Lattices of order ideals as monoidal intervals

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- $\mathscr{C}$ contains the projections and
- $\mathscr{C}$ closed under composition.

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But... $Cl(X)$ is too complicated.

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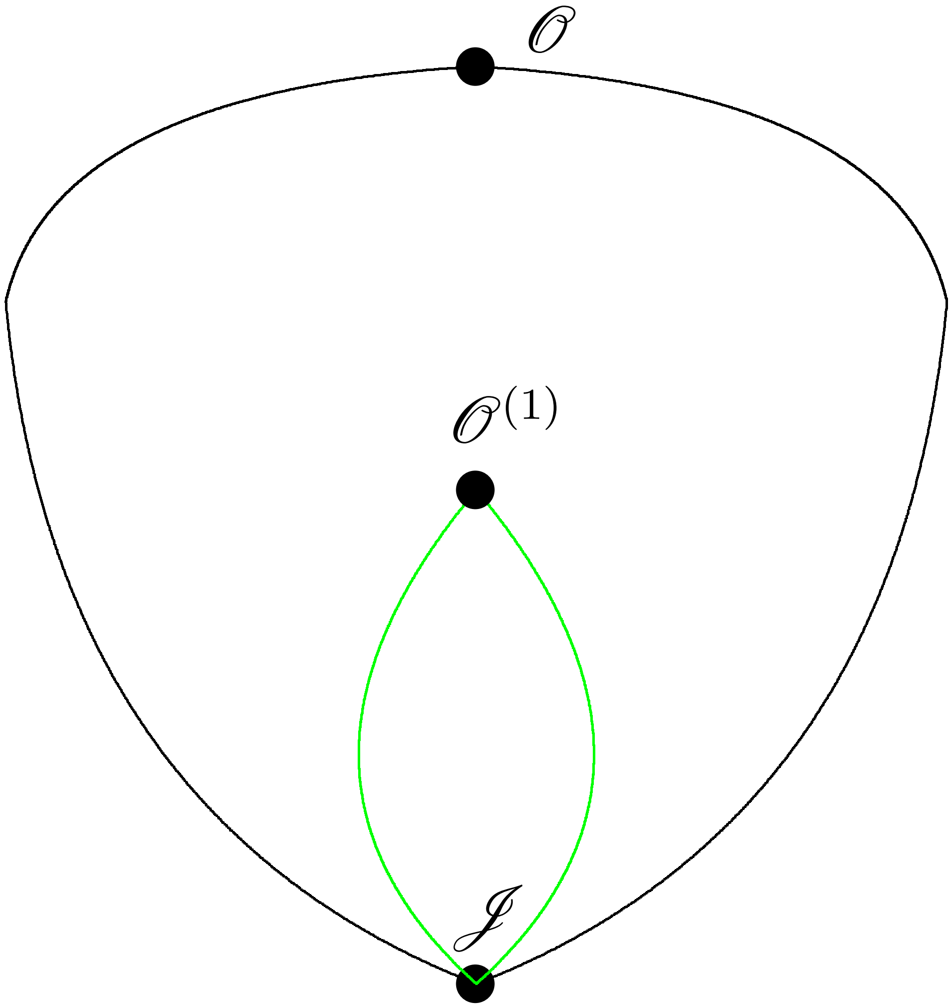
The monoidal intervals are a partition of the clone lattice.

