TBA

Michael Pinsker

2nd Workshop on Homogeneous Structures

Prague 2012

Topological Birkhoff & Applications

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Outline

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Topological Birkhoff

by Manuel Bodirsky and Michael Pinsker

on arXiv since March 2012.

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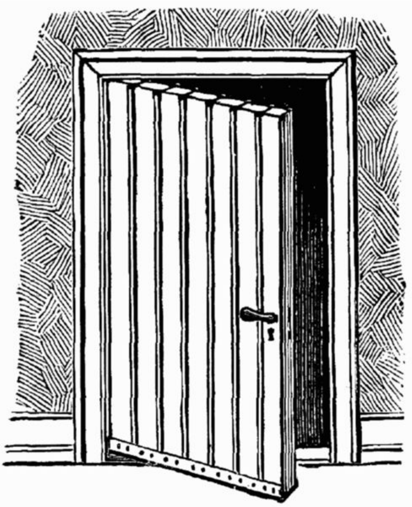
Implication chain: $\downarrow$

Motivation chain: $\uparrow$



Part I: Birkhoff's theorem

Varieties and pseudovarieties

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Fact (Birkhoff)

- The variety generated by $\mathfrak{A}$ equals $\mathsf{HSP}(\mathfrak{A})$.
- The pseudovariety generated by $\mathfrak{A}$ equals $\mathsf{HSP}^{\mathrm{fin}}(\mathfrak{A})$.

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Different abstract τ-terms s, t might induce the same function:

$$s^{\mathfrak{A}} = t^{\mathfrak{A}}$$

Those are the *equations* that hold in $\mathfrak{A}$.

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Bad for *aesthetic* and *computational* reasons.

Troubling

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- An algebra $\mathfrak{A}$ is locally oligomorphic $\leftrightarrow$ Clo($\mathfrak{A}$) is locally oligomorphic.

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= set of all homomorphisms from some Δ^n to Δ.

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Theorem "Topological Birkhoff" (Bodirsky + MP)

Let $\mathfrak{A}$, $\mathfrak{B}$ be locally oligomorphic or finite.

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- $\mathfrak{A}$ is locally oligomorphic;
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Set $\tau := \{f_i\}_{i\in\omega} \cup \{g_i\}_{i\in\omega}$, all function symbols unary.

Let $\mathfrak{A}$ be any τ-algebra on ω such that

- the functions $f_i^{\mathfrak{A}}$ form a locally oligomorphic permutation group;
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Let $\mathfrak{B}$ be the τ-algebra on $\{0,1\}$ such that

- $f_i^{\mathfrak{B}}$ is the identity function for all $i \in \omega$;
- $g_i^{\mathfrak{B}}$ is the constant function with value 0.

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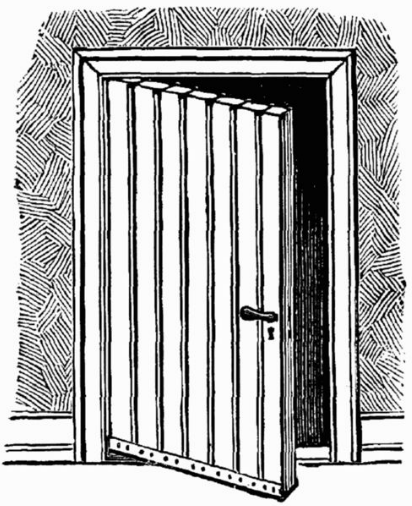
Interpretations!



Part II: Topological clones and interpretations

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for all atomic σ-formulas $\phi(y_1, \ldots, y_k)$ and all $\bar{a}_1, \ldots, \bar{a}_k \in \delta(\Gamma^d)$

$$\Delta \models \phi(h(\bar{a}_1), \ldots, h(\bar{a}_k)) \leftrightarrow \Gamma \models \phi'(\bar{a}_1, \ldots, \bar{a}_k)$$

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The interpretation is called primitive positive (pp) iff
all involved formulas are primitive positive, i.e., of the form

$$\exists\, v_1, \ldots v_r.\ \psi_1 \wedge \ldots \wedge \psi_l,$$

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Interpretations

A σ-structure Δ as an interpretation in a τ-structure Γ iff there exist

- $d \geq 1$ (the *dimension*),
- a τ-formula $\delta(x_1, \ldots, x_d)$ (the *domain formula*),
- for every atomic σ-formula $\phi(y_1, \ldots, y_k)$ a τ-formula $\phi'(\bar{u}_1, \ldots, \bar{u}_k)$,
- a surjective map $h : \delta(\Gamma^d) \rightarrow \Delta$, such that

for all atomic σ-formulas $\phi(y_1, \ldots, y_k)$ and all $\bar{a}_1, \ldots, \bar{a}_k \in \delta(\Gamma^d)$

$$\Delta \models \phi(h(\bar{a}_1), \ldots, h(\bar{a}_k)) \leftrightarrow \Gamma \models \phi'(\bar{a}_1, \ldots, \bar{a}_k)$$

The interpretation is called primitive positive (pp) iff
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Example: $(\mathbb{Q}; +, \cdot)$ has a pp interpretation in $(\mathbb{Z}; +, \cdot)$.

