C$ in $O$ has reconstruction;
- the clone of unary functions of $C$ has reconstruction;
- $C$ has reconstruction.

Theorem (Bodirsky + MP + Pongrácz '13)

Let $G$ be the random graph. The following have automatic homeomorphicity:
- $\text{End}(G)$;
- $\text{Pol}(G)$;
- Various other famous clones containing $\text{Aut}(G)$. 

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Method III: Rubin’s interpretations

Interpret structure \( \Delta \) in the algebraic structure of its clone \( \text{Pol}(\Delta) \).

Theorem (Maissel + Rubin '15)

Let \( \text{Pol}(\Delta) \), \( \text{Pol}(\Delta') \) contain all transpositions on their domain \( \omega \).

Then any clone isomorphism \( \text{Pol}(\Delta) \to \text{Pol}(\Delta') \) is induced by a permutation of \( \omega \).
Interpret structure $\Delta$ in the algebraic structure of its clone $\text{Pol}(\Delta)$.

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Part IV
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Let \( \mathbf{1} \) be the clone containing only projections – the smallest clone.
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Problem

Let \( \Delta \) be \( \omega \)-categorical.
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**Problem**

Let $\Delta$ be $\omega$-categorical.

- If $\text{Pol}(\Delta) \to \mathbf{1}$ via a clone homomorphism, then also continuously?
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Let 1 be the clone containing only projections – the smallest clone.

Problem

Let Δ be ω-categorical.

- If Pol(Δ) → 1 via a clone homomorphism, then also continuously?
- 1 ∈ HSP(Pol(Δ)) implies 1 ∈ HSP^{fin}(Pol(Δ))?
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Let $\Delta$ be $\omega$-categorical.

- If $\text{Pol}(\Delta) \rightarrow 1$ via a clone homomorphism, then also continuously?
- $1 \in \text{HSP}(\text{Pol}(\Delta))$ implies $1 \in \text{HSP}^{\text{fin}}(\text{Pol}(\Delta))$?

Theorem (Barto + Kompatscher + Olšák + Van Pham + MP ’17)

Let $\Delta$ be $\omega$-categorical, with less than double exponential type growth. TFAE:
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**Theorem (Barto + Kompatscher + Olšák + Van Pham + MP ’17)**

Let $\Delta$ be $\omega$-categorical, with less than double exponential type growth.

TFAE:

- There is no linear uniformly continuous homomorphism $\text{Pol}(\Delta) \to \mathbf{1}$;
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Let \( \mathbf{1} \) be the clone containing only projections – the smallest clone.

**Problem**

Let \( \Delta \) be \( \omega \)-categorical.

- If \( \text{Pol}(\Delta) \rightarrow \mathbf{1} \) via a clone homomorphism, then also continuously?
- \( \mathbf{1} \in \text{HSP}(\text{Pol}(\Delta)) \) implies \( \mathbf{1} \in \text{HSP}^\text{fin}(\text{Pol}(\Delta)) \)?

**Theorem (Barto + Kompatscher + Olšák + Van Pham + MP ’17)**

Let \( \Delta \) be \( \omega \)-categorical, with less than double exponential type growth.

TFAE:

- There is no linear uniformly continuous homomorphism \( \text{Pol}(\Delta) \rightarrow \mathbf{1} \);
- \( \text{Pol}(\Delta) \) contains functions \( u, v \) (unary) and \( s \) (6-ary) such that

\[
\forall x, y, z \ (u \circ s(x, y, x, z, y, z) = v \circ s(y, x, z, x, z, y))
\]
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Thank you!