Reconstructing structures from their abstract clones

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Outline

Reconstructing structures from their automorphism groups and polymorphism clones

The topology of algebras

Reconstruction the topology of function clones

Reconstructing sheep from clones

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Reconstructing structures from their automorphism groups and polymorphism clones
Reconstructing structures from their automorphism groups and polymorphism clones

The topology of algebras
Outline

- Reconstructing structures from their automorphism groups and polymorphism clones
- The topology of algebras
- Reconstruction the topology of function clones
Part I

Reconstructing structures from their automorphism groups and polymorphism clones
Reconstructing structures up to first-order . . .

Theorem (Ryll-Nardzewski) Let $\Delta$, $\Gamma$ be $\omega$-categorical structures on the same domain. Then $\text{Aut}(\Delta) = \text{Aut}(\Gamma)$ if and only if $\Delta$, $\Gamma$ are first-order interdefinable.

$\text{Aut}(\Delta)$ as a topological group first-order bi-interpretable with $\text{Aut}(\Gamma)$.

Theorem (Ahlbrandt + Ziegler '86) Let $\Delta$, $\Gamma$ be $\omega$-categorical structures. Then $\text{Aut}(\Delta) \sim = \text{T}(\text{Aut}(\Gamma))$ if and only if $\Delta$, $\Gamma$ are first-order bi-interpretable.
Reconstructing structures up to first-order . . .

Theorem (Ryll-Nardzewski)
Let $\Delta, \Gamma$ be $\omega$-categorical structures on the same domain. Then $\text{Aut}(\Delta) = \text{Aut}(\Gamma)$ iff $\Delta, \Gamma$ are first-order interdefinable.

Aut as a topological group

Theorem (Ahlbrandt + Ziegler '86)
Let $\Delta, \Gamma$ be $\omega$-categorical structures. Then $\text{Aut}(\Delta) \cong \text{Aut}(\Gamma)$ iff $\Delta, \Gamma$ are first-order bi-interdefinable.

countable
Reconstructing structures up to first-order . . .

countable, $\omega$-categorical
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Reconstructing structures up to first-order . . .

\[ \text{Aut} (\text{\includegraphics{house}}) \]
Reconstructing structures up to first-order . . .

\[ \text{Aut}(\mathbb{H}) \rightarrow \]
Reconstructing structures up to first-order . . .

\[ \text{Aut(} \text{\ includes the image of a house symbol} \text{)} \rightarrow \text{\ includes the image of a house symbol} \] \text{first-order interdefinable with} \text{\ includes the image of a house symbol}
Reconstructing structures up to first-order . . .

\[ \text{Aut}(\Delta) \rightarrow \text{first-order interdefinable with } \Gamma \]

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\[ \text{Aut(\[\text{\textbullet}\])} \rightarrow \text{\[\text{\textbullet}\)} \text{ first-order interdefinable with } \text{\[\text{\textbullet}\)} \]

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\[ \text{Aut(\[\text{\textbullet}\]) as a topological group} \]

Reconstructing sheep from clones
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\[ \text{Aut}(\Delta) \rightarrow \Delta \text{ first-order interdefinable with } \Gamma \]

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\[ \text{Aut}(\Delta) \text{ as a topological group} \rightarrow \]
Reconstructing structures up to first-order . . .

\[ \text{Aut}(\Delta) \rightarrow \text{\quad} \quad \text{first-order interdefinable with} \quad \text{Aut}(\Gamma) \]

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\[ \text{Aut}(\Delta) \text{ as a topological group} \rightarrow \quad \text{first-order bi-interpretable with} \quad \text{Aut}(\Gamma) \]
Reconstructing structures up to first-order . . .

