Minimal functions on the random graph

Michael Pinsker

joint work with Manuel Bodirsky

ÉLM Université Denis-Diderot Paris 7

Logic Colloquium 2010
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For all $A, B \subseteq \Gamma$ finite, for all isomorphisms $i : A \rightarrow B$ there exists $\alpha \in \text{Aut}(\Gamma)$ extending $i$.
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Reducts of homogeneous structures

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**Definition**

A *reduct* of $\Gamma$ is a structure with a first-order (f.o.) definition in $\Gamma$. 
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**Definition**

A *reduct* of $\Gamma$ is a structure with a first-order (f.o.) definition in $\Gamma$.

**Problem**

Classify the reducts of $\Gamma$. 
Possible classifications

Consider two reducts $\Delta, \Delta'$ of $\Gamma$ equivalent iff $\Delta$ is also a reduct of $\Delta'$ and vice-versa.
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Finer classifications of the reducts of $\Gamma$, e.g. up to

Existential interdefinability
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Example: The random graph

Let $G = (V; E)$ be the random graph, and set for all $k \geq 2$

$$R(k) = \{ (x_1, \ldots, x_k) \subseteq V^k : x_i \text{ distinct, number of edges odd} \}.$$

Theorem (Thomas '91)

Let $\Gamma$ be a reduct of $G$. Then:

1. $\Gamma$ is first-order interdefinable with $(V; E)$,
2. $\Gamma$ is first-order interdefinable with $(V; R(3))$,
3. $\Gamma$ is first-order interdefinable with $(V; R(4))$,
4. $\Gamma$ is first-order interdefinable with $(V; R(5))$,
5. $\Gamma$ is first-order interdefinable with $(V; =)$.
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5. $\Gamma$ is first-order interdefinable with $(V; =)$. 
Further examples

Theorem (Thomas '91)
The homogeneous $K_n$-free graph has 2 reducts, up to f.o.-interdefinability.

Theorem (Thomas '96)
The homogeneous $k$-graph has $2k+1$ reducts, up to f.o.-interdefinability.

Theorem (Cameron '76)
$\langle \mathbb{Q}; < \rangle$ has 5 reducts, up to f.o.-interdefinability.

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$\langle \mathbb{Q}; <, 0 \rangle$ has 116 reducts, up to f.o.-interdefinability.
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Conjecture (Thomas ’91)

Let $\Gamma$ be homogeneous in a finite language.
Then $\Gamma$ has finitely many reducts up to f.o.-interdefinability.
A formula is existential iff it is of the form $\exists x_1, \ldots, x_n. \psi$, where $\psi$ is quantifier-free.

A formula is existential positive iff it is existential and does not contain negations.

A formula is primitive positive iff it is existential positive and does not contain disjunctions.

Theorem (Bodirsky, Chen, P.'08) For the structure $\Gamma := (X; =)$, there exist:

1. $\aleph_0$ reducts up to first order / existential interdefinability
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M. Pinsker (Paris 7)
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- $2^{\aleph_0}$ reducts up to primitive positive interdefinability
Groups, Monoids, Clones

Theorem

The mapping $\Delta \mapsto \text{Aut}(\Delta)$ is a one-to-one correspondence between the first-order closed reducts of $\Gamma$ and the closed supergroups of $\text{Aut}(\Gamma)$.

The mapping $\Delta \mapsto \text{End}(\Delta)$ is a one-to-one correspondence between the existential positive closed reducts of $\Gamma$ and the closed supermonoids of $\text{Aut}(\Gamma)$.

The mapping $\Delta \mapsto \text{Pol}(\Delta)$ is a one-to-one correspondence between the primitive positive closed reducts of $\Gamma$ and the closed superclones of $\text{Aut}(\Gamma)$.

Pol$(\Delta)$ is the set of finitary operations which contains all projections and which is closed under composition.
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The reducts of the random graph, revisited

Let $G := (V; E)$ be the random graph.
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Let $\text{sw}_c : V \rightarrow V$ be an isomorphism between $G$ and $G_c$. 

