On Decidability of Primitive Positive Definability

Manuel Bodirsky
École Polytechnique
LIX (CNRS UMR 7161)

Michael Pinsker
Équipe de Logique Mathématique
Paris 7, France

Todor Tsankov
Équipe de Logique Mathématique
Paris 7, France

Abstract

For a fixed infinite structure with finite signature Γ, we study the following computational problem: given quantifier-free first-order formulas $\phi_0, \phi_1, \ldots, \phi_n$ that define relations $R_0, R_1, \ldots, R_n$ over Γ, is the relation R_0 *primitive positive definable* in the structure $(D; R_1, \ldots, R_n)$, i.e., definable by a first-order formula that uses only relation symbols for $R_1, \ldots, R_n$, equality, conjunctions, and existential quantification (disjunction, negation, and universal quantification are forbidden).

We show decidability of our problem for a large class of homogeneous structures Γ. The assumptions on Γ are that the class $\mathcal{C}$ of finite induced substructures of Γ can be described by a finite set of finite forbidden substructures and that $\mathcal{C}$ is a Ramsey class (examples for structures with these properties are $(\mathbb{Q}; <)$, and ordered versions of the random graph, the homogeneous universal poset, the random tournament, the homogeneous universal C-relation, and many more). Our proof makes use of universal-algebraic concepts, Ramsey theory, and a characterization of Ramsey classes in topological dynamics (KPT05).

1 Introduction and Result

An algorithm for primitive positive definability has theoretical and practical consequences in the study of the computational complexity of CPSs. It is motivated by the fundamental fact that expansions of structures Δ by primitive positive relations do not change the complexity of CSP(Δ). On a practical side, it turns out that hardness of a CSP can usually be shown by presenting primitive positive definitions of relations for which it is known that the CSP is hard. Therefore, a procedure that decides primitive positive definability of a given relation might be a useful tool to determine the computational complexity of CSPs.

The computational problem as described in the abstract above will be denoted by Expr(Γ). For *finite* structures Γ the problem Expr(Γ) has recently shown to be co-NEXPTIME-hard (Wil10). For general structures Γ, the problem Expr(Γ) is clearly undecidable: for $\Gamma = (\mathbb{Z}; +, *)$, this follows from Matiyasevich's theorem (Mat93).

We present assumptions for Γ that imply that Expr(Γ) is decidable. Due to lack of space, and since this is an abstract for a constraint satisfaction workshop, we assume familiarity with mathematical logic (such as provided in (Hod97)), and the universal-algebraic approach to constraint satisfaction (in particular, with the concept of a polymorphism). Only basic knowledge about Ramsey theory and topology is required.

The *age* of a relational structure Γ is the class of all finite substructures with the same signature as Γ that embed into Γ. We say that a class $\mathcal{C}$ of structures (or a structure with age $\mathcal{C}$) is

- *finitely bounded* (we use the same terminology as in (Mac09)) if there exists a finite set of finite structures $\mathcal{F}$ such that $A \in \mathcal{C}$ iff no structure from $\mathcal{F}$ embeds into A;
- *Ramsey* if for all $k \geq 1$ and for all $H, P \in \mathcal{C}$ there exists a G such that $G \to (H)^P_k$ (for background in Ramsey theory see (GRS90).
- *ordered* if the signature contains a binary relation that denotes a total linear order in every $A \in \mathcal{C}$.

Theorem 1. *Let Γ be ultrahomogeneous, finitely bounded, ordered, Ramsey, and with finite signature (are a structure definable in such a structure). Then Expr(Γ) is decidable.*

Examples of structures that satisfy the assumptions of Theorem 1 are $(\mathbb{Q}; <)$, the Fraisse limit of ordered finite graphs (or tournaments (Nes05)), the Fraisse limit of finite partial orders with a linear extension (Nes05), the homogeneous universal 'naturally ordered' C-relation (BP08), just to name a few. CSPs for templates that are definable in such structures are abundant in particular for qualitative reasoning calculi in Artificial Intelligence.