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\text{Aut}(\mathcal{H}) \rightarrow \mathcal{H} \quad \text{first-order interdefinable with} \quad \mathcal{H}
\]

**Theorem (Ryll-Nardzewski)**

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\text{Aut}(\mathcal{H}) \text{ as a topological group} \rightarrow \mathcal{H} \quad \text{first-order bi-interpretable with} \quad \mathcal{H}
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**Theorem (Ahlbrandt + Ziegler ’86)**

Let \( \Delta, \Gamma \) be \( \omega \)-categorical structures. Then
\[
\text{Aut}(\Delta) \cong^T \text{Aut}(\Gamma) \iff \Delta, \Gamma \text{ are first-order bi-interpretable}.
\]
Reconstruction from the abstract group

$\text{Aut}(\text{house})$ as an abstract group $\rightarrow$ ?
Reconstruction from the abstract group

$\text{Aut}(\Delta)$ as an abstract group $\rightarrow ?$

- Can we reconstruct an $\omega$-categorical structure $\Delta$
  from the algebraic group structure of $\text{Aut}(\Delta)$?
Reconstruction from the abstract group

\( \text{Aut(\sqcap)} \) as an abstract group \( \rightarrow ? \)

- Can we reconstruct an \( \omega \)-categorical structure \( \Delta \) from the algebraic group structure of \( \text{Aut}(\Delta) \)?
- Can we reconstruct the topological structure of \( \text{Aut}(\Delta) \) from its algebraic structure?
Reconstruction from the abstract group

$\text{Aut}(\Delta)$ as an abstract group $\rightarrow \square$

- Can we reconstruct an $\omega$-categorical structure $\Delta$ from the algebraic group structure of $\text{Aut}(\Delta)$?

- Can we reconstruct the topological structure of $\text{Aut}(\Delta)$ from its algebraic structure?

The automorphism groups of $\omega$-categorical structures are the closed oligomorphic permutation groups:
Reconstruction from the abstract group

\( \text{Aut}(\Delta) \) as an abstract group \( \rightarrow \) ?

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The automorphism groups of \( \omega \)-categorical structures are the closed oligomorphic permutation groups: their coordinatewise action on \( n \)-tuples has finitely many orbits for all \( n \geq 1 \).
Reconstruction from the abstract group

\( \text{Aut}(\Delta) \) as an abstract group \( \rightarrow ? \)

- Can we reconstruct an \( \omega \)-categorical structure \( \Delta \) from the algebraic group structure of \( \text{Aut}(\Delta) \)?

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The automorphism groups of \( \omega \)-categorical structures are the closed oligomorphic permutation groups:
their coordinatewise action on \( n \)-tuples has finitely many orbits for all \( n \geq 1 \).

- Can we reconstruct the topological structure of closed oligomorphic permutation groups from their algebraic structure?
Better reconstruction plans

\[ \Delta \text{ is a structure.} \]

\[ \text{Aut}(\Delta) \text{ is the automorphism group of } \Delta. \]

\[ \text{End}(\Delta) \text{ is the endomorphism monoid of } \Delta. \]

\[ \text{Pol}(\Delta) \text{ is the polymorphism clone of } \Delta. \]

\[ \text{End}(\Delta) \text{ contains all homomorphisms } f : \Delta \to \Delta. \]

\[ \text{Pol}(\Delta) \text{ is a function clone: closed under composition, contains projections.} \]

Observe:

\[ \text{Pol}(\Delta) \supseteq \text{End}(\Delta) \supseteq \text{Aut}(\Delta). \]
Better reconstruction plans

Let $\Delta$ be a structure.
Better reconstruction plans

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- $\text{Aut}(\Delta)$... automorphism group of $\Delta$
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Reconstructing sheep from clones

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Better reconstruction plans

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$\text{End}(\Delta)$... all homomorphisms $f: \Delta \to \Delta$.

$\text{Pol}(\Delta)$... all homomorphisms $f: \Delta^n \to \Delta$, where $1 \leq n < \omega$. 

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**Observe:** $\text{Pol}(\Delta) \supseteq \text{End}(\Delta) \supseteq \text{Aut}(\Delta)$. 
Reconstruction up to primitive positive definitions

Theorem (Bodirsky + Nešetřil '03)

Let $\Delta$, $\Gamma$ be $\omega$-categorical structures on the same domain. Then:

$$\text{Pol}(\Delta) = \text{Pol}(\Gamma) \iff \Delta, \Gamma \text{ are primitive positive interdefinable.}$$

Why primitive positive definitions?

Let $\Delta$ be a structure with a finite relational signature $\tau$.

Definition (Constraint Satisfaction Problem)

$\text{CSP}(\Delta)$ is the computational problem to decide whether a given primitive positive $\tau$-sentence holds in $\Delta$. 

Reconstructing sheep from clones

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Reconstruction up to primitive positive definitions

\[ \text{Pol(\text{sheep})} \rightarrow ? \]
Reconstruction up to primitive positive definitions

Pol(\(\mathcal{H}\)) → ?

