NON-PROPERTIES OF CANONICAL EXTENSIONS OF LOCAL CLONES

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Abstract. To every locally closed clone, one can assign a larger clone in a canonical way. We examine properties of such extensions, and show that unfortunately, the most desirable properties do not hold.

1. Local clones and their canonical extensions

Let X be an infinite set, denote for every $n \geq 1$ the set of n-ary operations on X, and set $\mathscr{O} := \bigcup_{n\geq 1} \mathscr{O}^{(n)}$ to be the set of all finitary operations on X. A *clone* is a subset of $\mathscr{O}$ which is closed under composition and which contains all projections. This notion is a generalization of a *transformation monoid*, i.e., a subset of $\mathscr{O}^{(1)}$ which is closed under composition and which contains the identity operation.

If X is equipped with the discrete topology, then we naturally obtain a topology on $\mathscr{O}$ if we view it as the sum space of the product spaces $\mathscr{O}^{(n)} = X^{X^n}$. A clone is *locally closed*, or *local*, or simply *closed* iff it is a closed set in this topology. (The first two notions are used among universal algebraists; the last one is confusing since there are two closure operators acting on $\mathscr{O}$, the topological and the algebraic one, but it is used for topologically closed permutation groups.)

Since intersections of arbitrary sets of clones are again clones, the set of all clones on X forms a complete lattice $\mathrm{Cl}(X)$ with respect to inclusion. For the same reason, the set of all local clones forms a lattice $\mathrm{Cl}_{\mathrm{loc}}$, which is a subset (but not a sublattice) of $\mathrm{Cl}(X)$. Most of what is known on the lattice $\mathrm{Cl}(X)$ has been summarized in [GP08], and the structure of $\mathrm{Cl}_{\mathrm{loc}}(X)$ has been further investigated in [Pin]. The lattices $\mathrm{Cl}(X)$ and $\mathrm{Cl}_{\mathrm{loc}}(X)$ do not seem to have too much in common; they already differ considerably in size ($|\mathrm{Cl}(X) = 2^{2^{|X|}}$ whereas $|\mathrm{Cl}_{\mathrm{loc}}(X)| = 2^{|X|}$).

A clone is called *maximal* iff it is a dual atom of $\mathrm{Cl}(X)$. A local clone is called *locally maximal* iff it is a dual atom of $\mathrm{Cl}_{\mathrm{loc}}(X)$. Observe that local clones can, at least in theory, be maximal, and that they can be locally

Date: January 31, 2010.

2000 *Mathematics Subject Classification.* Primary 08A40; secondary 08A05.

Key words and phrases. clone lattice, local clones, maximal clones, locally maximal clones.

The second author is grateful for support through Erwin-Schrödinger-Fellowship J2742-N18 of the Austrian Science Foundation.

maximal but not maximal. Clearly, maximality implies local maximality. In this paper, we will be concerned with (global) maximality of locally maximal local clones.

We note that $\mathrm{Cl}_{\mathrm{loc}}(X)$ is not dually atomic ([RS82]; see also [GP08], [GSb] or [BCP]), i.e., there exists a local clone which is not contained in a locally maximal one. It is still unknown whether $\mathrm{Cl}(X)$ is dually atomic, although it is at least not provable in ZFC that it is [GSa]. A set of local clones which is cofinal in $\mathrm{Cl}_{\mathrm{loc}}(X)$ has been found in [RS82], [RS84] and [RS00].

In order to understand local clones, one uses a characterization of such clones which depends on the following definition.

Definition 1. Let $R \subseteq X^m$ be a relation, where $m \geq 1$ is finite.

- Let $f \in \mathscr{O}^{(n)}$. We say that f *preserves* R iff whenever $x_1, \ldots, x_m \in X^n$ are so that they belong to R componentwise, then $(f(x_1), \ldots, f(x_m))$ also belongs to R.
- We write $\mathrm{Pol}(R)$ for the set of all $f \in \mathscr{O}$ which preserve the relation R; the elements of $\mathrm{Pol}(R)$ are called *polymorphisms* of R.
- Given a set of relations $\mathscr{R}$, we also write $\mathrm{Pol}(\mathscr{R})$ for the set of all $f \in \mathscr{O}$ which preserve all $R \in \mathscr{R}$ (so $\mathrm{Pol}(\mathscr{R}) = \bigcap_{R \in \mathscr{R}} \mathrm{Pol}(R)$).

