How complicated is the local clone lattice?

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\( X \) ... base set.

\[ O^{(n)} = X^{X^n} = \{ f : X^n \to X \} \ldots n\text{-ary functions on } X. \]

\[ O = \bigcup_{n \geq 1} O^{(n)} \ldots \text{finitary operations on } X. \]
Clones

\( X \ldots \) base set.

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**Definition**

\( C \subseteq \mathcal{O} \) clone iff
Clones

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**Definition**

$\mathcal{C} \subseteq \mathcal{O}$ clone iff

- $\mathcal{C}$ contains the projections and
- $\mathcal{C}$ closed under composition.
\( X \) ... base set.

\( \emptyset^{(n)} = X^{X^n} = \{ f : X^n \to X \} \ldots n\)-ary functions on \( X \).

\( \emptyset = \bigcup_{n \geq 1} \emptyset^{(n)} \ldots \) finitary operations on \( X \).

**Definition**

\( \mathcal{C} \subseteq \emptyset \) **clone** iff

- \( \mathcal{C} \) contains the projections and
- \( \mathcal{C} \) closed under composition.

**Definition**

\( \text{Cl}(X) = (\{ \text{Clones on } X \}, \subseteq) \ldots \) complete algebraic **lattice** of clones.
The clone lattice from the view of the uninitiated

\{ Projections \}

\emptyset
Locally closed (or: local) clones

Equip $X$ with the discrete topology. $O(n) = X \times X^n$ has the product topology.

Definition

A clone $C$ is locally closed or local $\leftrightarrow C$ is closed in this topology.

Fact

$C$ local $\leftrightarrow$ WHENEVER $g \in O$ AND for all finite $A \subseteq X$ there exists $f \in C$ which agrees with $g$ on $A$ THEN $g \in C$. 
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$C$ *local* $\iff$

- **WHENEVER** $g \in \emptyset$
- **AND for all finite** $A \subseteq X$
  - there exists $f \in C$ which agrees with $g$ on $A$
- **THEN** $g \in C$. 
Let \( f \in \emptyset \) be an operation, and \( R \subseteq X^m \) be a relation.

\( f \) preserves \( R \) if and only if \( f(r_1, \ldots, r_n) \in R \) for all \( r_1, \ldots, r_n \in R \).
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$f$ preserves $R$ $\iff$ $f(r_1, \ldots, r_n) \in R$ for all $r_1, \ldots, r_n \in R$. 

Let $\mathcal{F} \subseteq O$. $\text{Inv}(\mathcal{F}) := \{R : R$ is preserved by all $f \in \mathcal{F}\}$. 
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Fact

The local clones are exactly the $\text{Pol Inv}$-closed subsets of $\mathcal{O}$. 
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**Fact**

*The local closure of \( \mathcal{F} \subseteq \mathcal{O} \) is the topological closure of the term closure of \( \mathcal{F} \).*
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\textbf{Fact}

$\text{Cl}_{\text{loc}}(X) = (\{\text{Local clones on } X\}, \subseteq) \ldots$ complete lattice of local clones.
The LOCAL clone lattice from the view of the uninitiated
Reasons for studying local clones

Long train rides.

More natural generalization of the clone lattice on finite sets than the full clone lattice.

Constraint Satisfaction Problem (CSP):
Infinite domains make sense;
For certain relational structures $\Gamma$ one has $pp(\Gamma) = Inv Pol(\Gamma)$.

Model theory:
For certain $\Gamma$, the reducts up to pp-definability correspond to the local clones containing $Aut(\Gamma)$.
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Examples

Examples of local clones:

- The clone of all idempotent operations \( f(x, \ldots, x) = x \).
- The "clone" of all operations which take at most 5 values.
- The "clone" of all unary operations.

Examples of non-local clones:

- The "clone" of all bijective operations.
- The "clone" of all operations which grow not faster than \( G(x) \).

Example for the local closure:

\( \alpha, \beta \in O^1 \) permutations, \( \alpha \) one two-cycle, \( \beta \) one infinite cycle.

Then \( \{ \alpha, \beta \} \) generates all injective unary operations.

Observe:

\( \text{Cl}_{\text{loc}}(X) \) is NOT a sublattice of \( \text{Cl}(X) \)!
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\( \text{Cl}_{loc}(X) \) is NOT a sublattice of \( \text{Cl}(X) \)!
Sublattices of $\text{Cl}_{loc}(X)$

**Proposition**

$\text{Cl}_{loc}(X)$ is NOT algebraic.

It has no compact elements except \{Projections\}.

It is not upper continuous.

Corollary

$\text{Cl}_{loc}(X)$ does not satisfy any non-trivial lattice (quasi-)identities.

Theorem

Let $L$ be any algebraic lattice with $\omega$ compact elements.

Then $L$ embeds completely into $\text{Cl}_{loc}(X)$. 
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Let $\mathcal{L}$ be any algebraic lattice with $\omega$ compact elements.
Then $\mathcal{L}$ embeds completely into $\text{Cl}_{loc}(X)$. 
**Definition**

A *partial operation* is a function $f$ with FINITE domain $\subseteq X^n$ and range $\subseteq X$.

A *partial clone* is a set of partial operations containing all partial projections and closed under composition.

**Fact**

The set of partial clones $\text{Cl}_{\text{part}}(X)$ forms a complete algebraic lattice with $\omega$ compacts.

**Theorem**

The mapping $\text{Cl}_{\text{loc}}(X) \rightarrow \text{Cl}_{\text{part}}(X) \ C \mapsto \{\text{Restrictions of all } f \in C \text{ to finite sets}\}$ preserves arbitrary joins.
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$\text{Cl}_{loc}(X)$ order-embeds into the power set of $\omega$.

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Summary: Inclusions

$\text{Sub}(\text{Cl}(Y)) \subsetneq \text{Sub}(\text{Alg}(\omega)) \subseteq \text{Sub}(\text{Cl}_{loc}(X)) \subseteq \text{Jsub}(\text{Alg}(\omega))$. 
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Theorem

$\mathcal{M}_{2^{\aleph_0}}$ embeds into $\text{Cl}_{\text{loc}}(X).$
Larger sublattices

**Corollary**

\[ \text{Cl}_{loc}(X) \text{ order-embeds into the power set of } \omega. \]
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**Corollary**

\[ \text{Sub(Alg}(\omega)) \subsetneq \text{Sub(Cl}_{loc}(X)) \subseteq \text{Jsub(Alg}(\omega)). \]
The local clone lattice from the view of the attentive listener

\[ \{ \text{Projections} \} \]

\[ \mathcal{M}_{2^\aleph_0} \]

\[ \text{Subalg}(\omega, \mathcal{F}) \]

\[ \text{Cl}(2) \]

\[ \text{Cl}(2008) \]

\[ \emptyset \]
Problems

Let $L$ have a $\lor$-preserving embedding into an algebraic lattice with $\omega$ compacts. Does $L$ embed into $\text{Cl}_{\text{loc}}(X)$? Does it embed completely?

Attention lattice theorists

Let $L$ have an order-preserving embedding into the power set of $\omega$. Does $L$ have a $\lor$-preserving embedding into an algebraic lattice with $\omega$ compacts?

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THE problem

Let \( \mathcal{L} \) have a \( \vee \)-preserving embedding into an algebraic lattice with \( \omega \) compacts.
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Does $\mathcal{L}$ have a $\vee$-preserving embedding into an algebraic lattice with $\omega$ compacts?