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Let Δ be ω-categorical.

A relation R has a pp definition in $\Delta \leftrightarrow$
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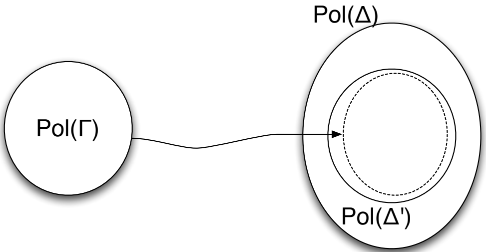
Let Γ be finite or ω-categorical, and let Δ be arbitrary. Tfae:

- *Δ has a pp interpretation in Γ;*
- *Δ is the reduct of a finite or ω-categorical structure Δ' such that there exists a continuous homomorphism from $\mathrm{Pol}(\Gamma)$ to $\mathrm{Pol}(\Delta')$ whose image is dense in $\mathrm{Pol}(\Delta')$.*



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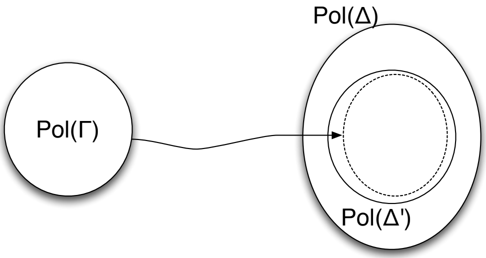

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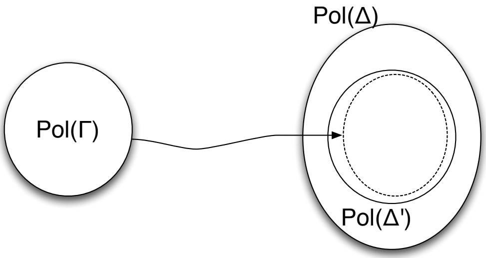

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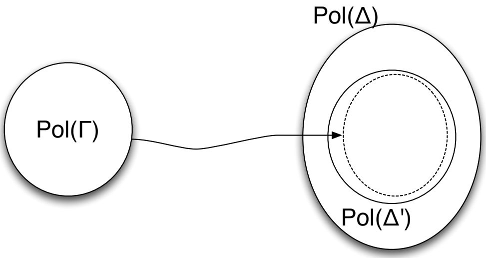

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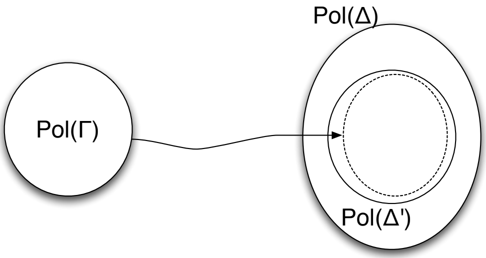

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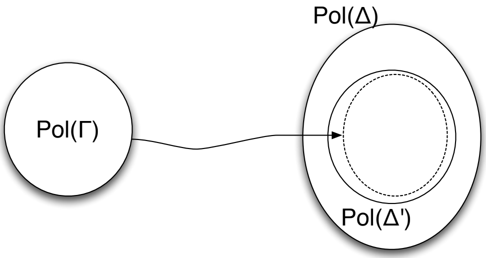

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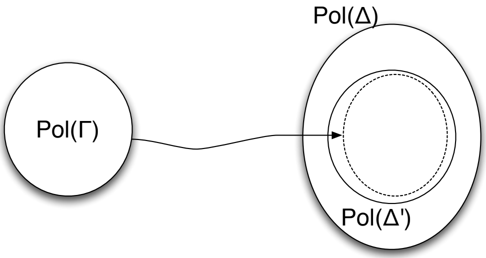

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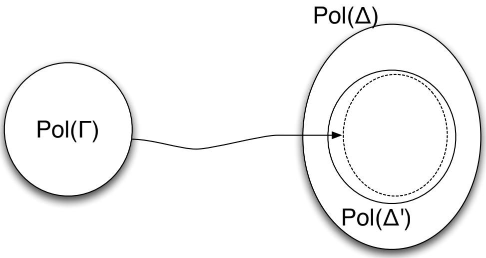

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ξ: $\text{Aut}(\Gamma) \to \text{Aut}(\Delta')$ defined by $f \mapsto f \restriction_S$.

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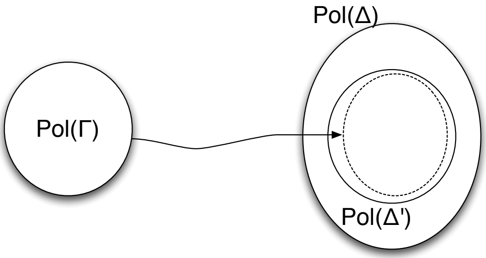
Two structures Δ and Γ are mutually pp interpretable iff Γ has a pp interpretation in Δ and vice versa.