It is easy to check that $\mathrm{Pol}(\mathscr{R})$ is always a local clone, and in fact

Theorem 2 ([Rom77])**.** *The local clones are exactly the sets of operations of the form* $\mathrm{Pol}(\mathscr{R})$.

Clearly every locally maximal clone is of the form $\mathrm{Pol}(R)$ for a single relation R. Inspired by the question

When is a locally maximal clone also maximal in $\mathrm{Cl}(X)$?

one can weaken the condition of preservation of R quite naturally as follows, obtaining a larger clone:

Definition 3. Let $R \subseteq X^m$ be a relation.

- Let $f \in \mathscr{O}^{(n)}$. We say that f *morally preserves* R iff there is an equivalence relation $\theta = \theta_f$ on X^n with finitely many classes such that for all $x_1, \ldots, x_m \in X^n$ which are equivalent with respect to θ_f it is true that if $(x_1, \ldots, x_m) \in R$ componentwise, then $(f(x_1), \ldots, f(x_m)) \in R$.
- We write $\mathrm{Pol}^*(R)$ for the set of all $f \in \mathscr{O}$ which morally preserve R.
- $\mathrm{Pol}^*(\mathscr{R})$, where $\mathscr{R}$ is a set of relations, is defined as $\mathrm{Pol}^*(\mathscr{R}) = \bigcap_{R \in \mathscr{R}} \mathrm{Pol}^*(R)$.

It is easy to see the following:

Fact 4. $\mathrm{Pol}^*(R)$ *(and hence,* $\mathrm{Pol}^*(\mathscr{R})$*) is a clone.*

In the notion of moral preservation, we do not demand a function to really preserve a relation, but to preserve it on each of the finitely many classes of θ_f. In particular, if a function f preserves a relation R, then it also morally preserves it, since we can set $\theta_f := (X^n)^2$. Therefore we have

Fact 5. $\mathrm{Pol}(R) \subseteq \mathrm{Pol}^*(R)$.

The following less obvious fact makes the concept interesting. It says that unless R is *essentially unary*, i.e., unless $\mathrm{Pol}(R) = \mathrm{Pol}(S)$ for a unary relation $S \subseteq X$, the clone $\mathrm{Pol}^*(R)$ properly contains $\mathrm{Pol}(R)$. We say that R is *pp-definable* from a structure Γ with domain X iff it can be defined using a primitive positive formula over the relations of Γ.

Proposition 6. *The following are equivalent:*

(1) $\mathrm{Pol}(R) = \mathrm{Pol}^*(R)$.
(2) *R is pp-definable from some relational structure (X, S), where $S \subseteq X$.*
(3) *R is essentially unary, i.e.,* $\mathrm{Pol}(R) = \mathrm{Pol}(S)$ *for some unary relation $S \subseteq X$.*

We remark that $\mathrm{Pol}(S)$ is a maximal clone for all proper non-empty subsets S of X (see [RS82]).

A first conjecture of the authors of this paper was that unless R is essentially unary, $\mathrm{Pol}^*(R)$ is a cover of $\mathrm{Pol}(R)$ in the lattice $\mathrm{Cl}(X)$, at least if $\mathrm{Pol}(R)$ is locally maximal. This belief was nourished by the observation that this was true for equivalence relations:

Proposition 7. *Let R be an equivalence relation. Then* $\mathrm{Pol}(R)$ *is maximal in* $\mathrm{Cl}(X)$ *iff* $\mathrm{Pol}^*(R) = \mathscr{O}$.
In fact, $\mathrm{Pol}^*(R)$ *is the unique cover of* $\mathrm{Pol}(R)$*, i.e.* $\mathrm{Pol}(R) \subsetneq \mathscr{C}$ *iff* $\mathrm{Pol}^*(R) \subseteq \mathscr{C}$ *for all (global) clones $\mathscr{C}$.*

We remark that $\mathrm{Pol}(R)$ is locally maximal for all non-trivial equivalence relations R (see [RS84]).

