How complicated is the local clone lattice?

Michael Pinsker

LMNO
Université de Caen
Caen, France

May 2008 / Mahdia

Clones

$X \ldots$ base set.

$\mathcal{O}^{(n)} = X^{X^n} = \{f : X^n \to X\} \ldots$ n-ary functions on X.

$\mathcal{O} = \bigcup_{n \geq 1} \mathcal{O}^{(n)} \ldots$ finitary operations on X.

Clones

$X \dots$ base set.

$\mathcal{O}^{(n)} = X^{X^n} = \{f : X^n \to X\} \dots$ n-ary functions on X.

$\mathcal{O} = \bigcup_{n \geq 1} \mathcal{O}^{(n)} \dots$ finitary operations on X.

Definition

$\mathcal{C} \subseteq \mathcal{O}$ clone iff

Clones

$X \ldots$ base set.

$\mathcal{O}^{(n)} = X^{X^n} = \{f : X^n \to X\} \ldots$ n-ary functions on X.

$\mathcal{O} = \bigcup_{n \geq 1} \mathcal{O}^{(n)} \ldots$ finitary operations on X.

Definition

$\mathcal{C} \subseteq \mathcal{O}$ clone iff

- $\mathcal{C}$ contains the projections and
- $\mathcal{C}$ closed under composition.

Clones

$X \ldots$ base set.

$\mathcal{O}^{(n)} = X^{X^n} = \{f : X^n \to X\} \ldots$ n-ary functions on X.

$\mathcal{O} = \bigcup_{n \geq 1} \mathcal{O}^{(n)} \ldots$ finitary operations on X.

Definition

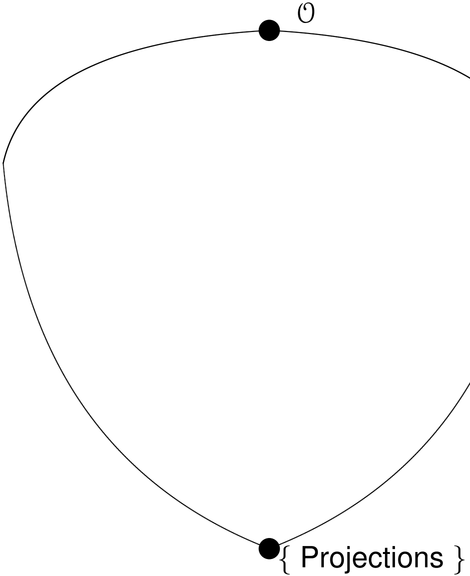
$\mathcal{C} \subseteq \mathcal{O}$ clone iff

- $\mathcal{C}$ contains the projections and
- $\mathcal{C}$ closed under composition.

Definition

$\mathrm{Cl}(X) = (\{\text{Clones on } X\}, \subseteq) \ldots$ complete algebraic lattice of clones.

The clone lattice from the view of the uninitiated



Locally closed (or: local) clones

Locally closed (or: local) clones

Equip X with the discrete topology.

$\mathcal{O}^{(n)} = X^{X^n}$ has the product topology.

$\mathcal{O} = \bigcup_n \mathcal{O}^{(n)}$ sum space of the $\mathcal{O}^{(n)}$.

Locally closed (or: local) clones

Equip X with the discrete topology.

$\mathcal{O}^{(n)} = X^{X^n}$ has the product topology.

$\mathcal{O} = \bigcup_n \mathcal{O}^{(n)}$ sum space of the $\mathcal{O}^{(n)}$.

Definition

A clone $\mathcal{C}$ is *locally closed* or *local* $\leftrightarrow$
$\mathcal{C}$ is closed in this topology.

Locally closed (or: local) clones

Equip X with the discrete topology.

$\mathcal{O}^{(n)} = X^{X^n}$ has the product topology.

$\mathcal{O} = \bigcup_n \mathcal{O}^{(n)}$ sum space of the $\mathcal{O}^{(n)}$.