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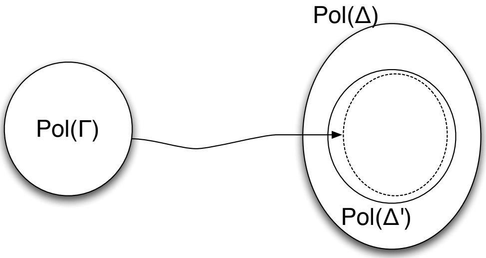

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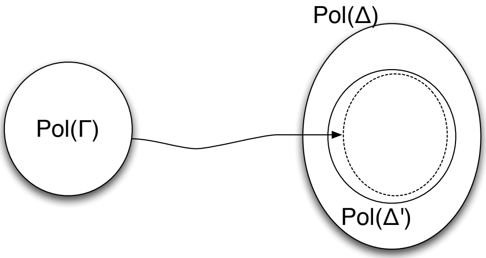

Stronger notion: Δ and Γ are pp bi-interpretable iff the coordinate maps h_1 and h_2 of the pp interpretations are so that

$$x = h_1(h_2(y_{1,1},\dots,y_{1,d_2}),\dots,h_2(y_{d_1,1},\dots,y_{d_1,d_2}))$$
$$x = h_2(h_1(y_{1,1},\dots,y_{d_1,1}),\dots,h_1(y_{1,d_2},\dots,y_{d_1,d_2}))$$

are pp definable in Δ and Γ, respectively.

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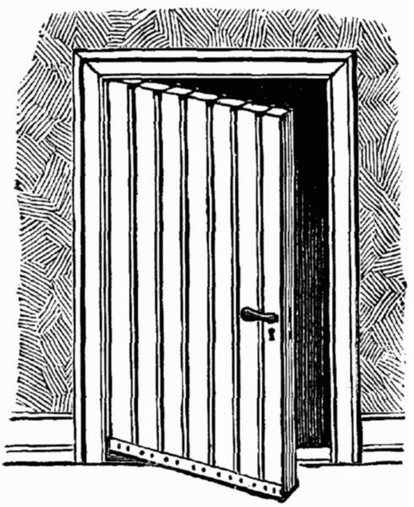

Part III: Constraint Satisfaction Problems

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For ω-categorical Δ, the complexity of CSP(Δ) *only depends on the topological polymorphism clone of Δ.*

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Straightforward: ξ is continuous homomorphism.

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- $(\mathbb{Q}; <)$ and the atomless Boolean algebra (Truss'89)
- the random graph (Hodges+Hodkinson+Lascar+Shelah'93)

Reconstruction

In which situations does the algebraic structure of the clone Pol(Δ) determine its topological structure? Always?

For Aut(Δ), this question has been studied.

Definition

Δ has the *small index property* iff
every subgroup of Aut(Δ) of index less than $2^{\aleph_0}$ is open.

Equivalent: every homomorphism from Aut(Δ) to Sym($\mathbb{N}$) is continuous.

Small index property has been verified for

- $(\mathbb{N}; =)$ (Dixon+Neumann+Thomas'86)
- $(\mathbb{Q}; <)$ and the atomless Boolean algebra (Truss'89)
- the random graph (Hodges+Hodkinson+Lascar+Shelah'93)
- and the Henson graphs (Herwig'98).

Automatic continuity

Non-reconstruction:

There are two ω-categorical structures whose automorphism groups are isomorphic as abstract groups but not as topological groups (Evans+Hewitt'90).

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- Every Baire measurable homomorphism between Polish groups is continuous.

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Non-reconstruction:

There are two ω-categorical structures whose automorphism groups are isomorphic as abstract groups but not as topological groups (Evans+Hewitt'90). (Assumes AC)

Automatic continuity:

- Every Baire measurable homomorphism between Polish groups is continuous.
- There exists a model of ZF+DC where every set is Baire measurable (Shelah'84).

Open problems

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- Do there exist ω-categorical Γ, Δ such that Pol(Γ), Pol(Δ) are isomorphic algebraically but not topologically?
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- When does the algebraic structure of Pol(Δ) determine the topological one? (e.g., “Small index property”)

Open problems

- Do there exist ω-categorical Γ, Δ such that $\mathrm{Pol}(\Gamma)$, $\mathrm{Pol}(\Delta)$ are isomorphic algebraically but not topologically? (Analogue of Evans+Hewitt).
- When does the algebraic structure of $\mathrm{Pol}(\Delta)$ determine the topological one? (e.g., “Small index property”)
- In negative cases: does the complexity of $\mathrm{CSP}(\Delta)$ only depend on the algebraic structure of $\mathrm{Pol}(\Delta)$? (Automatic continuity).

Reference

Topological Birkhoff

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