Reconstructing structures from their abstract clones

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Outline

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- Reconstructing structures from their automorphism groups and polymorphism clones

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- The topology of algebras

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- Reconstructing structures from their automorphism groups and polymorphism clones
- The topology of algebras
- Reconstruction the topology of function clones

Part I

Reconstructing structures from their automorphism groups and polymorphism clones

Reconstructing structures up to first-order . . .

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countable

Reconstructing structures up to first-order . . .

countable, ω-categorical

Reconstructing structures up to first-order . . .

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Aut()

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Aut() $\rightarrow$

Aut() $\to$  first-order interdefinable with 

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Let Δ, Γ be ω-categorical structures on the same domain. Then $\mathrm{Aut}(\Delta) = \mathrm{Aut}(\Gamma)$ iff Δ, Γ are first-order interdefinable.

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Theorem (Ahlbrandt + Ziegler '86)

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- Can we reconstruct the topological structure of closed oligomorphic permutation groups from their algebraic structure?

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Observe: $\mathrm{Pol}(\Delta) \supseteq \mathrm{End}(\Delta) \supseteq \mathrm{Aut}(\Delta)$.

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Theorem (Bodirsky + Nešetřil '03)

Let Δ, Γ be ω-categorical structures on the same domain. Then:
Pol(Δ) $=$ Pol(Γ) iff Δ, Γ are primitive positive interdefinable.

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Let Δ be a structure with a *finite* relational signature τ.

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Definition (Constraint Satisfaction Problem)

CSP(Δ) is the computational problem to decide whether a given primitive positive τ-sentence holds in Δ.

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Part II

The topology of algebras

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Structural conclusions about $\mathfrak{A}$ from abstract clone $\mathrm{Clo}(\mathfrak{A})$: Varieties.

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Theorem (Birkhoff 1935)

Let $\mathfrak{A}$, $\mathfrak{B}$ be finite.

$\mathfrak{B}$ is in $\mathsf{HSP}^{\mathsf{fin}}(\mathfrak{A}) \leftrightarrow$
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Part III

Reconstructing the topology

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Fact. For groups (3) $\implies$ (2).

Groups: the small index property

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Our favorite automorphism groups have automatic continuity:

- $(\mathbb{N}; =)$ (Dixon+Neumann+Thomas'86)
- $(\mathbb{Q}; <)$ and the atomless Boolean algebra (Truss'89)
- the random graph (Hodges+Hodkinson+Lascar+Shelah'93)
- the random K_n-free graphs (Herwig'98)

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  the random partial order
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  (Rubin '94)
- the random *k*-hypergraphs
  the Henson digraphs
  (Barbina+MacPherson '07).

Part IV

Negative results

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Important in constraint satisfaction:
"main reason" for NP-hardness of the CSP of a structure.

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- Also has a continuous homomorphism to **1**.

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There exists a closed oligomorphic clone **C** and $\xi\colon \mathbf{C} \to \mathbf{C}$ such that:

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Thus **C** does not have automatic homeomorphicity.

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Theorem (Evans + Hewitt '90)

There exists an closed oligomorphic group which does not have reconstruction.

Part V

Positive results

Method I: Automatic continuity via Birkhoff's theorem

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The algebra $(\omega; \xi[\mathbf{C}])$ is an HSP of the algebra $(\omega; \mathbf{C})$.

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Theorem (Bodirsky + MP + Pongrácz '13)

Any closed subclone of **O** containing ω^ω has automatic continuity and automatic homeomorphicity.

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Show:

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- the clone of unary functions of **C** has reconstruction;

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- Pol(G);
- All minimal tractable clones containing Aut(G).

Part VI

Open problems

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- Which topological clones are closed subclones of **O**?

Thank you!