Cooking

with model theory, universal algebra and Ramsey theory

in the complexity theory kitchen

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Outline

- **Part I**

  Graph-SAT problems

- **Part II**

  Making the finite infinite

  Homogeneous structures

- **Part III**

  Making the infinite finite

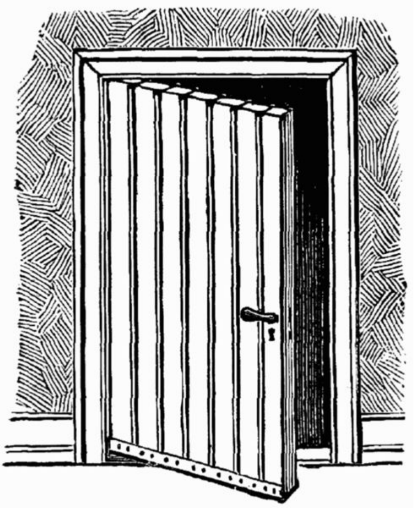
  Ramsey theory

- **Part IV**

  The Graph-SAT dichotomy

- **Part V**

  The future



Part I

Graph-SAT problems

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For which Ψ is Graph-SAT(Ψ) tractable?

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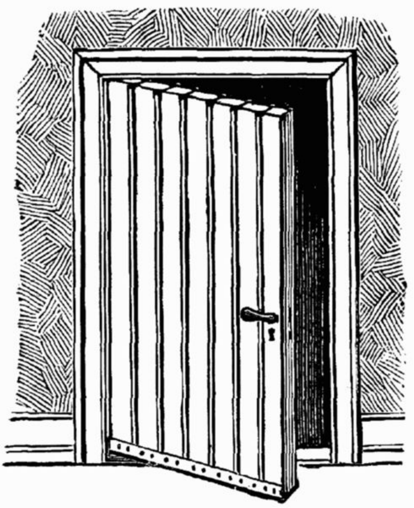
Graph-SAT(Ψ_2) is in P.



Part II

Making the finite infinite

(Homogeneous structures)

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Γ_Ψ is a *reduct of* the random graph, i.e.,
a structure with a first-order definition in G.

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- $W = \{w_1, \ldots, w_m\}$
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So Graph-SAT(Ψ) and CSP(Γ_Ψ) are one and the same problem.

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Let's study reducts of homogeneous structures!

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Problem

Classify the reducts of Δ.

We call Δ the *base structure*.

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We factor this quasiorder
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In fact:

The lattice corresponding to fo-definability is a factor of the lattice corresponding to pp-definability.

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This talk: Method for pp. Helps also for fo.

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For our method, we will need even “more” than homogeneity in a finite language:

The Ramsey property

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Denote by $(\mathbb{Q}; <)$ be the order of the rationals, and set

$$\begin{aligned}
\text{betw}(x,y,z) :=& \{(x,y,z) \in \mathbb{Q}^3 : x < y < z \text{ or } z < y < x\} \\
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Let $G = (V; E)$ be the random graph, and set for all $k \geq 2$

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Theorem (Junker, Ziegler '08)

$(\mathbb{Q}; <, 0)$ has 116 reducts up to fo-interdefinability.

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Depressing fact (Horváth, Pongrácz, MP '11)

The random graph with a constant has too many reducts up to fo-interdefinability.

Thomas' conjecture

Conjecture (Thomas '91)

Let Δ be homogeneous in a finite language.

Then Δ has finitely many reducts up to fo-interdefinability.

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- $2^{\aleph_0}$ reducts up to primitive positive interdefinability

Permutation groups - fo

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Theorem (Ryll-Nardzewski)

Let Δ be homogeneous, finite language.

The mapping

$$\Gamma \mapsto \mathrm{Aut}(\Gamma)$$

is a one-to-one correspondence between
the *first-order closed* reducts of Δ and
the *closed permutation groups* containing $\mathrm{Aut}(\Delta)$.

first order closed $=$ contains all fo-definable relations

group called closed iff it is closed in the convergence topology.

Clones - pp

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Theorem (Bodirsky, Nešetřil '03)

Let Δ be homogeneous, finite language. Then

$$\Gamma \mapsto \text{Pol}(\Gamma)$$

is a one-to-one correspondence between
the *primitive positive closed* reducts of Δ and
the *closed clones* containing $\text{Aut}(\Delta)$.

A clone is a set of finitary operations on Δ which

- contains all projections $\pi_i^n(x_1, \ldots, x_n) = x_i$, and
- is closed under composition.

$\text{Pol}(\Gamma)$ is the clone of all homomorphisms from finite powers of Γ to Γ.

A clone C is closed if for each $n \geq 1$, the set of n-ary operations in C is a closed subset of the Baire space Δ^{Δ^n}.

