The reducts of $(\mathbb{N}, =)$
up to primitive positive interdefinability

Michael Pinsker

Joint work with Manuel Bodirsky and Hubie Chen

Institute of Discrete Mathematics and Geometry
Vienna University of Technology
Wien, Austria

June 14, 2007 / OAL Nashville
Outline

1. Reducts of relational structures
2. First order definability and permutations
3. Primitive positive definability and operations
4. Primitive positive definability and local clones
5. The local clones above $S_\omega$
6. Connection to CSP and outlook
Reducts of relational structures

\[ \Gamma = (X, R) \ldots \text{relational structure}. \]

Problem

Determine the reducts of \( \Gamma \), i.e. all relational structures which are first-order definable from \( \Gamma \).
Reducts of relational structures

\[ \Gamma = (X, R) \ldots \text{relational structure.} \]

**Problem**

Determine the reducts of \( \Gamma \), i.e. all relational structures which are first-order definable from \( \Gamma \).

Usually done up to first-order interdefinability, i.e. two structures \( \Gamma_1, \Gamma_2 \) are considered equivalent iff \( \Gamma_1 \) has a first-order definition in \( \Gamma_2 \) and vice-versa.
Reducts of relational structures

Γ = (X, R) . . . relational structure.

Problem

Determine the reducts of Γ, i.e. all relational structures which are first-order definable from Γ.

Usually done up to first-order interdefinability, i.e. two structures Γ₁, Γ₂ are considered equivalent iff Γ₁ has a first-order definition in Γ₂ and vice-versa.

Examples
Reducts of relational structures

\[ \Gamma = (X, R) \ldots \text{relational structure.} \]

**Problem**

Determine the reducts of \( \Gamma \), i.e. all relational structures which are first-order definable from \( \Gamma \).

Usually done up to first-order interdefinability, i.e. two structures \( \Gamma_1, \Gamma_2 \) are considered equivalent iff \( \Gamma_1 \) has a first-order definition in \( \Gamma_2 \) and vice-versa.

**Examples**

- P. J. Cameron: There are 5 reducts of \((\mathbb{Q}, <)\) up to f.o.-interdefinability.
Reducts of relational structures

\[ \Gamma = (X, R) \ldots \text{relational structure.} \]

**Problem**

Determine the reducts of \( \Gamma \), i.e. all relational structures which are first-order definable from \( \Gamma \).

Usually done up to first-order interdefinability, i.e. two structures \( \Gamma_1, \Gamma_2 \) are considered equivalent iff \( \Gamma_1 \) has a first-order definition in \( \Gamma_2 \) and vice-versa.

**Examples**

- P. J. Cameron: There are 5 reducts of \((\mathbb{Q}, <)\) up to f.o.-interdefinability.
- M. Junker and M. Ziegler: There are 116 reducts of \((\mathbb{Q}, <, a)\) up to f.o.-interdefinability.
Definition

Γ ω-categorical ↔ its theory has (up to iso) one countable model.

Definition

\[
\text{Aut}(\Gamma) := \{ \text{automorphisms of } \Gamma \}.
\]

Let \( G \) be a permutation group.

\[
\text{Inv}(G) := \{ R : \text{all } g \in G \text{ are automorphisms of } (X, R) \}.
\]

Fact

\[
\text{Inv Aut} = \text{hull operator on the relational structures}.
\]

\[
\text{Aut Inv} = \text{hull operator on the sets of permutations}.
\]

Fact

Let \( \Gamma \) be \( \omega \)-categorical.

Then \( \text{Inv Aut}(\Gamma) = \text{fo}(\Gamma) \).
Definition

Γ is \( \omega \)-categorical if and only if its theory has (up to iso) exactly one countable model.

Definition

\[ \text{Aut}(\Gamma) := \{ \text{automorphisms of } \Gamma \} \]
Let \( \mathcal{G} \) be a permutation group.
\[ \text{Inv}(\mathcal{G}) := \{ R : \text{all } g \in \mathcal{G} \text{ are automorphisms of } (X, R) \} \]
First order definability and permutations

**Definition**

\( \Gamma \) \( \omega \)-categorical \( \iff \) its theory has (up to iso) one countable model.

**Definition**

\[ \text{Aut}(\Gamma) := \{ \text{automorphisms of } \Gamma \} \].

Let \( \mathcal{G} \) be a permutation group.

\[ \text{Inv}(\mathcal{G}) := \{ R : \text{ all } g \in \mathcal{G} \text{ are automorphisms of } (X, R) \} \].