**Theorem (Bodirsky + Nešetřil ’03)**

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\[ \text{Pol}(\text{\includegraphics{house.png}}) \rightarrow ? \]

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\[ \text{Pol}(\text{house}) \rightarrow ? \]

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Why primitive positive definitions?

Let \( \Delta \) be a structure with a *finite* relational signature \( \tau \).
Reconstruction up to primitive positive definitions

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**Definition (Constraint Satisfaction Problem)**

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Reconstructing sheep from clones

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Topological clones

Function clones carry:
- algebraic structure (laws of composition): multi-sorted algebra
- topological structure (pointwise convergence)

Let $C, D$ be function clones.

$\xi : C \to D$ is a (clone) homomorphism iff
- it preserves arities;
- sends every projection in $C$ to the corresponding projection in $D$;
- $\xi(f(g_1, \ldots, g_n)) = \xi(f)(\xi(g_1), \ldots, \xi(g_n))$ for all $f, g_1, \ldots, g_n \in C$.

Theorem (Bodirsky + MP '12)

Let $\Delta, \Gamma$ be $\omega$-categorical structures. Then:

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$\implies$ Topological clones
Topological clones

Function clones carry:

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Let \( \mathbf{C}, \mathbf{D} \) be function clones.

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\[ \implies \] Topological clones

Theorem (Bodirsky + MP ’12)

Let \( \Delta, \Gamma \) be \( \omega \)-categorical structures. Then:

\( \text{Pol}(\Delta) \cong^T \text{Pol}(\Gamma) \) iff \( \Delta, \Gamma \) are primitive positive bi-interpretable.
Can we reconstruct an $\omega$-categorical structure $\Delta$ from the algebraic clone structure $\text{Pol}(\Delta)$?

Can we reconstruct the topological structure of $\text{Pol}(\Delta)$ from its algebraic structure?

Polymorphism clones of $\omega$-categorical structures are the closed oligomorphic function clones: they contain an oligomorphic permutation group.

Can we reconstruct the topological structure of closed oligomorphic function clones from their algebraic structure?
Reconstruction from the abstract clone

Pol(ハウス) as an abstract clone $\rightarrow \ ?$

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Pol(ハウス) as an abstract clone → ?

- Can we reconstruct an $\omega$-categorical structure $\Delta$
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Reconstruction from the abstract clone

Pol(阊) as an abstract clone $\rightarrow$ ?

- Can we reconstruct an $\omega$-categorical structure $\Delta$ from the algebraic clone structure of Pol($\Delta$)?

- Can we reconstruct the topological structure of Pol($\Delta$) from its algebraic structure?
Reconstruction from the abstract clone

\( \text{Pol}(\Delta) \) as an abstract clone \( \rightarrow ? \)

- Can we reconstruct an \( \omega \)-categorical structure \( \Delta \) from the algebraic clone structure of \( \text{Pol}(\Delta) \)?

- Can we reconstruct the topological structure of \( \text{Pol}(\Delta) \) from its algebraic structure?

Polymorphism clones of \( \omega \)-categorical structures are the closed oligomorphic function clones: they contain an oligomorphic permutation group.
Reconstruction from the abstract clone

Pol(\(\mathbb{R}\)) as an abstract clone \(\rightarrow\) ?

- Can we reconstruct an \(\omega\)-categorical structure \(\Delta\) from the algebraic clone structure of Pol(\(\Delta\))?

- Can we reconstruct the topological structure of Pol(\(\Delta\)) from its algebraic structure?

Polymorphism clones of \(\omega\)-categorical structures are the closed oligomorphic function clones: they contain an oligomorphic permutation group.

- Can we reconstruct the topological structure of closed oligomorphic function clones from their algebraic structure?
Part II
The topology of algebras
Clones from algebras

Let $A$ be an algebra, and $\tau$ its signature.

Abstract

$\tau$-term $t \Rightarrow$ term function $t_A$ on $A$.

Term functions of $A$: function clone $\text{Clo}(A)$.

Structural conclusions about $A$ from abstract clone $\text{Clo}(A)$: Varieties.

Reconstructing sheep from clones

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Let $\mathfrak{A}$ be an algebra, and $\tau$ its signature.
Let $A$ be an algebra, and $\tau$ its signature.