Definition

A clone $\mathcal{C}$ is *locally closed* or *local* $\leftrightarrow$
$\mathcal{C}$ is closed in this topology.

Fact

$\mathcal{C}$ *local* $\leftrightarrow$

- *WHENEVER* $g \in \mathcal{O}$
- *AND for all finite* $A \subseteq X$
  there exists $f \in \mathcal{C}$ *which agrees with* g *on* A
- *THEN* $g \in \mathcal{C}$.

The Galois connection Inv-Pol

Let $f \in \mathcal{O}$ be an operation, and $R \subseteq X^m$ be a relation.

f preserves R $\leftrightarrow f(r_1, \ldots, r_n) \in R$ for all $r_1, \ldots, r_n \in R$.

The Galois connection Inv-Pol

Let $f \in \mathcal{O}$ be an operation, and $R \subseteq X^m$ be a relation.

f preserves $R \leftrightarrow f(r_1, \ldots, r_n) \in R$ for all $r_1, \ldots, r_n \in R$.

Let $\mathcal{F} \subseteq \mathcal{O}$. $\quad \mathrm{Inv}(\mathcal{F}) := \{R : R$ is preserved by all $f \in \mathcal{F}\}$.

The Galois connection Inv-Pol

Let $f \in \mathcal{O}$ be an operation, and $R \subseteq X^m$ be a relation.

f preserves R $\leftrightarrow f(r_1, \ldots, r_n) \in R$ for all $r_1, \ldots, r_n \in R$.

Let $\mathcal{F} \subseteq \mathcal{O}$. $\quad \mathrm{Inv}(\mathcal{F}) := \{R : R$ is preserved by all $f \in \mathcal{F}\}$.

Let $\mathcal{R}$ be a set of relations. $\quad \mathrm{Pol}(\mathcal{R}) := \{f : f$ preserves all $R \in \mathcal{R}\}$.

The Galois connection Inv-Pol

Let $f \in \mathcal{O}$ be an operation, and $R \subseteq X^m$ be a relation.

f preserves R $\leftrightarrow f(r_1, \ldots, r_n) \in R$ for all $r_1, \ldots, r_n \in R$.

Let $\mathcal{F} \subseteq \mathcal{O}$. $\quad \mathrm{Inv}(\mathcal{F}) := \{R : R \text{ is preserved by all } f \in \mathcal{F}\}$.

Let $\mathcal{R}$ be a set of relations. $\quad \mathrm{Pol}(\mathcal{R}) := \{f : f \text{ preserves all } R \in \mathcal{R}\}$.

Fact

The local clones are exactly the Pol Inv*-closed subsets of* $\mathcal{O}$*.*

The Galois connection Inv-Pol

Let $f \in \mathcal{O}$ be an operation, and $R \subseteq X^m$ be a relation.

f preserves R $\leftrightarrow f(r_1, \ldots, r_n) \in R$ for all $r_1, \ldots, r_n \in R$.

Let $\mathcal{F} \subseteq \mathcal{O}$. $\quad \mathrm{Inv}(\mathcal{F}) := \{R : R$ is preserved by all $f \in \mathcal{F}\}$.

Let $\mathcal{R}$ be a set of relations. $\quad \mathrm{Pol}(\mathcal{R}) := \{f : f$ preserves all $R \in \mathcal{R}\}$.

Fact

The local clones are exactly the Pol Inv-closed subsets of $\mathcal{O}$.

Fact

The local closure of $\mathcal{F} \subseteq \mathcal{O}$ is
the topological closure of the term closure of $\mathcal{F}$.

The Galois connection Inv-Pol

Let $f \in \mathcal{O}$ be an operation, and $R \subseteq X^m$ be a relation.

f *preserves* $R \leftrightarrow f(r_1, \ldots, r_n) \in R$ for all $r_1, \ldots, r_n \in R$.

Let $\mathcal{F} \subseteq \mathcal{O}$. $\quad \mathrm{Inv}(\mathcal{F}) := \{R : R$ is preserved by all $f \in \mathcal{F}\}$.