**Fact**

- \( \text{Inv Aut} = \) hull operator on the relational structures.
- \( \text{Aut Inv} = \) hull operator on the sets of permutations.
First order definability and permutations

**Definition**

Γ \(\omega\)-categorical \(\iff\) its theory has (up to iso) one countable model.

**Definition**

\[
\text{Aut}(\Gamma) := \{ \text{automorphisms of } \Gamma \}.
\]

Let \(\mathcal{G}\) be a permutation group.

\[
\text{Inv}(\mathcal{G}) := \{ R : \text{all } g \in \mathcal{G} \text{ are automorphisms of } (X, R) \}.
\]

**Fact**

- \(\text{Inv Aut} = \text{hull operator on the relational structures.}\)
- \(\text{Aut Inv} = \text{hull operator on the sets of permutations.}\)

**Fact**

Let \(\Gamma\) be \(\omega\)-categorical.

Then \(\text{Inv Aut}(\Gamma) = fo(\Gamma)\).
Definitions

Let $O(n) := X^n = \{ f : X^n \to X \}$. . . set of $n$-ary operations on $X$.

$O := \bigcup_{n \geq 1} O(n)$. . . finitary operations on $X$.

Let $f \in O(n)$ and $R \subseteq X^m$.

$f$ preserves $R$ $\iff$ $f(r_1, \ldots, r_n) \in R$ for all $r_1, \ldots, r_n \in R$.

$Pol(\Gamma) := \{ f \in O : f$ preserves all relations of $\Gamma \}$.

$Inv(F) := \{ R : R$ is preserved by all $f \in F \}$ (for $F \subseteq O$).

Fact Inv $Pol = hull$ operator on the relational structures

Inv $Pol = hull$ operator on the sets of operations

Observation

Let $\Gamma$ be $\omega$-categorical. Then Inv $Pol(\Gamma) = pp(\Gamma)$.
### Definitions

Let $O(n) := \{ f : X^n \to X \}$ be the set of $n$-ary operations on $X$. Then $O := \bigcup_{n \geq 1} O(n)$ is the set of finitary operations on $X$.

Let $f \in O(n)$ and $R \subseteq X^m$. Then $f$ preserves $R$ if and only if $f(r_1, \ldots, r_n) \in R$ for all $r_1, \ldots, r_n \in R$.

**Pol**$(Γ) := \{ f \in O : f$ preserves all relations of $Γ \}$.

**Inv**$(F) := \{ R : R$ is preserved by all $f \in F \}$ for $F \subseteq O$.

**Fact**

- **Inv Pol** is a hull operator on the relational structures.
- **Inv** is a hull operator on the sets of operations.

**Observation**

Let $Γ$ be $\omega$-categorical. Then $\text{Inv Pol}(Γ) = \text{pp}(Γ)$. 

M. Pinsker (TU Wien)
Definitions

Let $O^{(n)} := X^{X^n} = \{f : X^n \to X\} \ldots$ set of $n$ ary operations on $X$. 

$\mathcal{O} := \bigcup_{n \geq 1} O^{(n)} \ldots$ finitary operations on $X$. 

Observation

Let $\Gamma$ be ω-categorical. Then $\text{Inv Pol}(\Gamma) = \text{pp}(\Gamma)$. 

M. Pinsker (TU Wien)
Primitive positive definability and operations

Definitions

Let $\mathcal{O}(n) := X^{X^n} = \{f : X^n \to X\}$ ... set of $n$ ary operations on $X$.
$\mathcal{O} := \bigcup_{n \geq 1} \mathcal{O}(n)$ ... finitary operations on $X$.

Let $f \in \mathcal{O}(n)$ and $R \subseteq X^m$.

$f$ preserves $R \iff f(r_1, \ldots, r_n) \in R$ for all $r_1, \ldots, r_n \in R$. 
Definitions

Let $\mathcal{O}^{(n)} := X^X^n = \{f : X^n \to X\}$ ... set of $n$-ary operations on $X$.

$\mathcal{O} := \bigcup_{n \geq 1} \mathcal{O}^{(n)}$ ... finitary operations on $X$.

Let $f \in \mathcal{O}^{(n)}$ and $R \subseteq X^m$.

$f$ preserves $R \iff f(r_1, \ldots, r_n) \in R$ for all $r_1, \ldots, r_n \in R$.

$\text{Pol}(\Gamma) := \{f \in \mathcal{O} : f$ preserves all relations of $\Gamma\}$.