It turns out, however, that the "cover conjecture" does not hold.

Proposition 8. *There is a binary relation R such that*

- $\mathrm{Pol}(R)$ *is locally maximal, and*
- $\mathrm{Pol}^*(R)$ *is not a cover of* $\mathrm{Pol}(R)$ *in* $\mathrm{Cl}(X)$.

Although $\mathrm{Pol}^*(R)$ is not a cover of $\mathrm{Pol}(R)$ in general, it could still be true that the additional assumption $\mathrm{Pol}^*(R) = \mathscr{O}$ implies that there is nothing in between these two clones. So we were hoping for a while that the following "maximality conjecture" holds:

> If $\mathrm{Pol}(R)$ locally maximal but not (globally) maximal, then
> this is witnessed by $\mathrm{Pol}^*(R)$, i.e. $\mathrm{Pol}^*(R) \neq \mathscr{O}$.

If this were true, it would very much facilitate proving global maximality for locally maximal clones: Without that statement, one would have to show that for any function $f \notin \mathrm{Pol}(R)$, the only clone containing f and $\mathrm{Pol}(R)$ is $\mathscr{O}$, whereas with the statement one would only have to prove that $\mathrm{Pol}^*(R) = \mathscr{O}$, so we are given all functions $f \notin \mathrm{Pol}(R)$ for which we can find an equivalence relation showing $f \in \mathrm{Pol}^*(R)$ for free. Indeed, with this maximality criterion at hand, one would be spared the often technical procedure of composing functions and need only find a partition for every operation f which witnesses $f \in \mathrm{Pol}^*(R)$.

The statement turned out to be true for a number of nice relations.

Proposition 9. *Let R be a non-trivial equivalence relation, a locally bounded partial order, or the graph of a fixed point free permutation on X all of whose cycles have the same prime length. Then:*

- $\mathrm{Pol}(R)$ *is locally maximal, and*
- $\mathrm{Pol}(R)$ *is maximal iff* $\mathrm{Pol}^*(R) = \mathscr{O}$.

In the end, however, the maximality conjecture turned out to be false.

Proposition 10. *There exists a binary relation R such that*

- $\mathrm{Pol}^*(R) = \mathscr{O}$, *and*
- $\mathrm{Pol}(R)$ *is locally maximal, and*
- $\mathrm{Pol}(R)$ *is not maximal.*

The following sections contain the proofs of the results of this section.

1.1. **Notation.** When R is a binary relation, we will also write $a\ R\ b$ for $(a, b) \in R$. If the arity of $f \in \mathscr{O}$ has not yet been given a name, we may denote it by n_f. If $a \in X^n$ is a tuple, we write a_i for its i-th component, for all $1 \leq i \leq n$. For an m-ary relation $R \subseteq X^m$ and $n \geq 1$, we write R^n for the m-ary relation on X^n induced by R componentwise: If $x_1, \ldots, x_m \in X^n$, then $(x_1, \ldots, x_m) \in R^n$ iff they belong to R componentwise.

2. Basic properties of the canonical extension

This section contains the proof of Proposition 6. In a first lemma, we establish the equivalence between items (2) and (3) of that proposition.

Lemma 11. $\mathrm{Pol}(R) = \mathrm{Pol}(S)$ *for some subset $S \subseteq X$ iff R has a pp definition from S and vice-versa.*

Proof. It is clear that the structure (X, S) is ω-categorical, i.e., its first-order theory has precisely one countably infinite model up to isomorphisms. It then follows from [BN06] that a relation is pp-definable from S iff it is preserved by all functions in $\mathrm{Pol}(S)$.