Abstract $\tau$-term $t \mapsto$ term function $t^A$ on $A$. 
Let $\mathcal{A}$ be an algebra, and $\tau$ its signature.

Abstract $\tau$-term $t \implies$ term function $t^\mathcal{A}$ on $\mathcal{A}$.

Term functions of $\mathcal{A}$: function clone $\text{Clo}(\mathcal{A})$. 
Let $A$ be an algebra, and $\tau$ its signature.

Abstract $\tau$-term $t \mapsto$ term function $t_A$ on $A$.

Term functions of $A$: function clone $\text{Clo}(A)$.

Structural conclusions about $A$ from abstract clone $\text{Clo}(A)$: Varieties.
Garrett Birkhoff’s theorem: abstract clones

For an algebra $A$, write $\text{HSP}_{\text{fin}}(A)$ for the algebras obtained by taking homomorphic images, subalgebras, finite powers.

Let $A, B$ be $\tau$-algebras. If the mapping $t_A \mapsto t_B$ is well-defined, then it is a clone homomorphism $\xi : \text{Clo}(A) \to \text{Clo}(B)$ called the natural homomorphism.

Theorem (Birkhoff 1935) Let $A, B$ be finite. $B$ is in $\text{HSP}_{\text{fin}}(A) \iff$ the natural homomorphism from $\text{Clo}(A)$ to $\text{Clo}(B)$ exists.

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Garrett Birkhoff’s theorem: abstract clones

For an algebra $\mathfrak{A}$, write $\text{HSP}^{\text{fin}}(\mathfrak{A})$ for the algebras obtained by taking

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For an algebra $\mathfrak{A}$, write $\text{HSP}^{\text{fin}}(\mathfrak{A})$ for the algebras obtained by taking
- Homomorphic images
For an algebra $\mathcal{A}$, write $\text{HSP}^\text{fin}(\mathcal{A})$ for the algebras obtained by taking
- Homomorphic images
- Subalgebras
Garrett Birkhoff’s theorem: abstract clones

For an algebra $\mathcal{A}$, write $\text{HSP}^\text{fin}(\mathcal{A})$ for the algebras obtained by taking

- Homomorphic images
- Subalgebras
- finite Powers.
Garrett Birkhoff’s theorem: abstract clones

For an algebra $\mathcal{A}$, write $\text{HSP}^\text{fin}(\mathcal{A})$ for the algebras obtained by taking

- Homomorphic images
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Let $\mathcal{A}, \mathcal{B}$ be $\tau$-algebras.

Theorem (Birkhoff 1935)

Let $A, B$ be finite. $B$ is in $\text{HSP}^\text{fin}(A) \iff$ the natural homomorphism from $\text{Clo}(A)$ to $\text{Clo}(B)$ exists.
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For an algebra $\mathcal{A}$, write $\text{HSP}^{\text{fin}}(\mathcal{A})$ for the algebras obtained by taking
- Homomorphic images
- Subalgebras
- finite Powers.

Let $\mathcal{A}, \mathcal{B}$ be $\tau$-algebras. If the mapping

$$t^\mathcal{A} \mapsto t^\mathcal{B}$$

is well-defined,
Garrett Birkhoff’s theorem: abstract clones

For an algebra $\mathfrak{A}$, write $\text{HSP}^{\text{fin}}(\mathfrak{A})$ for the algebras obtained by taking
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$$\xi: \text{Clo}(\mathfrak{A}) \rightarrow \text{Clo}(\mathfrak{B})$$
Garrett Birkhoff’s theorem: abstract clones

For an algebra $A$, write $\text{HSP}^{\text{fin}}(A)$ for the algebras obtained by taking

- Homomorphic images
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Let $A, B$ be $\tau$-algebras. If the mapping

$$t^A \mapsto t^B$$

is well-defined, then it is a clone homomorphism

$$\xi: \text{Clo}(A) \to \text{Clo}(B)$$

called the natural homomorphism.
Garrett Birkhoff’s theorem: abstract clones

For an algebra \( \mathfrak{A} \), write \( \text{HSP}^{\text{fin}}(\mathfrak{A}) \) for the algebras obtained by taking

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Let \( \mathfrak{A}, \mathfrak{B} \) be \( \tau \)-algebras. If the mapping

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is well-defined, then it is a clone homomorphism

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called the natural homomorphism.