Groups and Clones

For homogeneous Δ in finite language:

Reducts up to **fo-interdefinability** $\leftrightarrow$

closed **permutation groups** $\supseteq \mathrm{Aut}(\Delta)$;

Reducts up to **pp-interdefinability** $\leftrightarrow$

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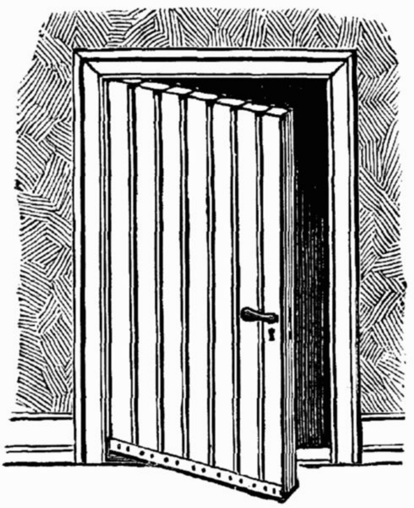

Part III

Making the infinite finite

(Ramsey theory)

How to classify all reducts up to . . .-interdefinability?

Climb up the lattice!

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Conclusion: Every finite graph has a copy in G on which f is canonical.

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Problem: Keeping some information on f when canonizing.

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Let $f : G \to G$.

If f violates a relation R, then there are $c_1, \ldots, c_n \in V$ witnessing this.

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By topological closure, f generates a function which:

- behaves like f on $\{c_1, \ldots, c_n\}$, and
- is canonical as a function from $(V; E, c_1, \ldots, c_n)$ to $(V; E)$.

The minimal clones on the random graph

Theorem (Bodirsky, MP '10)

Let f be a finitary operation on G which "is" not an automorphism.

Then f generates one of the following:

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More involved argument: Extend G by a random dense linear order.

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Definition

A class $\mathcal{C}$ of τ-structures is called a *Ramsey class* iff
for all $H, P \in \mathcal{C}$ there exists S in $\mathcal{C}$ such that $S \to (H)^P$.

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Thus: Any $f : \Delta \to \Delta$ generates a canonical function,

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Let Δ now be an arbitrary structure.

Definition

$f : \Delta \to \Delta$ is *canonical* iff
for all tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type
$(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type too.

Observation. If Δ is

- Ramsey
- homogeneous in a finite language
- (ordered),

then all finite substructures of Δ have a copy in Δ
on which f is canonical.

Thus: Any $f : \Delta \to \Delta$ generates a canonical function, but it could be the identity.

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Why don't you just do it?

Adding constants to Ramsey structures

Problem

If Δ is Ramsey, is $(\Delta, c_1, \ldots, c_n)$ still Ramsey?

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Theorem (Kechris, Pestov, Todorcevic '05)

An ordered homogeneous structure is Ramsey iff
its automorphism group is *extremely amenable*, i.e.,
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Corollary

If Δ is ordered, homogeneous, and Ramsey, then so is $(\Delta, c_1, \ldots, c_n)$.

Canonizing functions on Ramsey structures

Proposition

If Δ is ordered Ramsey homogeneous finite language, $f : \Delta^k \to \Delta$, and $c_1, \ldots, c_n \in \Delta$, then f generates a function which

- is canonical as a function from $(\Delta, c_1, \ldots, c_n)^k$ to Δ
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Let $\text{Aut}(\Delta, c_1, \ldots, c_n)^k$ act on $S/\sim$ by

$$(\beta_1, \ldots, \beta_k)([g(x_1, \ldots, x_k)]_\sim) := [g(\beta_1(x_1), \ldots, \beta_k(x_k))]_\sim$$

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Any element of the fixed point is canonical. □

Minimal clones above Ramsey structures

Theorem (Bodirsky, MP, Tsankov '10)

Let Γ be a reduct of a finite language homogeneous ordered Ramsey structure Δ. Then:

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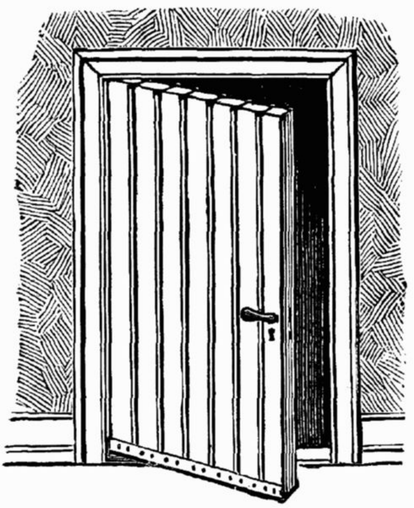
- If Γ has a finite language, then there are finitely many minimal closed superclones of $\mathrm{Pol}(\Gamma)$.
  (Arity bound!)
- Every closed superclone of $\mathrm{Pol}(\Gamma)$ contains a minimal closed superclone of $\mathrm{Pol}(\Gamma)$.



Part IV

The Graph-SAT dichotomy

The Graph Satisfiability Problem

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Let Ψ be a finite set of graph formulas.

