Schaefer's theorem for graphs

Why to consult the infinite at times

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Université Diderot - Paris 7

Tel Aviv University, May 2012

Outline

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- **Part I**
  Graph-SAT problems

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- **Part II**

  Making the finite infinite

  CSPs of reducts of the random graph

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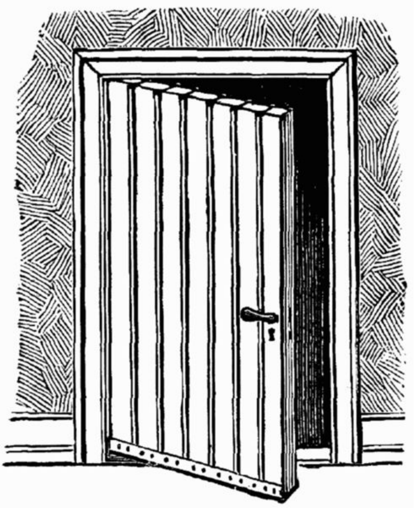

Part I

Graph-SAT problems

Boolean satisfiability problems

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Computational problem: Boolean-SAT(Ψ)

INPUT:

- A set W of propositional variables, and
- statements $\phi_1, \ldots, \phi_n$ about the variables in W, where each ϕ_i is taken from Ψ.

QUESTION: Is $\bigwedge_{1 \leq i \leq n} \phi_i$ satisfiable?

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Question

For which Ψ is Graph-SAT(Ψ) tractable?

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Example 1 Let Ψ_1 only contain

$$\psi_1(x,y,z) := (E(x,y) \wedge \neg E(y,z) \wedge \neg E(x,z)) \\ \vee (\neg E(x,y) \wedge E(y,z) \wedge \neg E(x,z)) \\ \vee (\neg E(x,y) \wedge \neg E(y,z) \wedge E(x,z)) \ .$$

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Graph-SAT(Ψ_2) is in P.

Part II

Making the finite infinite

CSPs over the random graph

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Γ_Ψ is a *reduct of* the random graph, i.e.,
a structure with a first-order definition in G.

Graph-SAT as CSP of a reduct of G

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An instance

- $W = \{w_1, \ldots, w_m\}$
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of Graph-SAT(Ψ) has a positive solution $\leftrightarrow$
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So Graph-SAT(Ψ) and CSP(Γ_Ψ) are one and the same problem.

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Let's study CSP(Γ) for reducts Γ of G!

Primitive positive (pp) definability and polymorphisms

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For finite $n \geq 1$, a function $f : \Gamma^n \to \Gamma$ is a *polymorphism* of Γ iff
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Theorem (Bodirsky, Nešetřil '03). $\Gamma \leq_{pp} \Delta \;\leftrightarrow\; \mathrm{Pol}(\Delta) \subseteq \mathrm{Pol}(\Gamma)$.

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Reducts of (ii) have polymorphisms violating the relations of (i).

Polymorphisms provide algorithms.

Part III

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Canonical polymorphisms

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Definition. A function $f : G \to G$ is *canonical* $\leftrightarrow$
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Generalization of *canonical* to functions $f : G^n \to G$ possible.

Example. edge-max: $G^2 \to G$.

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Canonical functions are finite objects: functions on types!

Part IV

The Graph-SAT dichotomy

Complexity of CSP for reducts of G

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Theorem (Bodirsky, MP '10)

Let Γ be a reduct of the random graph. Then:

- Either Γ has one out of 17 canonical polymorphisms, and CSP(Γ) is tractable,
- or CSP(Γ) is NP-complete.

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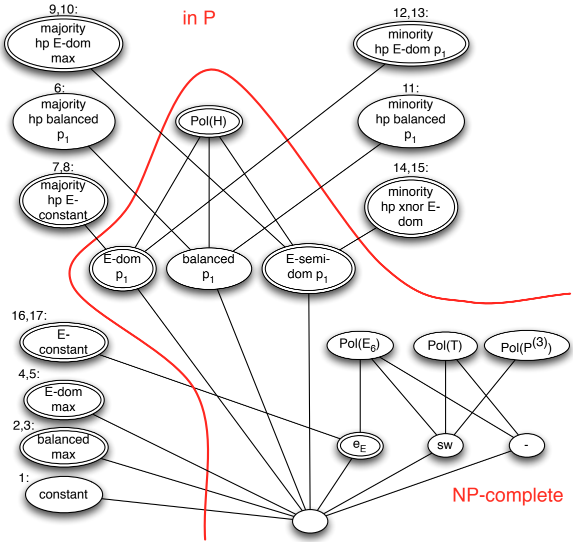
Let Γ be a reduct of the random graph. Then:

- Either Γ pp-defines one out of 4 hard relations, and CSP(Γ) is NP-complete,
- or CSP(Γ) is tractable.

The Graph-SAT dichotomy visualized



Theorem

The following 17 distinct clones are precisely the minimal tractable closed clones containing Aut(G):

1. The clone generated by a constant operation.
2. The clone generated by a balanced binary injection of type max.
3. The clone generated by a balanced binary injection of type min.
4. The clone generated by an E-dominated binary injection of type max.
5. The clone generated by an N-dominated binary injection of type min.
6. The clone generated by a function of type majority which is hyperplanely balanced and of type projection.
7. The clone generated by a function of type majority which is hyperplanely E-constant.
8. The clone generated by a function of type majority which is hyperplanely N-constant.
9. The clone generated by a function of type majority which is hyperplanely of type max and E-dominated.
10. The clone generated by a function of type majority which is hyperplanely of type min and N-dominated.

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Meta-Problem of Graph-SAT(Ψ)

INPUT: A finite set Ψ of graph formulas.

QUESTION: Is Graph-SAT(Ψ) in P?

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Theorem (Bodirsky, MP '10)

The Meta-Problem of Graph-SAT(Ψ) is decidable.

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Graph satisfiability problems

Let Ψ be a finite set of graph formulas.

Computational problem: Graph-SAT(Ψ)

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Theorem (Bodirsky, MP '10)

Graph-SAT(Ψ) is either in P or NP-complete, for all Ψ.

Part V

The future

CSPs over homogeneous structures

Amalgamation classes

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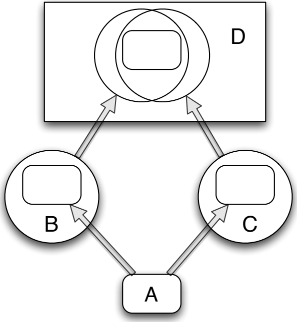
The classes of finite graphs and linear orders are *amalgamation classes*.



Amalgamation classes have homogeneous limit

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- Permutations

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   Modern method: exposing a continuous homomorphism from Pol(Γ) to the projection clone on $\{0, 1\}$. *Topological Birkhoff.*

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(d) Apply method to:
- finite partial orders – **Poset-SAT(Ψ)**
- finite Boolean algebras – “set constraints”
etc.

References

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