Let $\mathcal{R}$ be a set of relations. $\quad \mathrm{Pol}(\mathcal{R}) := \{f : f$ preserves all $R \in \mathcal{R}\}$.

Fact

The local clones are exactly the Pol Inv-closed subsets of $\mathcal{O}$.

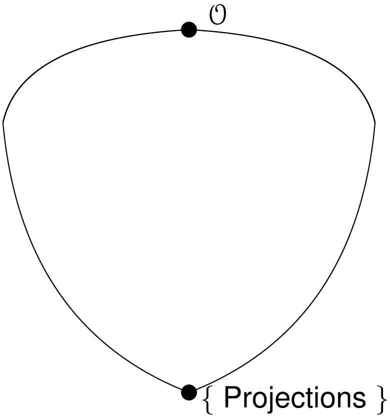
Fact

The local closure of $\mathcal{F} \subseteq \mathcal{O}$ is
the topological closure of the term closure of $\mathcal{F}$.

Fact

$\mathrm{Cl}_{loc}(X) = (\{\text{Local clones on } X\}, \subseteq) \ldots$ *complete lattice of local clones.*

The LOCAL clone lattice from the view of the uninitiated



Reasons for studying local clones

Reasons for studying local clones

- Long train rides.

Reasons for studying local clones

- Long train rides.
- More natural generalization of the clone lattice on finite sets than the full clone lattice.

Reasons for studying local clones

- Long train rides.
- More natural generalization of the clone lattice on finite sets than the full clone lattice. (???)

Reasons for studying local clones

- Long train rides.
- More natural generalization of the clone lattice on finite sets than the full clone lattice. (???)
- Constraint Satisfaction Problem (CSP):
  Infinite domains make sense;
  For certain relational structures Γ one has $\mathrm{pp}(\Gamma) = \mathrm{Inv}\,\mathrm{Pol}(\Gamma)$

Reasons for studying local clones

- Long train rides.
- More natural generalization of the clone lattice on finite sets than the full clone lattice. (???)
- Constraint Satisfaction Problem (CSP):
  Infinite domains make sense;
  For certain relational structures Γ one has $\mathrm{pp}(\Gamma) = \mathrm{Inv}\,\mathrm{Pol}(\Gamma)$
- Model theory:
  For certain Γ, the reducts up to pp-definability
  correspond to the local clones containing $\mathrm{Aut}(\Gamma)$.

Examples

Examples of local clones:

Examples

Examples of local clones:

- The clone of all idempotent operations ($f(x, \ldots, x) = x$).

Examples

Examples of local clones:

- The clone of all idempotent operations ($f(x, \ldots, x) = x$).
- The "clone" of all operations which take at most 5 values.

Examples

Examples of local clones:

- The clone of all idempotent operations ($f(x, \dots, x) = x$).
- The “clone” of all operations which take at most 5 values.
- The “clone” of all unary operations.

Examples

Examples of local clones:

- The clone of all idempotent operations ($f(x, \ldots, x) = x$).
- The "clone" of all operations which take at most 5 values.
- The "clone" of all unary operations.

Examples of non-local clones:

Examples

Examples of local clones:

- The clone of all idempotent operations ($f(x, \ldots, x) = x$).
- The "clone" of all operations which take at most 5 values.
- The "clone" of all unary operations.

Examples of non-local clones:

- The "clone" of all bijective operations.

Examples

Examples of local clones:

- The clone of all idempotent operations ($f(x, \ldots, x) = x$).
- The "clone" of all operations which take at most 5 values.
- The "clone" of all unary operations.

Examples of non-local clones:

- The "clone" of all bijective operations.
- The "clone" of all operations which grow not faster than $G(x)$.

Examples

Examples of local clones:

- The clone of all idempotent operations ($f(x, \ldots, x) = x$).
- The "clone" of all operations which take at most 5 values.
- The "clone" of all unary operations.

Examples of non-local clones:

- The "clone" of all bijective operations.
- The "clone" of all operations which grow not faster than $G(x)$.

