Algebraic lattices are complete sublattices of the clone lattice over an infinite set

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Outline

1. The clone lattice
2. How complicated is the clone lattice?
3. The clone lattice on finite $X$ is quite complicated
4. Monoidal intervals
5. The clone lattice on infinite $X$ is very complicated
6. Remarks and outlook
The clone lattice

\( X \ldots \) base set.

\( \mathcal{O}^{(n)} = \mathcal{X}^{\mathcal{X}^n} = \{ f : \mathcal{X}^n \to \mathcal{X} \} \ldots n\text{-ary functions on } \mathcal{X}.

\( \mathcal{O} = \bigcup_{n \geq 1} \mathcal{O}^{(n)} \ldots \) finitary operations on \( \mathcal{X} \).
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$\mathcal{C} \subseteq \mathcal{O}$ clone iff

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Post’s theorem

$|X| = 2 \to Cl(X)$ completely known ($|Cl(X)| = \aleph_0$).
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The clone lattice is large:

$$|Cl(X)| = 2^\aleph_0$$ if $3 \leq |X| < \aleph_0$

$$|Cl(X)| = 2^{2^{|X|}}$$ if $|X| \geq \aleph_0$

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|\(|X| \geq 3 \rightarrow\) the subsemigroup lattice of the additive semigroup of the natural numbers embeds into \(Cl(X)\).|

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|\(|X| \geq 3 \rightarrow\) \(Cl(X)\) does not satisfy any non-trivial identity.|
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\[ |X| \geq 4 \rightarrow Cl(X) \text{ does not satisfy any non-trivial quasi-identity.} \]
Monoidal intervals

For any monoid \( G \subseteq \mathcal{O}^{(1)} \),

\[ J_G = \{ C \in \text{Cl}(X) : C \cap \mathcal{O}^{(1)} = G \} \]

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**Theorem (P. 2005)**

\( X \) infinite, \( L \) completely distributive algebraic with at most \( 2^{|X|} \) compact elements \( \rightarrow \)

\( 1 + L \) is a monoidal interval of \( Cl(X) \).
Monoidal intervals

\[ \emptyset \]

\[ \emptyset^{(1)} \]

\[ \mathcal{M} \]

\[ \emptyset \]

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\[ \mathcal{O}, \mathcal{J} \]
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- $X$ finite $\rightarrow$ $Cl(X)$ has $\aleph_0 = |\emptyset|$ compact clones.
- $X$ infinite $\rightarrow$ $Cl(X)$ has $2^{\aleph_0} = |\emptyset|$ compact clones.

Theorem (P. 2006)

$X$ infinite $\rightarrow$ Every algebraic lattice with at most $2^{\aleph_0}$ compact elements is a complete sublattice of $Cl(X)$. 
Remarks and Outlook

Theorem (Bulatov)

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Problem

$X$ infinite. Is every algebraic lattice with at most $2^{|X|}$ compact elements an interval of $Cl(X)$? Even a monoidal interval?