We want to point out that that our decidability result is already non-trivial when Γ is trivial from a model-theoretic perspective: for the case that Γ is the structure $(\mathbb{N}; =)$, whose age is clearly finitely bounded and Ramsey, the decidability of Expr(Γ) has been posed as an open problem in (BCP10).

2 Proof Ideas

Our approach rests on the following characterization of primitive positive definability.

Theorem 2 (from (BN06)). *A relation R is pp definable in a (finite or) ω-categorical structure Γ if and only if R is preserved by all polymorphisms of Γ.*

For finite structures Γ, the proof of Theorem 2 straightforwardly leads to a proof of decidability for Expr(Γ). For infinite structures Γ we cannot use polymorphisms in the same straightforward way to obtain a decidability result.

To state how we use Ramsey theory in our proof, we need the following concepts. Let D be the domain of Γ. The *type* $\text{tp}(t)$ of a tuple $t \in D^k$ is the set of first-order formulas with free variables $x_1, \ldots, x_k$ that hold on t in Γ. The type of a sequence of tuples $t^1, \ldots, t^l \in D^k$, denoted by $\text{tp}(t^1, \ldots, t^l)$, is the cartesian product of the types of $(a_i^1, \ldots, a_i^n)$ in Γ.

Definition 3. Let $F_i \subseteq D$, for $1 \leq i \leq m$. Set $F := F_1 \times \cdots \times F_m$. An operation $g : D^m \to D$ is *n-canonical on F* iff for all $a^1, \ldots, a^n, b^1, \ldots, b^n$ in F with $\text{tp}(a^1, \ldots, a^n) = \text{tp}(b^1, \ldots, b^n)$ we have

$$\text{tp}(f(a^1), \ldots, f(a^n)) = \text{tp}(f(b^1), \ldots, f(b^n)) \ .$$

It is *canonical on F* iff it is n-canonical on F for all $n \geq 1$. It is called *canonical* (*n-canonical*) if it is canonical (n-canonical) on D^m.

We make use of the following recent landmark result.

Theorem 4 (from (KPT05)). *An ordered structure is Ramsey if and only if its automorphism group is* extremely amenable, *i.e., if any continuous action of the group on a compact Hausdorff space has a fixed point.*

The following is crucial for our approach. Despite the significance and elegance of the statement even within pure mathematics, it has not yet been published. Its proof is due to the third author; apparently, it uses only techniques that are standard in the theory of group actions (but not standard in computer science).

Proposition 5. *Let G be an extremely amenable group, and let H be an open subgroup of G. Then H is extremely amenable.*

Using Theorem 4, this has the following consequence.

Lemma 6. *Let $f : D \to D$, and let $c_1, \ldots, c_n \in D$. Then f together with Aut(Γ) generates an operation which behaves like f on $\{c_1, \ldots, c_n\}$ and which is a canonical operation for $(\Gamma; c_1, \ldots, c_n)$.*

To apply this technique to polymorphisms, and not just to unary operations, we need the following well-known fact.

Lemma 7 (Product Ramsey Theorem). *When Γ is Ramsey, then Γ^m is also Ramsey.*

Proof Sketch for Theorem 1. Let Δ be a relational structure that is first-order definable in Γ, and let R_0 be a k-ary relation that is first-order definable in Γ. Since Δ is ω-categorical, Theorem 2 asserts that it suffices to decide whether there exists a polymorphism of Δ that violates R_0. It has been observed in (BK09) that such a polymorphism exists if and only if there exists an m-ary polymorphism of Δ that violates R_0, where m is the number of orbits of R_0 (by ω-categoricity of Δ, m is finite).