**Theorem (Birkhoff 1935)**

Let \( \mathfrak{A}, \mathfrak{B} \) be finite.

\( \mathfrak{B} \) is in \( \text{HSP}^{\text{fin}}(\mathfrak{A}) \) if and only if the natural homomorphism from \( \text{Clo}(\mathfrak{A}) \) to \( \text{Clo}(\mathfrak{B}) \) exists.
Call a countable algebra $A$ oligomorphic iff $\text{Clo}(A)$ is. Theorem ('Topological Birkhoff'; Bodirsky + MP '12) Let $A, B$ be oligomorphic or finite. $B$ is in $\text{HSP}_{\text{fin}}(A)$ $\iff$ the natural homomorphism from $\text{Clo}(A)$ to $\text{Clo}(B)$ exists and is continuous.

Problem. Can it be discontinuous? Can we reconstruct the topological structure of closed oligomorphic function clones from their algebraic structure?
Topological Birkhoff’s theorem: topological clones

Call a countable algebra \( \mathcal{A} \) **oligomorphic** iff \( \text{Clo}(\mathcal{A}) \) is.

Theorem ('Topological Birkhoff'; Bodirsky + MP '12)

Let \( \mathcal{A}, \mathcal{B} \) be oligomorphic or finite.

\( \mathcal{B} \) is in \( \text{HSP}_{\text{fin}}(\mathcal{A}) \) \( \iff \) the natural homomorphism from \( \text{Clo}(\mathcal{A}) \) to \( \text{Clo}(\mathcal{B}) \) exists and is continuous.

Problem.

Can it be discontinuous?

Can we reconstruct the topological structure of closed oligomorphic function clones from their algebraic structure?
Call a countable algebra $\mathcal{A}$ oligomorphic iff $\text{Clo}(\mathcal{A})$ is.

**Theorem** (‘Topological Birkhoff’; Bodirsky + MP ’12)

Let $\mathcal{A}$, $\mathcal{B}$ be oligomorphic or finite.

$\mathcal{B}$ is in $\text{HSP}^{\text{fin}}(\mathcal{A}) \iff$

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$\mathcal{B}$ is in $\text{HSP}^\text{fin}(\mathcal{A})$ ↔

the natural homomorphism from $\text{Clo}(\mathcal{A})$ to $\text{Clo}(\mathcal{B})$ exists and is continuous.

**Problem.**

- Can it be discontinuous?
Call a countable algebra $\mathcal{A}$ oligomorphic iff $\text{Clo}(\mathcal{A})$ is.

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**Problem.**

- Can it be discontinuous?
- Can we reconstruct the topological structure of closed oligomorphic function clones from their algebraic structure?
Part III
Reconstructing the topology
Reconstruction notions

Let $O$ be the largest function clone on $\omega$, and $C$ be a closed subclone.

Definition

$C$ has reconstruction iff $C \sim D \Rightarrow C \sim T D$ for all closed subclones $D$ of $O$;

$C$ has automatic homeomorphicity iff every clone isomorphism between $C$ and a closed subclone of $O$ is a homeomorphism;

$C$ has automatic continuity iff every clone homomorphism from $C$ into $O$ is continuous.

Observation.

$(2) \Rightarrow (1)$.

Fact.

For groups $(3) = (2)$.

Reconstructing sheep from clones

Michael Pinsker
Reconstruction notions

Let \( \mathcal{O} \) be the largest function clone on \( \omega \), and \( \mathcal{C} \) be a closed subclone.

**Definition**
\( \mathcal{C} \) has reconstruction iff \( \mathcal{C} \cong D \) implies \( \mathcal{C} \cong T \) for all closed subclones \( D \) of \( \mathcal{O} \);
\( \mathcal{C} \) has automatic homeomorphicity iff every clone isomorphism between \( \mathcal{C} \) and a closed subclone of \( \mathcal{O} \) is a homeomorphism;
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**Observation.** (2) \( \Rightarrow \) (1).

**Fact.** For groups (3) \( \Rightarrow \) (2).
Let $O$ be the largest function clone on $\omega$, and $C$ be a closed subclone.