Therefore, if R has a pp definition from S, then it is invariant under $\mathrm{Pol}(S)$, implying $\mathrm{Pol}(R) \supseteq \mathrm{Pol}(S)$. Also, (X, R) is ω-categorical if R is definable from S, so by the same argument, with the roles of S and R reversed, we get $\mathrm{Pol}(R) \subseteq \mathrm{Pol}(S)$.

For the other direction, observe that $\mathrm{Pol}(R) = \mathrm{Pol}(S)$ implies that R is preserved by $\mathrm{Pol}(S)$ and vice-versa. Since S is ω-categorical, we get that R is pp-definable from S. But then R is ω-categorical as well, implying that S is pp-definable from R by the same argument. $\square$

We now show that (2) implies (1) in Proposition 6.

Lemma 12. *If R has a pp definition from (X, S) for some set $S \subseteq X$, then* $\mathrm{Pol}(R) = \mathrm{Pol}^*(R)$.

Proof. We may assume that $\mathrm{Pol}(R) \neq \mathscr{O}$ since otherwise the statement is trivial; in particular, this implies that S is non-empty and does not equal X. Since R has a pp definition from S, a quick check shows that for all $x \in X$, R contains the tuple $(x, \ldots, x)$ if and only if $x \in S$. Now let $f \in \mathrm{Pol}^*(R)$ be n-ary, and let θ be an equivalence relation on X^n such that f preserves R on every θ-class. If f did not preserve R, then, since $\mathrm{Pol}(R) = \mathrm{Pol}(S)$, it would not preserve S. Hence there would exist $s_1, \ldots, s_n \in S$ such that $f(s_1, \ldots, s_n) =: s \notin S$. Denoting the arity of R by m, we define tuples $x_i := (s_1, \ldots, s_n)$, for all $1 \leq i \leq m$. Being identical, any two x_i are related with respect to θ. Also, $(x_1, \ldots, x_m) \in R$ componentwise, and so $(f(x_1), \ldots, f(x_m)) = (s, \ldots, s) \notin R$, a contradiction. Hence $\mathrm{Pol}(R) = \mathrm{Pol}^*(R)$.

□

Finally, we prove that (1) implies (2) in Proposition 6.

Lemma 13. *If R is not pp-definable from any structure (X, S), where $S \subseteq X$, then* $\mathrm{Pol}(R) \neq \mathrm{Pol}^*(R)$.

Proof. Let m be the arity of R. Set $T = \{b \in X : (b, \ldots, b) \in R\}$, and $S = X \setminus T$.

Assume there exist $b_1, \ldots, b_m \in S$ such that $(b_1, \ldots, b_m) \in R$. Let $f \in \mathscr{O}^{(1)}$ map all b_i to b_1 and be the identity otherwise. Then $f \notin \mathrm{Pol}(R)$ as $(f(b_1), \ldots, f(b_m)) = (b_1, \ldots, b_1) \notin R$. However, $f \in \mathrm{Pol}^*(R)$ as is ensured by the equivalence relation $\{\{b_1\}, \ldots, \{b_m\}, X \setminus \{b_1, \ldots, b_m\}\}$. This proves the lemma for this case, and we may henceforth assume that there are no such $b_1, \ldots, b_m$; in particular, we may assume that T is non-empty.

Let $i_1, \ldots, i_r$ be those numbers i_j with $1 \leq i_j \leq m$ which have the property that $t \in R$ implies $t_{i_j} \in T$. Now there exists a tuple $t \notin R$ such that $t_{i_j} \in T$ for all $1 \leq j \leq r$; otherwise, we would have $t \in R$ iff $t_{i_j} \in T$ for all $1 \leq j \leq r$, and R would be pp-definable from T, in contradiction with our assumption.

