Lattices of subgroups of the symmetric group

Michael Pinsker (Paris 7)

joint work with

Saharon Shelah (Jerusalem)

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Outline

Part I
Lattices of: Groups - Monoids - Clones

Part II
(Locally) closed groups - monoids - clones

Lattices of permutation groups
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Part I

Lattices of: Groups - Monoids - Clones
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Part II
(Locally) closed groups - monoids - clones
Three lattices

Let $X$ be an infinite set.
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$\text{Gr}(X)$ . . . lattice of all permutation groups on $X$.
(order = containment, meet = intersection)
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(order = containment, meet = intersection)
Alternatively: \( \text{Gr}(X) \) subgroup lattice of the symmetric group \( \text{Sym}(X) \).

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\text{Mo}(X) \ldots \text{lattice of all transformation monoids on } X.
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(order = containment, meet = intersection)
Alternatively: \( \text{Mo}(X) \) submonoid lattice of the monoid \( X \times X \).

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\text{Cl}(X) \ldots \text{lattice of all (concrete) clones on } X.
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Alternatively: \( \text{Cl}(X) \) subclone lattice of the full clone \( \bigcup_{n \in \mathbb{N}} X \times X \).
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Alternatively: $\text{Mo}(X)$ submonoid lattice of the monoid $X^X$. 
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Alternatively: Cl($X$) subclone lattice of the full clone $\varnothing := \bigcup_n X^{X^n}$. 
Lattice worries

What do these lattices look like?

(lattices: Gr\(_X\), Mo\(_X\), Cl\(_X\))

Are they ugly?

What do they contain?

Is this vegetarian?

Lattices of permutation groups

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Is this vegetarian?
Comparing the lattices

Gr(\(X\)) is a sublattice of Mo(\(X\)) is a sublattice of Cl(\(X\)).
Comparing the lattices

Gr($X$) is a sublattice of Mo($X$) is a sublattice of Cl($X$).

**Problem**
Converse true?
Size of the lattices

Observations.

\[
\text{Sym}(X) = X \times X = \mathbb{O} = 2 \times X.
\]

\[
\text{Gr}(X) = \text{Mo}(X) = \text{Cl}(X) = 2^2 \times X.
\]

But:

number of finitely generated groups / monoids / clones: 2 \times X.

\[G \text{ finitely generated } \leftrightarrow G \text{ is compact}, \text{i.e., whenever } G \leq \bigvee_{i \in I} G_i, \text{ then also } G \leq \bigvee_{i \in J} G_i \text{ for some } J \subseteq I \text{ finite.}\]

Every group (monoid, clone) is the join of compact elements.

So \(\text{Gr}(X), \text{Mo}(X), \text{Cl}(X)\) are algebraic.

Fact: A complete sublattice of an algebraic lattice is algebraic and cannot have more compact elements.
Size of the lattices

Observations.

\[ |\text{Sym}(X)| = |X^X| = |\emptyset| = 2^{|X|}. \]
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**But:** number of *finitely generated* groups / monoids / clones: $2^{|X|}$.

If $\mathcal{G}$ finitely generated $\iff$ $\mathcal{G}$ is *compact*, i.e., whenever $\mathcal{G} \leq \bigvee_{i \in I} \mathcal{G}_i$, then also $\mathcal{G} \leq \bigvee_{i \in J} \mathcal{G}_i$ for some $J \subseteq I$ finite.
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$G$ finitely generated $\iff G$ is compact, i.e., whenever $G \leq \bigsqcup_{i \in I} G_i$, then also $G \leq \bigsqcup_{i \in J} G_i$ for some $J \subseteq I$ finite.

Every group (monoid, clone) is the join of compact elements.

So $\text{Gr}(X), \text{Mo}(X), \text{Cl}(X)$ are algebraic.

**Fact:** A *complete sublattice* of an algebraic lattice is algebraic and cannot have more compact elements.
Theorem (MP '06) \( \text{Cl}(X) \) is universal, i.e., every algebraic lattice with at most \( |X| \) compact elements is a complete sublattice of \( \text{Cl}(X) \).

Theorem (MP + Shelah '11) \( \text{Mo}(X) \) is universal.

Theorem (MP + Shelah '12) \( \text{Gr}(X) \) is universal.

So \( \text{Cl}(X) \) is a sublattice of \( \text{Mo}(X) \) which is a sublattice of \( \text{Gr}(X) \).
Non-vegetarian lattices

Theorem (MP ’06)

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Embeddings of algebraic lattices into subgroup lattices of (abstract) groups already known.
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Size of group equals number of compact elements.
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Size of group equals number of compact elements.

Stone representation as permutation group acting on set of size $2^{|X|}$, and not $|X|$.
Local closure

A permutation group $G$ is called (locally) closed if for all $\alpha \in \text{Sym}(X)$, if $\alpha$ can be interpolated by permutations from $G$ on all finite sets, then $\alpha \in G$.

Analogous definitions for transformation monoids, clones.

A group is closed if it is the automorphism group of a relational structure with domain $X$.

A monoid is closed if it is the endomorphism monoid of a relational structure with domain $X$.

A clone is closed if it is the polymorphism clone of a relational structure with domain $X$.

Topological structure of groups / monoids / clones important even for universal algebraists!

Topological Birkhoff (with M. Bodirsky)

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Lattices of closed groups / monoids / clones

Closed groups / monoids / clones form complete lattices (meet = intersection): $\text{Gr}_c(X) / \text{Mo}_c(X) / \text{Cl}_c(X)$. 
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Size: $2^{|X|}$. Non-algebraic!
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**Theorem (MP ’09)**

\( M_{2^{\aleph_0}} \) embeds into \( \text{Cl}_c(\mathbb{N}) \).

**Theorem (MP + Shelah ’12)**

Assume that \( \lambda < \lambda = \lambda \) and cofinality \( (\lambda) > \aleph_0 \).

Then \( M_{2^{\aleph_0}} \) embeds into \( \text{Gr}_c(\lambda) \).
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$M_{2^{\aleph_0}}$ embeds into $\text{Gr}(\mathbb{N})$ in such a way that the groups are $F_\sigma$ (= unions of closed groups).
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**Theorem (MP + Shelah ’12)**

Assume that $\lambda^{<\lambda} = \lambda$ and $\text{cofinality}(\lambda) \geq \aleph_0$.
Then $M_{2^\lambda}$ embeds into $\text{Gr}_c(\lambda)$. 
Open problems

Problem

Does $M_2 \bigotimes \aleph_0$ embed into $Gr(N)$?

Problem

Does $M_2 \bigotimes \aleph_0$ embed into $Mo(N)$?

Problem

Do all lattices (excluding impossible ones) embed into $Gr(N)/Mo(N)/Cl(N)$?

Example of impossible: chain of length 2 $\aleph_0$. 

Lattices of permutation groups

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Lattices of permutation groups

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