$\text{Inv}(\mathcal{F}) := \{R : R$ is preserved by all $f \in \mathcal{F}\}$ (for $\mathcal{F} \subseteq \mathcal{O}$)
Primitive positive definability and operations

Definitions

Let $\mathcal{O}^{(n)} := X^{X^n} = \{ f : X^n \to X \}$ \ldots set of $n$ ary operations on $X$.

$\mathcal{O} := \bigcup_{n \geq 1} \mathcal{O}^{(n)}$ \ldots finitary operations on $X$.

Let $f \in \mathcal{O}^{(n)}$ and $R \subseteq X^m$.

$f$ preserves $R \iff f(r_1, \ldots, r_n) \in R$ for all $r_1, \ldots, r_n \in R$.

$\text{Pol}(\Gamma) := \{ f \in \mathcal{O} : f$ preserves all relations of $\Gamma \}$.

$\text{Inv}(\mathcal{F}) := \{ R : R$ is preserved by all $f \in \mathcal{F} \}$ (for $\mathcal{F} \subseteq \mathcal{O}$)

Fact

- $\text{Inv Pol}$ = hull operator on the relational structures
- $\text{Pol Inv}$ = hull operator on the sets of operations
**Definitions**

Let \( \mathcal{O}^{(n)} := X^X \) be the set of \( n \) ary operations on \( X \).

\[ \mathcal{O} := \bigcup_{n \geq 1} \mathcal{O}^{(n)} \]  
... finitary operations on \( X \).

Let \( f \in \mathcal{O}^{(n)} \) and \( R \subseteq X^m \).

\( f \) preserves \( R \) if
\[ f(r_1, \ldots, r_n) \in R \]  
for all \( r_1, \ldots, r_n \in R \).

\( \text{Pol}(\Gamma) := \{ f \in \mathcal{O} : f \) preserves all relations of \( \Gamma \} \).  
\( \text{Inv}(\mathcal{F}) := \{ R : R \) is preserved by all \( f \in \mathcal{F} \} \) \( \text{ (for } \mathcal{F} \subseteq \mathcal{O}) \)

**Fact**

- \( \text{Inv Pol} \) = hull operator on the relational structures
- \( \text{Pol Inv} \) = hull operator on the sets of operations

**Observation**

Let \( \Gamma \) be \( \omega \)-categorical. Then \( \text{Inv Pol}(\Gamma) = pp(\Gamma) \).
Definition

A set $\mathcal{C} \subseteq \mathcal{O}$ is a clone $\iff$

- $\mathcal{C}$ is closed under composition, i.e. $f(g_1, \ldots, g_n) \in \mathcal{C}$ for all $f, g_1, \ldots, g_n \in \mathcal{C}$, and

- $\mathcal{C}$ contains the projections, i.e. for all $1 \leq k \leq n$ the operation $\pi^n_k(x_1, \ldots, x_n) = x_k$. 

M. Pinsker (TU Wien)

The reducts of $(\mathbb{N}, =)$

June 14, 2007 / OAL Nashville
pp-definability and local clones

Definition

A set $C \subseteq O$ is a clone $\leftrightarrow$

1. $C$ is closed under composition, i.e. $f(g_1, \ldots, g_n) \in C$ for all $f, g_1, \ldots, g_n \in C$, and
2. $C$ contains the projections, i.e. for all $1 \leq k \leq n$ the operation $\pi^n_k(x_1, \ldots, x_n) = x_k$.

Definition

A clone $C$ is locally closed or local $\leftrightarrow$

$C$ is closed in the product topology on $X^X$ (where $X$ is discrete) $\leftrightarrow$
$C$ contains all operations that can (on finite sets) be approximated by operations from $C$. 

Fact

The local clones are exactly the Inv Pol-closed subsets of $O$. 

M. Pinsker (TU Wien)
June 14, 2007 / OAL Nashville
**pp-definability and local clones**

**Definition**

A set $\mathcal{C} \subseteq \mathcal{O}$ is a **clone** $\iff$

- $\mathcal{C}$ is closed under composition, i.e. $f(g_1, \ldots, g_n) \in \mathcal{C}$ for all $f, g_1, \ldots, g_n \in \mathcal{C}$, and
- $\mathcal{C}$ contains the projections, i.e. for all $1 \leq k \leq n$ the operation $\pi^n_k(x_1, \ldots, x_n) = x_k$.