Theorem (Thomas '91)

The closed groups containing $\text{Aut}(G)$ are the following:

1. $\text{Aut}(G)$
2. $\langle \{-\} \cup \text{Aut}(G) \rangle$
3. $\langle \{\text{sw}_c\} \cup \text{Aut}(G) \rangle$
4. $\langle \{-, \text{sw}_c\} \cup \text{Aut}(G) \rangle$
5. The full symmetric group $S_V$. 

M. Pinsker (Paris 7)
The reducts of the random graph, revisited

Let $G := (V; E)$ be the random graph. Let $\bar{G}$ be the graph that arises by switching edges and non-edges. Let $\sigma : V \to V$ be an isomorphism between $G$ and $\bar{G}$.

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Let $\tilde{G}$ be the graph that arises by switching edges and non-edges.
Let $\overline{\cdot} : V \to V$ be an isomorphism between $G$ and $\tilde{G}$.
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How to find all reducts up to \ldots-interdefinability?

Climb up the lattice!
Canonical functions

Definition

$f: \Gamma \rightarrow \Gamma$ is canonical iff for all tuples $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ of the same type in $\Gamma$, $(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type in $\Gamma$.

Examples on the random graph.
The identity is canonical.
$-\,$ is canonical on $V$.
$sw\,c$ is canonical for $(V; E, c)$.

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**Examples on the random graph.**
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sw$_c$ is canonical for $(V; E, c)$.
Ramsey classes

Let $N, H, P$ be structures in the same language. $N \rightarrow (H)P$ means:

For all colorings of the copies of $P$ in $N$ with 2 colors there exists a copy of $H$ in $N$ such that all the copies of $P$ in $H$ have the same color.

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A class $C$ of structures of the same signature is called a Ramsey class iff for all $H, P \in C$ there is $N \in C$ such that $N \rightarrow (H)P$. 
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A class $\mathcal{C}$ of structures of the same signature is called a Ramsey class iff for all $H, P \in \mathcal{C}$ there is $N$ in $\mathcal{C}$ such that $N \to (H)^P$. 
Observation. Let $\Gamma$ be ordered Ramsey (i.e., its age is an ordered Ramsey class). Let $H$ be a finite structure in the age of $\Gamma$. Then there is a copy of $H$ in $\Gamma$ on which $f$ is canonical.

Refining this idea, one can show: If $\Gamma$ is a reduct of an ordered Ramsey structure, then every non-trivial function $g$ generates a non-trivial function which is canonical with respect to $(\Gamma, c_1, \ldots, c_n)$ for constants $c_1, \ldots, c_n$. 

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Minimal functions

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Theorem (Thomas ’96)

Let $f : V \to V$, $f \notin \text{Aut}(G)$.
Then $f$ generates one of the following:

- A constant operation
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**Corollary.** All reducts of the random graph are model-complete.
Theorem (Bodirsky, P. ’09)

Let \( f : V^n \to V, \ f \notin \text{Aut}(G). \)

Then \( f \) generates one of the following:

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Application.

Constraint Satisfaction in theoretical computer science.

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**Application.** Constraint Satisfaction in theoretical computer science.
Theorem (Bodirsky, P., Tsankov '10)

Let $\Gamma$ be a finite language reduct of an ordered Ramsey structure. Then:

- There are finitely many minimal closed supermonoids of $\text{Aut}(\Gamma)$.
- Every closed supermonoid of $\text{Aut}(\Gamma)$ contains a minimal closed supermonoid of $\text{Aut}(\Gamma)$.
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Theorem (Bodirsky, P., Tsankov '10)

Let \( \Gamma \) be a finite language reduct of an ordered Ramsey structure which is finitely bounded. Then the following problem is decidable:

Input: First-order formulas \( \psi \) and \( \phi_1, \ldots, \phi_n \) over \( \Gamma \).

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Most important problem

Does Thomas’ conjecture hold for Ramsey structures?