The outer loop in our algorithm enumerates all possible sequences $O_0, O_1, \ldots, O_m$ of orbits of k-tuples in Γ such that $O_1, \ldots, O_m$ are contained in R_0 and O_0 is not. By combination of Lemma 6 and 7 there exists a polymorphism of Δ that violates R_0 if and only if for some choice of orbits $O_0, O_1, \ldots, O_m$, and some choice of tuples $\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_m$ of tuples from those orbits, there exists a polymorphism of Δ that is canonical for $(\Gamma^m, \bar{c}_1 \times \cdots \times \bar{c}_m)$, and that maps $(c_1^i, \ldots, c_m^i)$ to c_0^i for all i. Note that expansions of ultrahomogeneous structures by finitely many constants are ultrahomogeneous, and therefore the well-known correspondence between orbits of n-tuples, n-types, and induced n-element substructures extends to expansions of ultrahomogeneous structures by constants. In particular, if n is the maximal arity of the relations in Γ, then being n-canonical is the same as being canonical.

We decide the existence of such a canonical polymorphism of Δ for a specific choice of orbits $O_0, O_1, \ldots, O_m$ by reduction to the following finite-domain constraint satisfaction problem. The domain of the CSP is the set of all orbits of n-types in $(\Gamma; \bar{c}_0)$. The variables of the CSP consist of all n-types of $(\Gamma^m; \bar{c}_1 \times \cdots \times \bar{c}_m)$. The constraints are as follows.

- (Compatibility.) There are binary constraints that exclude that two variables S_1 and S_2 that coincide on the subtype induced by some index set $I \subset \{1, \ldots, n\}$, are mapped to two n-types that do not conincide on the subtypes induced by I.
- (Violation.) In any solution of the CSP, the value of the type of $\bar{c}_1 \times \cdots \times \bar{c}_m$ in $(\Gamma; \bar{c}_1 \times \cdots \times \bar{c}_m)$ must be equal to the n-type of $\bar{c}_0$ in $(\Gamma; \bar{c}_0)$.
- (Preservation.) For every p-ary relation R from Δ, and every list $S_1, \ldots, S_m$ of p-types in $(\Gamma; \bar{c}_0)$ that are from R, we add constraints as follows. Let q be $\binom{p}{n}$. For all $j \leq m$, the p-type S_j is uniquely given by its sub-n-types in $(\Gamma; \bar{c}_0)$, listed by $S_j^1, \ldots, S_j^q$. For all $i \leq q$, let S^i be $S_1^i \times \cdots \times S_m^i$. Then we add the at most q-ary constraint that forces that the p-type described uniquely by the values of $S^1, \ldots, S^q$ in the CSP (which are n-types in $(\Gamma; \bar{c}_0)$) is in R.
- (Realizability.) For each substructure $N \in \mathcal{N}$, we have the following constraints. Assume without loss of generality that the size of N equals s. Let r be $\binom{s}{n}$. Let $S_1, \ldots, S_m$ be s-types of $(\Gamma; \bar{c}_0)$. Again, for all $j \leq m$ the p-type S_j is uniquely given by its sub-n-types in $(\Gamma; \bar{c}_0)$, listed by $S_j^1, \ldots, S_j^r$. For all $i \leq r$, let S^i be $S_1^i \times \cdots \times S_m^i$. Then we add the at most r-ary constraint that forces that at least one of the values of $S^1, \ldots, S^r$ in the CSP (which are n-types

in $(\Gamma; \bar{c}_0)$) does not correspond to the substructure of N induced by the respective n-subset of vertices.

We now prove that there is a canonical m-ary polymorphism f that violates R_0 by mapping $(c^i_1, \dots, c^i_m)$ to c^i_0 for all i if and only if the described CSP instance has a satisfying assignment, which will conclude the proof.

First, suppose that there exists such a polymorphism f. For each variable in the CSP, which is an n-type $S = S_1 \times S_m$ in $(\Gamma^m; \bar{c}_1 \times \cdots \times \bar{c}_m)$, pick witnesses $t_1, \dots, t_m$ from $S_1, \dots, S_m$, respectively. Then S will be mapped to the type of $f(t_1, \dots, t_m)$ in $(\Gamma; \bar{c}_0)$, which clearly satisfies compatibility, violation, preservation, and realizability constraints.