**Definition**

A clone $\mathcal{C}$ is **locally closed** or **local** $\iff$

- $\mathcal{C}$ is closed in the product topology on $X^X$ (where $X$ is discrete)
- $\mathcal{C}$ contains all operations that can (on finite sets) be approximated by operations from $\mathcal{C}$

**Fact**

The local clones are exactly the Inv Pol-closed subsets of $\mathcal{O}$. 
Problem
Given a structure $\Gamma$, determine its reducts up to primitive positive interdefinability.
Problem
Given a structure \( \Gamma \), determine its reducts up to primitive positive interdefinability.

First step
Try with the simplest structure, \( \Gamma := (\mathbb{N}, =) \).

Observations
Via \( \text{Pol}^{-1} \), those reducts correspond to local clones. \( \text{Aut}(\Gamma) = S_\omega \), so those clones contain all permutations. Conversely, if a clone contains \( S_\omega \), then it induces a reduct of \( \Gamma \).

Conclusion
\( \text{Inv} \) (or \( \text{Pol} \)) is an antiisomorphism between the lattice of local clones above \( S_\omega \) and the reducts of \( (\mathbb{N}, =) \).
Problem
Given a structure $\Gamma$, determine its reducts \emph{up to primitive positive interdefinability}.

First step
Try with the simplest structure, $\Gamma := (\mathbb{N}, \equiv)$.

Observations
Via $\text{Pol} – \text{Inv}$, those reducts correspond to local clones.
Problem
Given a structure $\Gamma$, determine its reducts up to primitive positive interdefinability.

First step
Try with the simplest structure, $\Gamma := (\mathbb{N}, =)$.

Observations
Via Pol $\rightarrow$ Inv, those reducts correspond to local clones. $\text{Aut}(\Gamma) = S_\omega$, so those clones contain all permutations.
## Reducts up to pp-interdefinability

### Problem
Given a structure \( \Gamma \), determine its reducts *up to primitive positive interdefinability*.

### First step
Try with the simplest structure, \( \Gamma := (\mathbb{N}, =) \).

### Observations
Via \( \text{Pol} - \text{Inv} \), those reducts correspond to local clones. \( \text{Aut}(\Gamma) = S_\omega \), so those clones contain all permutations. Conversely, if a clone contains \( S_\omega \), then it induces a reduct of \( \Gamma \).
Reducts up to pp-interdefinability

Problem
Given a structure $\Gamma$, determine its reducts up to primitive positive interdefinability.

First step
Try with the simplest structure, $\Gamma := (\mathbb{N}, =)$.

Observations
Via $\text{Pol} − \text{Inv}$, those reducts correspond to local clones. $\text{Aut}(\Gamma) = S_\omega$, so those clones contain all permutations. Conversely, if a clone contains $S_\omega$, then it induces a reduct of $\Gamma$.

Conclusion
$\text{Inv}$ (or $\text{Pol}$) is an antiisomorphism between the lattice of local clones above $S_\omega$ and the reducts of $(\mathbb{N}, =)$!
The local clones above $S_\omega$

Theorem
The local clones above $S_\omega$

Theorem
Constraint Satisfaction Problem

Fixed: A structure $\Gamma$ ("template").
Input: A finite structure $\Delta$.
Question: Does there exist a homomorphism $\Delta \rightarrow \Gamma$?
### Constraint Satisfaction Problem

**Fixed:** A structure $\Gamma$ ("template").

**Input:** A finite structure $\Delta$.

**Question:** Does there exist a homomorphism $\Delta \to \Gamma$?

### Fact

Complexity of CSP (polynomial time-) invariant under pp-definitions.
Connection to CSP and outlook

**Constraint Satisfaction Problem**

Fixed: A structure $\Gamma$ (“template”).
Input: A finite structure $\Delta$.
Question: Does there exist a homomorphism $\Delta \rightarrow \Gamma$?

**Fact**

Complexity of CSP (polynomial time-) invariant under pp-definitions.

**Consequence**

For $\omega$-categorical $\Gamma$, the Galois connection Inv-Pol can be used.
## Constraint Satisfaction Problem

Fixed: A structure $\Gamma$ ("template").  
Input: A finite structure $\Delta$.  
Question: Does there exist a homomorphism $\Delta \rightarrow \Gamma$?

## Fact

Complexity of CSP (polynomial time-) invariant under pp-definitions.

## Consequence

For $\omega$-categorical $\Gamma$, the Galois connection Inv-Pol can be used.

## Future work

Determine (up to pp interdefinability) the reducts of other $\omega$-categorical structures.  
Example: Random graph.