Computational problem: Graph-SAT(Ψ)

INPUT:

- A set W of variables (vertices), and
- statements $\phi_1, \ldots, \phi_n$ about the elements of W, where each ϕ_i is taken from Ψ.

QUESTION: Is $\bigwedge_{1 \leq i \leq n} \phi_i$ satisfiable in a graph?

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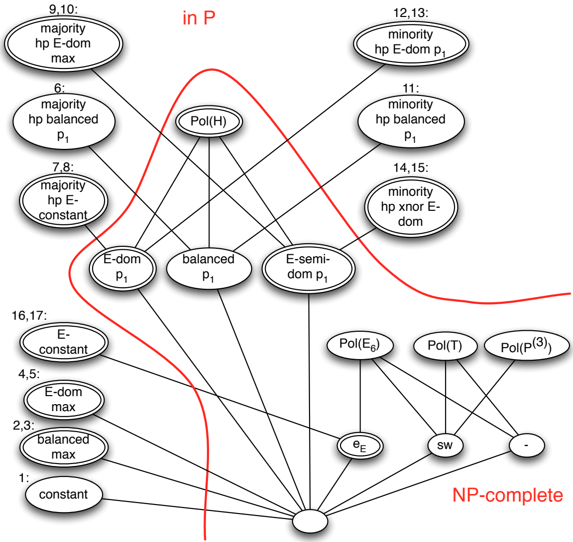
Theorem

Graph-SAT(Ψ) is either in P or NP-complete, for all Ψ.

The Graph-SAT dichotomy visualized



Theorem

The following 17 distinct clones are precisely the minimal tractable closed clones containing Aut(G):

1. The clone generated by a constant operation.
2. The clone generated by a balanced binary injection of type max.
3. The clone generated by a balanced binary injection of type min.
4. The clone generated by an E-dominated binary injection of type max.
5. The clone generated by an N-dominated binary injection of type min.
6. The clone generated by a function of type majority which is hyperplanely balanced and of type projection.
7. The clone generated by a function of type majority which is hyperplanely E-constant.
8. The clone generated by a function of type majority which is hyperplanely N-constant.
9. The clone generated by a function of type majority which is hyperplanely of type max and E-dominated.
10. The clone generated by a function of type majority which is hyperplanely of type min and N-dominated.

The Meta Problem

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Meta-Problem of Graph-SAT(Ψ)

INPUT: A finite set Ψ of graph formulas.

QUESTION: Is Graph-SAT(Ψ) in P?

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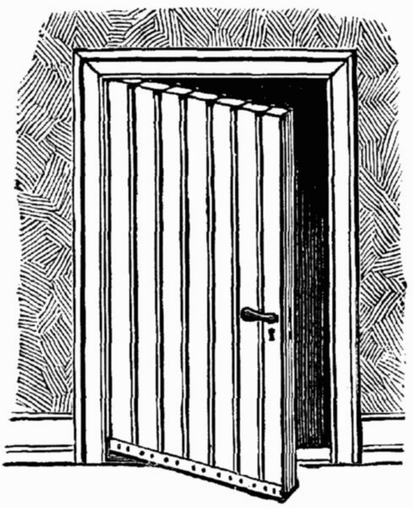
Theorem (Bodirsky, MP '10)

The Meta-Problem of Graph-SAT(Ψ) is decidable.



Part V

The future

Other homogeneous structures

Graph-SAT(Ψ): Is there a finite graph such that... (constraints)

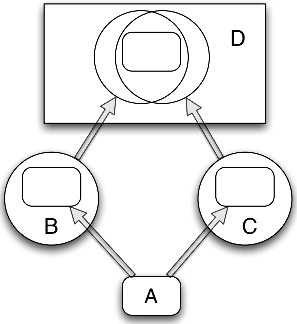
Temp-SAT(Ψ): Is there a linear order such that...

Other homogeneous structures

Graph-SAT(Ψ): Is there a finite graph such that... (constraints)

Temp-SAT(Ψ): Is there a linear order such that...

The classes of finite graphs and linear orders are *amalgamation classes*.



Amalgamation classes

Further amalgamation classes.

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- Partial orders

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- Lattices

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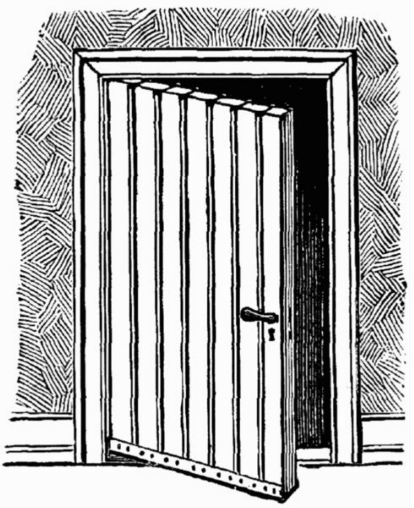
Homogeneous digraphs classified by Cherlin.



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