For the opposite direction, suppose that α is a solution to the CSP, i.e., a mapping from the n-types of $(\Gamma^m; \bar{c}_1 \times \cdots \times \bar{c}_m)$ to the n-types in $(\Gamma; \bar{c}_0)$. We show that there is a canonical m-ary polymorphism f that violates R_0 by mapping $(c^i_1, \dots, c^i_m)$ to c^i_0 for all i, in three steps.

We first construct an infinite structure Π with domain D^m and the same signature as Γ as follows. When the n-tuples $t_1, \dots, t_m \in D^n$ have the types $S_1, \dots, S_m$ in $(\Gamma; \bar{c}_0)$, then the substructure of Π induced by the n-tuple $t_1 \times \cdots \times t_m$ equals the induced substructure of Γ that corresponds to the n-type of $\alpha(S_1 \times \cdots \times S_m)$ in $(\Gamma; \bar{c}_0)$. This is well-defined by the compatibility constraints.

Next, we show that there exists a homomorphism from Π to Γ. By ω-categoricity of Γ and a standard compactness argument, it suffices to verify that Π does not contain any induced substructure from $\mathcal{N}$. But this is implied by the realizability constraints of the CSP.

Finally, observe that any homomorphism from Π to Γ must map the k-tuple $\bar{c}_1 \times \cdots \times \bar{c}_m$ to a k-tuple that lies in the same orbit as c^i_0 in Γ, by the violation constraints in the CSP and the construction of Π. Moreover, any homomorphism from Π to Γ must preserve all relations in Δ, which follows from the preservation constraints of the CSP and the construction of Π. By composition with an automorphism, we then obtain a polymorphism of Δ that maps $(c^i_1, \dots, c^i_m)$ to c^i_0. $\square$

Note that our method is non-constructive: the algorithm does not produce a primitive positive definition in case that there is one. It is an interesting open problem to come up with bounds on the number of existential variables that suffice to pp-define R_0 over $(D; R_1, \dots, R_n)$. For many structures Γ of practical interest, such as $(\mathbb{Q}; <)$ or the Random graph, our algorithm can certainly be tuned so that $\mathrm{Expr}(\Gamma)$ becomes feasible for reasonable input size; in particular, the gigantic Ramsey constants involved in the proofs of our results do not affect the running time of our procedure.

Finally, let us remark that the same ideas also give an algorithm that decides existential, or existential positive definability in structures that satisfy our assumptions.

References

[BCP10] Manuel Bodirsky, Hubie Chen, and Michael Pinsker. The reducts of equality up to primitive positive interdefinability. *To appear in the Journal of Symbolic Logic*, 2010.

[BK09] Manuel Bodirsky and Jan Kára. The complexity of temporal constraint satisfaction problems. *Journal of the ACM*, 57(2), 2009. An extended abstract appeared in the proceedings of STOC'08.

[BN06] Manuel Bodirsky and Jaroslav Nešetřil. Constraint satisfaction with countable homogeneous templates. *Journal of Logic and Computation*, 16(3):359–373, 2006.

[BP08] Manuel Bodirsky and Diana Piguet. Finite trees are ramsey with respect to topological embeddings. Preprint, arXiv:1002:1557, 2008.

[GRS90] R. L. Graham, Bruce L. Rothschild, and Joel H. Spencer. *Ramsey theory.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1990. Second edition.

[Hod97] Wilfrid Hodges. *A shorter model theory.* Cambridge University Press, Cambridge, 1997.

[KPT05] Kechris, Pestov, and Todorcevic. Fraissé limits, Ramsey theory, and topological dynamics of automorphism groups. *Geometric and Functional Analysis*, 15(1):106–189, 2005.

[Mac09] H.D. Macpherson. A survey of homogeneous structures. Preprint, available from the homepage of the author, 2009.

[Mat93] Yuri V. Matiyasevich. *Hilbert's Tenth Problem.* MIT Press, Cambridge, Massachusetts, 1993.

[Nes05] Jaroslav Nesetril. Ramsey classes and homogeneous structures. *Combinatorics, Probability & Computing*, 14(1-2):171–189, 2005.

[Wil10] Ross Willard. Testing expressibility is hard. In *Proceedings of CP'10*, 2010.