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$C$ has **reconstruction** iff $C \cong D$ implies $C \cong^T D$ for all closed subclones $D$ of $O$;

**Observation.** $(2) \implies (1)$.

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Reconstructing sheep from clones
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Groups: the small index property

Our favorite automorphism groups have automatic continuity: $(\mathbb{N}; =)$ (Dixon+Neumann+Thomas'86), $(\mathbb{Q}; <)$ and the atomless Boolean algebra (Truss'89), the random graph (Hodges+Hodkinson+Lascar+Shelah'93), the random $K_n$-free graphs (Herwig'98).
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Groups: Rubin’s forall-exists interpretations

Our favorite automorphism groups have automatic homeomorphicity: the random graph $\langle Q; < \rangle$ of all homogeneous countable graphs various $\omega$-categorical semilinear orders the random partial order the random $k$-hypergraphs the Henson digraphs (Rubin ’94).

Reconstructing sheep from clones
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Part IV
Negative results
Automatic continuity for clones

Proposition

If $\Delta$ is $\omega$-categorical, then $\text{Emb}(\Delta)$ does not have automatic continuity.

Thus concentrate on isomorphisms (i.e., automatic homeomorphicity) homomorphisms to special clones – in particular to the projection clone $1$.

Important in constraint satisfaction: "main reason" for NP-hardness of the CSP of a structure.

Reconstructing sheep from clones

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Theorem (Bodirsky + MP + Pongrácz '13)

There exists a closed oligomorphic clone with a discontinuous homomorphism to 1. Involves non-principal ultrafilter: unfair in the CSP context. Also has a continuous homomorphism to 1.
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Automatic homeomorphicity + reconstruction

Theorem (Bodirsky + MP + Pongrácz '13)

There exists a closed oligomorphic clone $C$ and $\xi : C \to C$ such that:

$\xi$ is an isomorphism;

$\xi$ is not continuous.

Thus $C$ does not have automatic homeomorphicity.

Theorem (Evans + Hewitt '90)

There exists a closed oligomorphic group which does not have reconstruction.

Reconstructing sheep from clones

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Part V
Positive results
Method I: Automatic continuity via Birkhoff’s theorem

Let $C$ be a closed subclone of $O$, and $\xi : C \to O$ be a homomorphism. Theorem (Birkhoff '35) The algebra $(\omega ; \xi \mid C)$ is an HSP of the algebra $(\omega ; C)$. The only possibly discontinuous step is an infinite product. Theorem (Bodirsky + MP + Pongrácz '13) Any closed subclone of $O$ containing $\omega \omega$ has automatic continuity and automatic homeomorphicity.
Let $\mathcal{C}$ be a closed subclone of $\mathcal{O}$, and $\xi: \mathcal{C} \to \mathcal{O}$ be a homomorphism.
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**Theorem (Birkhoff ’35)**

The algebra $(\omega; \xi[\mathbf{C}])$ is an HSP of the algebra $(\omega; \mathbf{C})$. 
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Theorem (Bodirsky + MP + Pongrácz ’13)
Any closed subclone of $\mathbf{O}$ containing $\omega^\omega$ has automatic continuity and automatic homeomorphicity.
Method II: Automatic homeomorphicity via groups

Let $C$ be a closed subclone of $O$ whose group $G_C$ of invertibles has automatic homeomorphicity. Show:

- the closure of $G_C$ in $O$ has reconstruction;
- the clone of unary functions of $C$ has reconstruction;
- $C$ has reconstruction.

Theorem (Bodirsky + MP + Pongrácz '13)

Let $G$ be the random graph. The following have automatic homeomorphicity:

- $\text{End}(G)$;
- $\text{Pol}(G)$;
- All minimal tractable clones containing $\text{Aut}(G)$.

Reconstructing sheep from clones

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1. The closure of $G_\mathcal{C}$ in $\mathcal{O}$ has reconstruction;
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Part VI
Open problems
Open problems

Which oligomorphic closed subclones of \( O \) have automatic homeomorphicity?

Is there an oligomorphic closed subclone of \( O \) which does not have reconstruction?

Is there an oligomorphic closed subclone of \( O \) which has a homomorphism to the projection clone \( 1 \), but no continuous one?

In ZF?

Which topological clones are closed subclones of \( O \)?
Open problems

- Which oligomorphic closed subclones of $O$ have automatic homeomorphicity?

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Thank you!