The Equivalence of Two Dichotomy Conjectures for Infinite Domain CSPs

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Constraint Satisfaction Problems (CSPs)

Let $\Gamma = (D; R_1, \ldots, R_n)$ be a relational structure.

Definition CSP $\Gamma$

**INPUT:** A primitive positive sentence $\phi \equiv \exists x_1 \cdots \exists x_n R_i^1(\ldots) \land \cdots \land R_i^m(\ldots)$

**QUESTION:** $\Gamma |_\omega = \phi$??

$\Gamma$ (i.e., its domain) can be finite or infinite.

Number of relations finite.

Any computational problem can be modeled as CSP $(\Gamma)$.

$\omega$-categorical $\Rightarrow$ "algebraic-topological approach".

$\omega$-categorical: countable and $\Gamma_n/\text{Aut}(\Gamma)$ is finite for all $n \geq 1$.

One conjecture for infinite CSPs

Michael Pinsker
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- Any computational problem can be modeled as CSP($\Gamma$).
- $\Gamma$ $\omega$-categorical $\implies$ “algebraic-topological approach".
Constraint Satisfaction Problems (CSPs)

Let \( \Gamma = (D; R_1, \ldots, R_n) \) be a relational structure.

**Definition CSP(\( \Gamma \))**

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**QUESTION:** \( \Gamma \models \phi \) ???

- \( \Gamma \) (i.e., its domain) can be finite or infinite.
- Number of relations finite.
- Any computational problem can be modeled as \( \text{CSP}(\Gamma) \).
- \( \Gamma \) \( \omega \)-categorical \( \implies \) "algebraic-topological approach".

\( \omega \)-categorical: countable and \( \Gamma^n/\text{Aut}(\Gamma) \) is finite for all \( n \geq 1 \).
The algebraic-topological approach: clones

\[ \text{CSP}(\Gamma) \downarrow \text{Pol}(\Gamma) = \{ h : \Gamma^* \rightarrow \Gamma \mid n \geq 1, h \text{ homomorphism} \} \]

"Polymorphism clone" poor-\(\Gamma\) \iff "rich-\(\Gamma\) \iff "poor-\(\text{Pol}(\Gamma)\)

In the following:

- \(\Gamma\) poor-\(\iff\) \text{CSP}(\Gamma) \text{ in P}
- \(\Gamma\) rich-\(\iff\) \text{CSP}(\Gamma) NP-hard

Goal: Characterize these by structural properties of \(\text{Pol}(\Gamma)\).

When is \(\text{Pol}(\Gamma)\) "rich" / "poor"?

\[\text{ONE conjecture for infinite CSPs}\]
The algebraic-topological approach: clones

\[ \text{CSP}(\Gamma) \]

In the following:

\( \Gamma \) poor: \( \Leftrightarrow \) CSP(\( \Gamma \)) in P

\( \Gamma \) rich: \( \Leftrightarrow \) CSP(\( \Gamma \)) NP-hard

Goal: Characterize these by structural properties of \( \text{Pol}(\Gamma) \).

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The algebraic-topological approach: clones

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\text{CSP}(\Gamma) \downarrow \quad \text{Pol}(\Gamma) = \{ h: \Gamma^n \to \Gamma \mid n \geq 1, \; h \text{ homomorphism} \}
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“Polymorphism clone"
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“Polymorphism clone”

“poor” \( \Gamma \) ⇔ “rich” \( \text{Pol}(\Gamma) \)

“rich” \( \Gamma \) ⇔ “poor” \( \text{Pol}(\Gamma) \)
The algebraic-topological approach: clones

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The algebraic-topological approach: clones

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“poor" \( \Gamma \) \( \iff \) “rich" \( \text{Pol}(\Gamma) \)

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Goal:

ONE conjecture for infinite CSPs

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The algebraic-topological approach: clones

\[
\begin{align*}
\text{CSP}(\Gamma) \\
\downarrow \\
\text{Pol}(\Gamma) = \{h : \Gamma^n \to \Gamma \mid n \geq 1, \ h \text{ homomorphism}\}
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“poor” \(\Gamma\) \(\iff\) “rich” \(\text{Pol}(\Gamma)\)

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The algebraic-topological approach: clones

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"poor" \( \Gamma \) \( \iff \) "rich" \( \text{Pol}(\Gamma) \)

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Goal:

- Characterize these by \textbf{structural properties} of \( \text{Pol}(\Gamma) \).
- When is \( \text{Pol}(\Gamma) \) "rich" / "poor"?
Structure of $\text{Pol}(\Gamma)$