A partition



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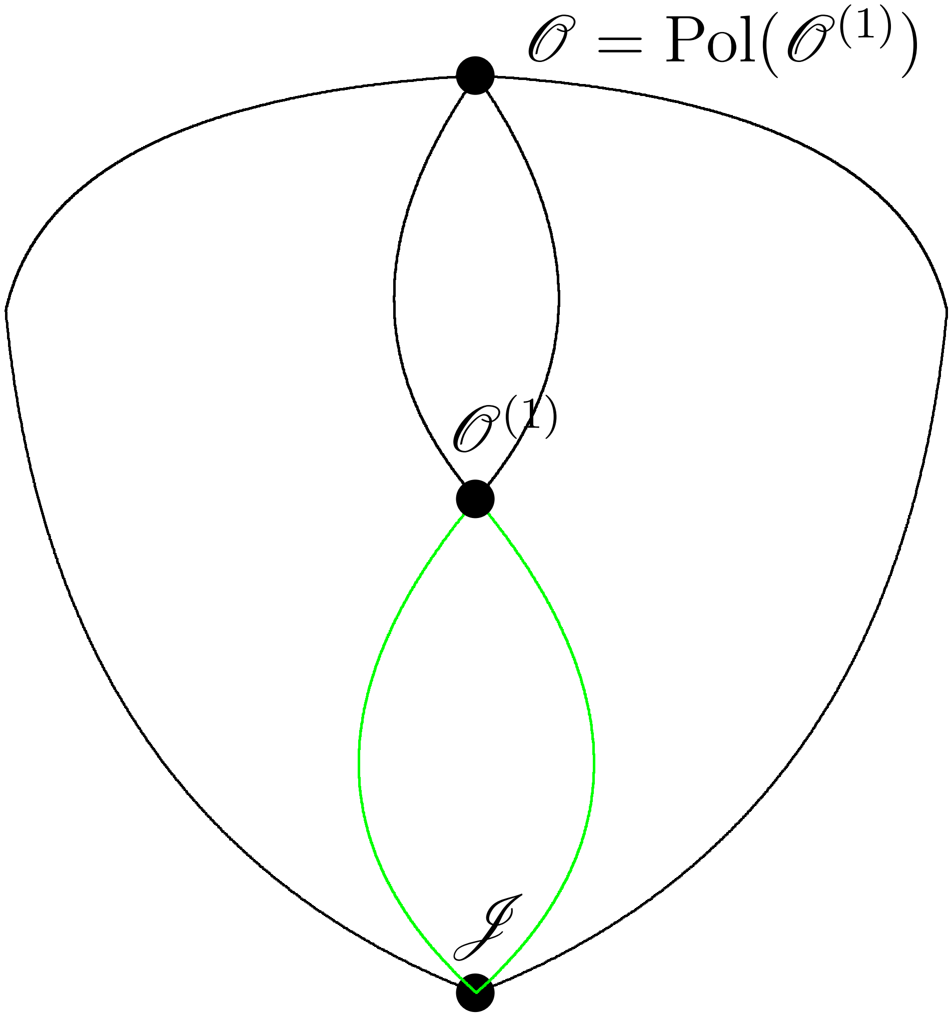

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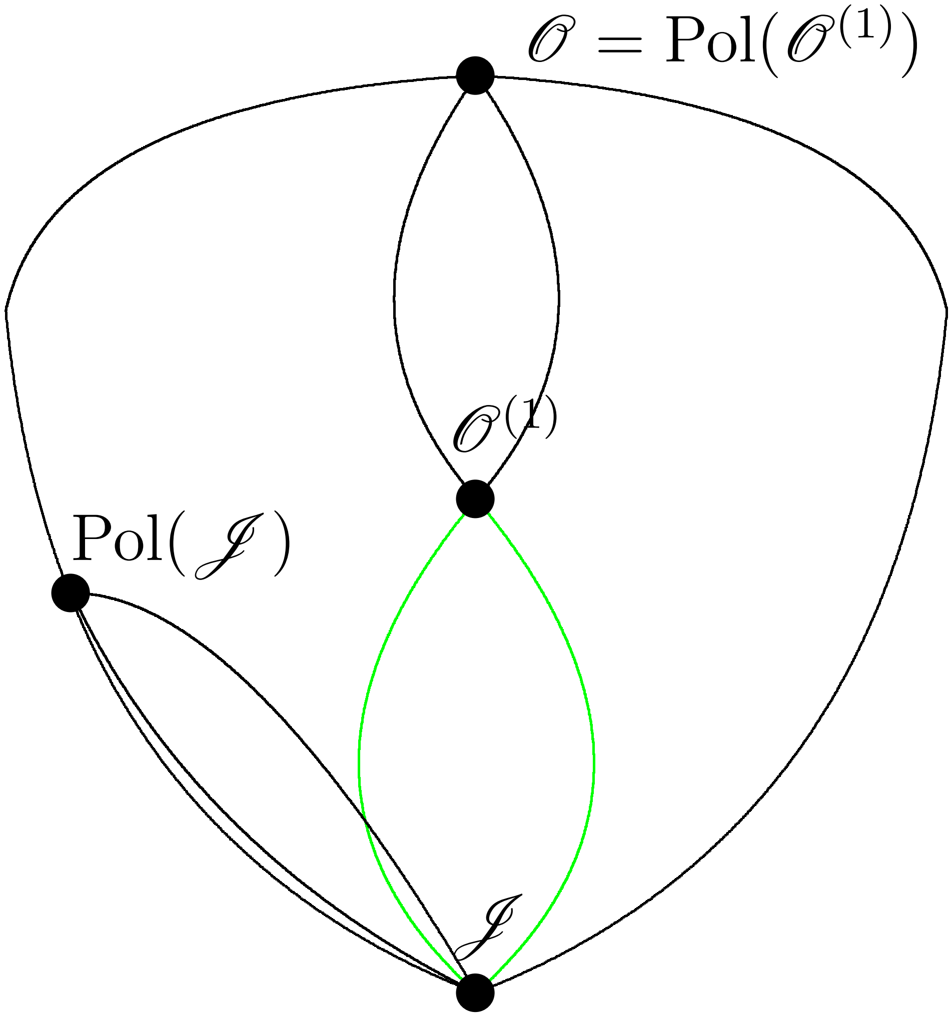