Example for the local closure:

Examples

Examples of local clones:

- The clone of all idempotent operations ($f(x, \ldots, x) = x$).
- The "clone" of all operations which take at most 5 values.
- The "clone" of all unary operations.

Examples of non-local clones:

- The "clone" of all bijective operations.
- The "clone" of all operations which grow not faster than $G(x)$.

Example for the local closure:

- $\alpha, \beta \in \mathcal{O}^{(1)}$ permutations, α one two-cycle, β one infinite cycle. Then $\{\alpha, \beta\}$ generates all injective unary operations.

Examples

Examples of local clones:

- The clone of all idempotent operations ($f(x, \ldots, x) = x$).
- The "clone" of all operations which take at most 5 values.
- The "clone" of all unary operations.

Examples of non-local clones:

- The "clone" of all bijective operations.
- The "clone" of all operations which grow not faster than $G(x)$.

Example for the local closure:

- $\alpha, \beta \in \mathcal{O}^{(1)}$ permutations, α one two-cycle, β one infinite cycle. Then $\{\alpha, \beta\}$ generates all injective unary operations.

Observe:

$\mathrm{Cl}_{loc}(X)$ is NOT a sublattice of $\mathrm{Cl}(X)$!

Sublattices of $\mathrm{Cl}_{loc}(X)$

Proposition

$\mathrm{Cl}_{loc}(X)$ is NOT algebraic.
It has no compact elements except {Projections}.
It is not upper continuous.

Sublattices of $\mathrm{Cl}_{loc}(X)$

Proposition

$\mathrm{Cl}_{loc}(X)$ is NOT algebraic.
It has no compact elements except {Projections}.
It is not upper continuous.

Proposition

$\mathrm{Cl}_{loc}(X)$ contains all clone lattices over finite Y as intervals.

Sublattices of $\mathrm{Cl}_{loc}(X)$

Proposition

$\mathrm{Cl}_{loc}(X)$ is NOT algebraic.
It has no compact elements except {Projections}.
It is not upper continuous.

Proposition

$\mathrm{Cl}_{loc}(X)$ contains all clone lattices over finite Y as intervals.

Corollary

$\mathrm{Cl}_{loc}(X)$ does not satisfy any non-trivial lattice (quasi-)identities.

Sublattices of $\mathrm{Cl}_{loc}(X)$

Proposition

$\mathrm{Cl}_{loc}(X)$ is NOT algebraic.
It has no compact elements except $\{$Projections$\}$.
It is not upper continuous.

Proposition

$\mathrm{Cl}_{loc}(X)$ contains all clone lattices over finite Y as intervals.

Corollary

$\mathrm{Cl}_{loc}(X)$ does not satisfy any non-trivial lattice (quasi-)identities.

Theorem

Let $\mathfrak{L}$ be any algebraic lattice with ω compact elements.
Then $\mathfrak{L}$ embeds completely into $\mathrm{Cl}_{loc}(X)$.

Clones of partial operations

Definition

A *partial operation* is a function f
with FINITE domain $\subseteq X^n$ and range $\subseteq X$.

A *partial clone* is a set of partial operations
containing all partial projections and closed under composition.

Clones of partial operations

Definition

A *partial operation* is a function f
with FINITE domain $\subseteq X^n$ and range $\subseteq X$.

A *partial clone* is a set of partial operations
containing all partial projections and closed under composition.

Fact

The set of partial clones $\text{Cl}_{part}(X)$ *forms a complete algebraic lattice with* ω *compacts.*

Clones of partial operations

Definition

A *partial operation* is a function f
with FINITE domain $\subseteq X^n$ and range $\subseteq X$.

A *partial clone* is a set of partial operations
containing all partial projections and closed under composition.

Fact

The set of partial clones $\mathrm{Cl}_{part}(X)$ *forms a complete algebraic lattice with* ω *compacts.*

Theorem

The mapping

$\mathrm{Cl}_{loc}(X) \to \mathrm{Cl}_{part}(X)$

$\mathcal{C} \mapsto \{\textit{Restrictions of all } f \in \mathcal{C} \textit{ to finite sets}\}$

preserves arbitrary joins.