Example:

$$\forall x, y, z. u(s(x, y, x, z, y, z)) = v(s(y, x, z, x, z, y))$$

topological / metric structure:

$$(f_i)_{i \in \omega} \rightarrow f: \leftrightarrow \forall c (f_i(c) = f(c))$$

Theorem (Bodirsky + P '11)

Let $\Gamma$, $\Delta$ be $\omega$-categorical. Suppose $\text{Pol}(\Gamma)$, $\text{Pol}(\Delta)$ have identical structure:

$$\exists \xi: \text{Pol}(\Gamma) \rightarrow \text{Pol}(\Delta), \text{bijective, preserving identities, uniformly cont.}$$

Then $\text{CSP}(\Gamma)$ and $\text{CSP}(\Delta)$ are polynomial-time equivalent.

Henceforth assume $\omega$-categoricity.

One conjecture for infinite CSPs

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Structure of $\text{Pol}(\Gamma)$

- **algebraic structure**: identities (universally quantified equations)
  Example: $\forall x, y, z. \ u(s(x, y, x, z, y, z)) = v(s(y, x, z, x, z, y))$
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- **topological / metric structure**:
  $(f_i)_{i \in \omega} \rightarrow f : \iff \forall \bar{c} \ (f_i(\bar{c}) = f(\bar{c}) \text{ eventually})$.

Theorem (Bodirsky + P ’11)

Let $\Gamma, \Delta$ be $\omega$-categorical. Suppose $\text{Pol}(\Gamma), \text{Pol}(\Delta)$ have identical structure: $\exists \xi: \text{Pol}(\Gamma) \rightarrow \text{Pol}(\Delta)$, bijective, preserving identities, uniformly cont.

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- **algebraic structure**: identities (universally quantified equations)
  Example: \(\forall x, y, z.\ u(s(x, y, x, z, y, z)) = v(s(y, x, z, x, z, y))\)

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Let \(\Gamma, \Delta\) be \(\omega\)-categorical.
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- **algebraic structure:** identities (universally quantified equations)
  
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Let \( \Gamma, \Delta \) be \( \omega \)-categorical.

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  Example: \( \forall x, y, z. \ u(s(x, y, x, z, y, z)) = v(s(y, x, z, x, z, y)) \)

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**Theorem (Bodirsky + P ’11)**

Let \( \Gamma, \Delta \) be \( \omega \)-categorical.

Suppose Pol(Γ), Pol(Δ) have **identical structure**:
\( \exists \xi: \text{Pol}(\Gamma) \rightarrow \text{Pol}(\Delta), \text{bijective, preserving identities, uniformly cont.} \)
Structure of \( \text{Pol}(\Gamma) \)

- **algebraic structure: identities** (universally quantified equations)
  
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Let \(\Gamma, \Delta\) be \(\omega\)-categorical.

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Then CSP(\(\Gamma\)) and CSP(\(\Delta\)) are polynomial-time equivalent.

Henceforth assume \(\omega\)-categoricity.
Richness and poverty of Pol(Γ)

Poorest polymorphism clone: \( \text{Clone} \overset{P}{\rightarrow} \text{Pol}(\Gamma) \) of projections on domain \( \{0, 1\} \).

Polymorphism clone of a structure with NP-complete CSP. \( P \overset{\rightarrow}{\rightarrow} \text{Pol}(\Gamma) \) (preserving structure: identities + topology) for any \( \Gamma \).

Theorem (Bodirsky + P '11) If \( \text{Pol}(\Gamma) \overset{\rightarrow}{\rightarrow} P \), then CSP(\( \Gamma \)) is NP-hard.

Richness: Theorem (Barto + P '16) \( \text{Pol}(\Gamma) \not\rightarrow P \), even after some preprocessing \( \Leftrightarrow \) Pol(\( \Gamma \)) contains \( u, v, s \):

\[
\begin{align*}
u(s(x, y, x, z, y, z)) &= v(s(y, x, z, x, z, y))
\end{align*}
\]

"Pseudo-Siggers."

Dichotomy Conjecture (Bodirsky + P '11) For a certain class of \( \Gamma \), richness of \( \text{Pol}(\Gamma) \) forces CSP(\( \Gamma \)) into P.