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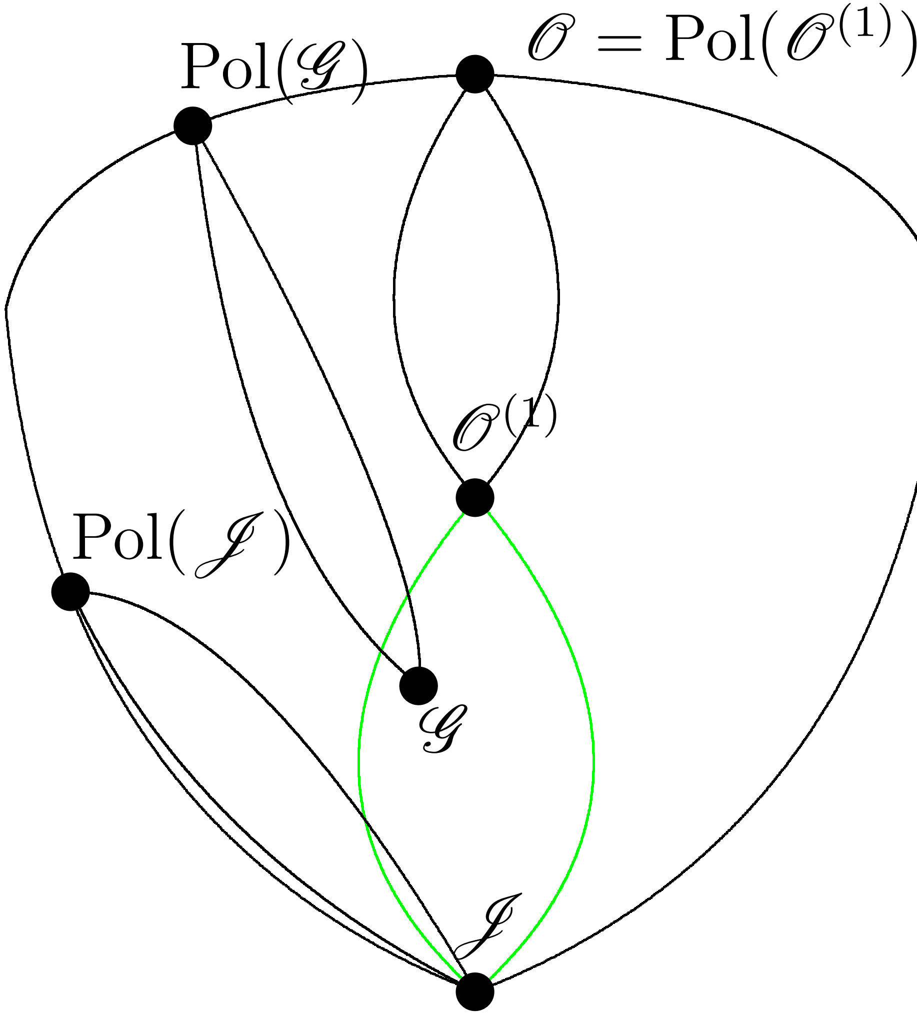
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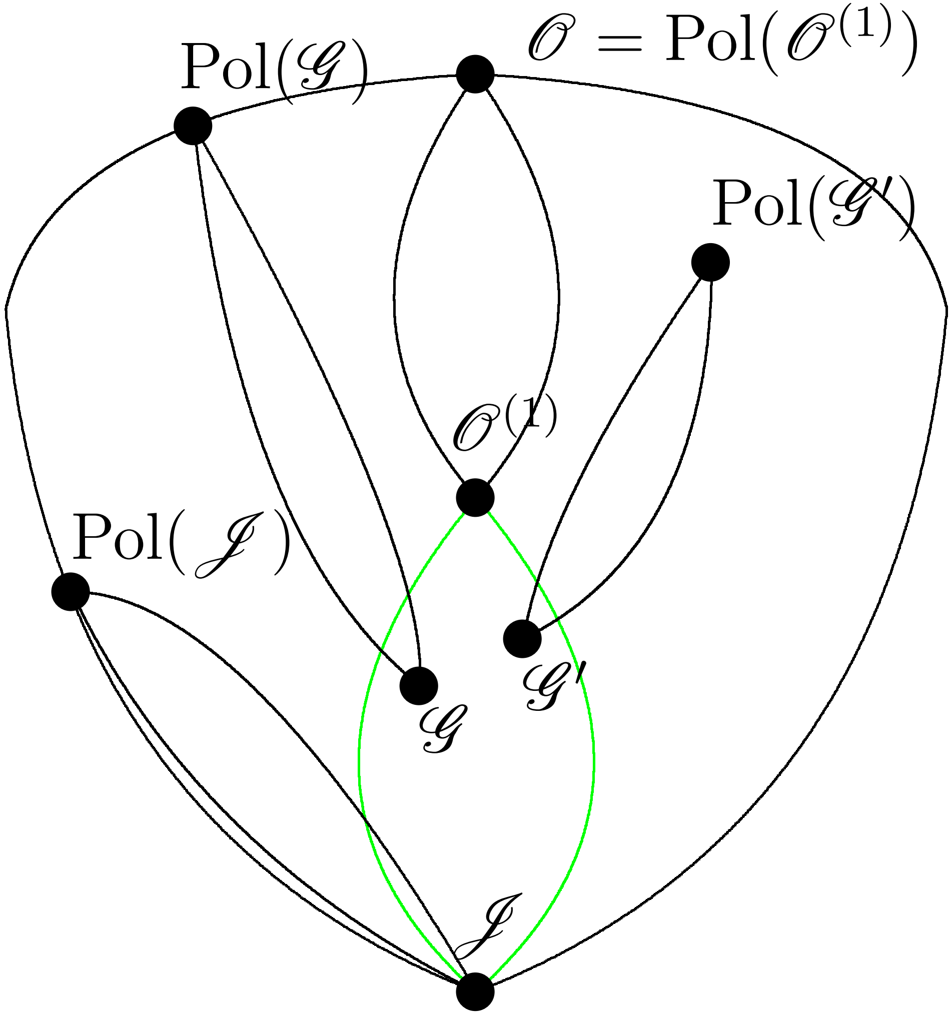
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Defn. For $\mathfrak{P}$ a partial order, the set of order ideals on $\mathfrak{P}$ form a lattice (meet= intersection, join=union). Denote it by $\mathfrak{L}(\mathfrak{P})$.

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Remark. $\mathscr{G}$ is a monoid of linear functions on a vector space of dimension $|X|$ on X.

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Remark. A chain $\mathfrak{L}$ is an algebraic lattice iff for all $p, q \in \mathfrak{L}$ with $p < q$ there is a successor $r \in \mathfrak{L}$ with $q \leq r \leq p$.

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Open. Given λ such that $2^{|X|} < \lambda < 2^{2^{|X|}}$ and λ is not of the form 2^{ξ}, does there exist a monoidal interval of size λ?