Larger sublattices

Corollary

$\mathrm{Cl}_{loc}(X)$ order-embeds into the power set of ω.
$|\mathrm{Cl}_{loc}(X)| = 2^{\aleph_0}$.

Larger sublattices

Corollary

$\mathrm{Cl}_{loc}(X)$ order-embeds into the power set of ω.
$|\mathrm{Cl}_{loc}(X)| = 2^{\aleph_0}$.

Summary: Inclusions

$\mathrm{Sub}(\mathrm{Cl}(Y)) \subsetneq \mathrm{Sub}(\mathrm{Alg}(\omega)) \subseteq \mathrm{Sub}(\mathrm{Cl}_{loc}(X)) \subseteq \mathrm{Jsub}(\mathrm{Alg}(\omega))$.

Larger sublattices

Corollary

$\mathrm{Cl}_{loc}(X)$ order-embeds into the power set of ω.
$|\mathrm{Cl}_{loc}(X)| = 2^{\aleph_0}$.

Summary: Inclusions

$\mathrm{Sub}(\mathrm{Cl}(Y)) \subsetneq \mathrm{Sub}(\mathrm{Alg}(\omega)) \subseteq \mathrm{Sub}(\mathrm{Cl}_{loc}(X)) \subseteq \mathrm{Jsub}(\mathrm{Alg}(\omega))$.

Theorem

$\mathcal{M}_{2^{\aleph_0}}$ embeds into $\mathrm{Cl}_{loc}(X)$.

Larger sublattices

Corollary

$\mathrm{Cl}_{loc}(X)$ order-embeds into the power set of ω.
$|\mathrm{Cl}_{loc}(X)| = 2^{\aleph_0}$.

Summary: Inclusions

$\mathrm{Sub}(\mathrm{Cl}(Y)) \subsetneq \mathrm{Sub}(\mathrm{Alg}(\omega)) \subseteq \mathrm{Sub}(\mathrm{Cl}_{loc}(X)) \subseteq \mathrm{Jsub}(\mathrm{Alg}(\omega))$.

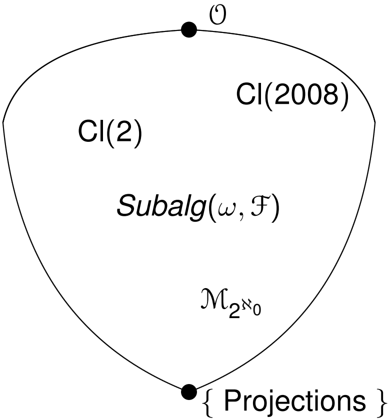
Theorem

$\mathcal{M}_{2^{\aleph_0}}$ *embeds into* $\mathrm{Cl}_{loc}(X)$.

Corollary

$\mathrm{Sub}(\mathrm{Alg}(\omega)) \subsetneq \mathrm{Sub}(\mathrm{Cl}_{loc}(X)) \subseteq \mathrm{Jsub}(\mathrm{Alg}(\omega))$.

The local clone lattice from the view of the attentive listener



Problems

Problems

THE problem

Let $\mathfrak{L}$ have a $\bigvee$-preserving embedding into an algebraic lattice with ω compacts.
Does $\mathfrak{L}$ embed into $\mathrm{Cl}_{loc}(X)$?
Does it embed completely?

Problems

THE problem

Let $\mathfrak{L}$ have a $\bigvee$-preserving embedding into an algebraic lattice with ω compacts.
Does $\mathfrak{L}$ embed into $\mathrm{Cl}_{loc}(X)$?
Does it embed completely?

Attention lattice theorists

Let $\mathfrak{L}$ have an order-preserving embedding into the power set of ω.
Does $\mathfrak{L}$ have a $\vee$-preserving embedding into an algebraic lattice with ω compacts?