ONE conjecture for infinite CSPs

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Richness and poverty of $\text{Pol}(\Gamma)$

Poorest polymorphism clone:
Richness and poverty of \( \text{Pol}(\Gamma) \)

Poorest polymorphism clone:
- Clone \( \mathbf{P} \) of projections on domain \( \{0, 1\} \).
Richness and poverty of Pol(Γ)

Poorest polymorphism clone:

- Clone $P$ of projections on domain $\{0, 1\}$.
- Polymorphism clone of a structure with NP-complete CSP.

Theorem (Bodirsky + P '11)

If $Pol(Γ) \rightarrow P$, then CSP$(Γ)$ is NP-hard.

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$Pol(Γ) \not\rightarrow P$, even after some preprocessing $\iff$ $Pol(Γ)$ contains $u, v, s$:

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Dichotomy Conjecture (Bodirsky + P '11)

For a certain class of $Γ$, richness of $Pol(Γ)$ forces CSP$(Γ)$ into P.
Richness and poverty of $\text{Pol}(\Gamma)$

**Poorest polymorphism clone:**

- Clone $\mathbf{P}$ of projections on domain $\{0, 1\}$.
- Polymorphism clone of a structure with NP-complete CSP.
- $\mathbf{P} \rightarrow \text{Pol}(\Gamma)$ (preserving structure: identities + topology) for any $\Gamma$. 

Theorem (Bodirsky + P ’11)

If $\text{Pol}(\Gamma) \rightarrow \mathbf{P}$, then $\text{CSP}(\Gamma)$ is NP-hard.

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$\text{Pol}(\Gamma) \not\rightarrow \mathbf{P}$, even after some preprocessing $\iff \text{Pol}(\Gamma)$ contains $u, v, s$:

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Dichotomy Conjecture (Bodirsky + P ’11)

For a certain class of $\Gamma$, richness of $\text{Pol}(\Gamma)$ forces $\text{CSP}(\Gamma)$ into P.
Richness and poverty of Pol(Γ)

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$\text{Pol}(\Gamma) \not\rightarrow \mathbf{P}$, even after some preprocessing $\iff$ $\text{Pol}(\Gamma)$ contains $u, v, s$:

$$u(s(x, y, x, z, y, z)) = v(s(y, x, z, x, z, y)) \text{ "Pseudo-Siggers".}$$

"Pseudo-Siggers".
Richness and poverty of $\text{Pol}(\Gamma)$

**Poorest polymorphism clone:**
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If $\text{Pol}(\Gamma) \rightarrow \mathbf{P}$, then $\text{CSP}(\Gamma)$ is NP-hard.

**Richness:**

**Theorem (Barto + P ’16)**

$\text{Pol}(\Gamma) \leftrightarrow \mathbf{P}$, even after some preprocessing $\iff$ $\text{Pol}(\Gamma)$ contains $u, v, s$: $u(s(x, y, x, z, y, z)) = v(s(y, x, z, x, z, y))$ “Pseudo-Siggers”.

**Dichotomy Conjecture (Bodirsky + P ’11)**

For a certain class of $\Gamma$, richness of $\text{Pol}(\Gamma)$ forces $\text{CSP}(\Gamma)$ into $\mathbf{P}$.
The wonderland of the new rich

Alternative poverty:

If there exists a mapping preserving linear identities (no nesting), uniformly continuous.

Theorem (Barto + Opršal + P '15)

If \( \text{Pol} \ (\Gamma) \not\rightarrow \text{P} \), then \( \text{CSP} \ (\Gamma) \) is NP-hard.

New Dichotomy Conjecture (Barto + Opršal + P '15)

For a certain class of \( \Gamma \), new richness of \( \text{Pol} \ (\Gamma) \) forces \( \text{CSP} \ (\Gamma) \) into P.

\[ \text{Pol} \ (\Gamma) \rightarrow \text{P} \Rightarrow \text{CSP} \ (\Gamma) \text{NP-hard} \]

\[ \text{Pol} \ (\Gamma) = \text{P} \Rightarrow \text{CSP} \ (\Gamma) \text{NP-hard} \]

\[ \text{Pol} \ (\Gamma) \not\rightarrow \text{P} \] (even after preprocessing) \Rightarrow Pseudo-Siggers?

\[ \text{Pol} \ (\Gamma) \not= \text{P} \] \Rightarrow \text{CSP} \ (\Gamma) \text{in P} 

ONE conjecture for infinite CSPs

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Alternative poverty:
Pol(Γ) → P if there exists a mapping preserving linear identities (no nesting), uniformly continuous.
The wonderland of the new rich

**Alternative poverty:**
Pol(Γ) → P if there exists a mapping preserving linear identities (no nesting), uniformly continuous.

**Theorem (Barto + Opršal + P ’15)**
If Pol(Γ) → P, then CSP(Γ) is NP-hard.
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If Pol(Γ) → P, then CSP(Γ) is NP-hard.

**New Dichotomy Conjecture (Barto + Opršal + P ’15):**
For a certain class of Γ, new richness of Pol(Γ) forces CSP(Γ) into P.
The wonderland of the new rich

**Alternative poverty:**
Pol(\(\Gamma\)) \(\rightarrow\) P if there exists a mapping preserving linear identities (no nesting), uniformly continuous.

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If Pol(\(\Gamma\)) \(\rightarrow\) P, then CSP(\(\Gamma\)) is NP-hard.

**New Dichotomy Conjecture (Barto + Opršal + P ’15)**
For a certain class of \(\Gamma\), new richness of Pol(\(\Gamma\)) forces CSP(\(\Gamma\)) into P.

- Pol(\(\Gamma\)) \(\rightarrow\) P \(\implies\) CSP(\(\Gamma\)) NP-hard
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New Dichotomy Conjecture (Barto + Opršal + P ’15)
For a certain class of Γ, new richness of Pol(Γ) forces CSP(Γ) into P.

- Pol(Γ) → P  ⇒  CSP(Γ) NP-hard
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New Dichotomy Conjecture (Barto + Opršal + P ’15)
For a certain class of Γ, new richness of Pol(Γ) forces CSP(Γ) into P.

- Pol(Γ) → P ⇒ CSP(Γ) NP-hard
- Pol(Γ) → P ⇒ CSP(Γ) NP-hard
- Pol(Γ) /→ P (even after preprocessing) ⇒ Pseudo-Siggers ? CSP(Γ) in P
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**New Dichotomy Conjecture (Barto + Opršal + P ’15)**
For a certain class of Γ, new richness of Pol(Γ) forces CSP(Γ) into P.

- Pol(Γ) → P  ⟹  CSP(Γ) NP-hard
- Pol(Γ) → P  ⟹  CSP(Γ) NP-hard
- Pol(Γ) ↳ P (even after preprocessing)  ⟹  Pseudo-Siggers ↳ CSP(Γ) in P
- Pol(Γ) ↳ P  ⟹  CSP(Γ) in P

ONE conjecture for infinite CSPs

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Comparing the rich and the new rich

Theorem

Let $\Gamma$ be the countable atomless Boolean algebra. Then $\text{Pol}(\Gamma)$ has $P$, but $\text{Pol}(\Gamma)$ is not $\rightarrow P$ after preprocessing.

Theorem

Any such $\Gamma$ must have at least double exponential orbit growth: for every $n \geq 1$, $\frac{\Gamma^n}{\text{Aut}(\Gamma)}$ has at least $2^{2^n}$ elements asymptotically.
Comparing the rich and the new rich

Preprocessing:

Theorem
Let $\Gamma$ be the countable atomless Boolean algebra. Then $\text{Pol}(\Gamma)$ is $\exists \Phi_9$, but $\text{Pol}(\Gamma) \not\rightarrow \Phi_9$ after preprocessing.

Theorem
Any such $\Gamma$ must have at least double exponential orbit growth: For every $n \geq 1$, $\Gamma_n/\text{Aut}(\Gamma)$ has at least $2^{2^n}$ elements asymptotically.
Comparing the rich and the new rich

Preprocessing:
- replacing \( \Gamma \) by its model-complete core
  (obtaining \( \text{Aut}(\Gamma) = \text{End}(\Gamma) \))
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- replacing $\Gamma$ by its model-complete core (obtaining $\text{Aut}(\Gamma) = \text{End}(\Gamma)$)
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Irrelevant for $\text{Pol}(\Gamma)$, but not for $\text{Pol}(\Gamma) \rightarrow P$.

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ONE conjecture for infinite CSPs

Michael Pinsker
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Theorem

Let $\Gamma$ be first-order definable in a finitely bounded homogeneous structure. Then the following are equivalent:

1. $\text{Pol}(\Gamma) \not\rightarrow \mathcal{P}$ after preprocessing.
2. $\text{Pol}(\Gamma) \not\rightarrow \mathcal{P}$. (Equivalently, $\text{Pol}(\Gamma)$ satisfies the Pseudo-Siggers identity.)

The Open Problem

Are the above equivalent to the satisfaction of linear identities?

Examples:

- Temp-SAT problems (rational order)
- Graph-SAT problems (random graph)
- Poset-SAT problems (random partial order)

ONE conjecture for infinite CSPs

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Thank you!