Fix such a tuple t. Write $\{j_1, \ldots, j_p\} := \{1, \ldots, m\} \setminus \{i_1, \ldots, i_r\}$. For every j_q in that set, fix a tuple $s^{j_q} \in R$ with $s^{j_q}_{j_q} \notin T$. Fix moreover any tuple $v \in R$. Set $n := p + 1$. Now for all $1 \leq w \leq m$, write $s_j := (s^{j_1}_w, \ldots, s^{j_p}_w, v_w) \in X^n$. Set $f(s_j) = t_j$, for all $1 \leq j \leq m$. Fix $c \in T$, and let f send all other tuples of X^n to c. Clearly, $f \notin \mathrm{Pol}(R)$ since the $s_1, \ldots, s_m$ are in R componentwise and since $(f(s_1), \ldots, f(s_m)) = t \notin R$. But $f \in \mathrm{Pol}^*(R)$, as is witnessed by the equivalence relation $\theta = \{\{s_1\}, \ldots, \{s_m\}, X^n \setminus \{s_1, \ldots, s_m\}\}$: For any $1 \leq i \leq m$, if $x_1, \ldots, x_m$ all equal s_i, then the x_i are not in R componentwise, so $f \in \mathrm{Pol}^*(R)$ cannot be violated there. If on the other hand $x_1, \ldots, x_m \in X^n \setminus \{s_1, \ldots, s_m\}$, then $(f(x_1), \ldots, f(x_n)) = (c, \ldots, c) \in R$. □

3. The cover conjecture

We will prove Proposition 7 which states that if R is an equivalence relation, then $\mathrm{Pol}^*(R)$ is the unique cover of $\mathrm{Pol}(R)$ in $\mathrm{Cl}(X)$, and which inspired the "cover conjecture", i.e., the statement that this is true in general. After that, we prove Proposition 8, disproving the conjecture.

Proof of Proposition 7. Let $f \notin \mathrm{Pol}(R)$ be n-ary. We can assume that f is unary: There exist $a, b \in X^n$ equivalent with respect to R componentwise and such that $(f(a), f(b)) \notin R$. Pick $0, 1 \in X$ such that $(0,1) \in R$ and define $f_i \in \mathscr{O}^{(1)}$ by $f_i(0) = a_i$ and $f_i(x) = b_i$, for all $1 \leq i \leq n$. Clearly, all f_i preserve R. Now set $f' := f(f_1, \ldots, f_n)$. Then $f'(0) = f(a)$ is not related to $f'(1) = f(b)$, and we can replace f by f'.

Let $g \in \mathscr{O}^{(k)}$ be any function in $\mathrm{Pol}^*(R)$. We will write g as a combination of functions in $\mathrm{Pol}(R)$ with the function f, thus proving that the smallest clone containing f and $\mathrm{Pol}(R)$ must contain $\mathrm{Pol}^*(R)$.

Assume that $g \in \mathrm{Pol}^*(R)$ is witnessed by an equivalence relation θ on X^k. We can find an $n \geq 1$ and a function $c : X^k \to \{0,1\}^n$ inducing θ. As the components of c take only the values 0 and 1, and $0 \; R \; 1$, we have that the component functions c_i, which send a tuple x from X^k to the i-th component of $c(x)$, are in $\mathrm{Pol}(R)$, for all $1 \leq i \leq n$.

Now define $G(x,y) : X^k \times X^n \to X$ distinguishing three cases (so $x \in X^k$, and $y \in X^n$):

- (a) There is $x' \in X^k$, equivalent to x componentwise, such that $(y_i, f(c_i(x'))) \in R$ for all i.
  - (a1) We can choose $x' = x$. Then $G(x,y) := g(x)$.
  - (a2) We cannot choose $x' = x$. Then we choose any x', and let $G(x,y) := g(x')$.
- (b) Case (a) does not hold. Let $G(x,y) := x_1$.

We claim that G preserves R. Note that for all i and all $x \in X^k$ we have $f(c_i(x)) \in \{f(0), f(1)\}$. Therefore, given $a, b \in X^k$, the statement $\forall i \, (f(c_i(a)), f(c_i(b))) \in R$ implies $\forall i \, c_i(a) = c_i(b)$, which in turn implies $(a,b) \in \theta$.

To prove that G preserves R, take $x_1, x_2 \in X^k$ and $y_1, y_2 \in X^n$ such that x_1, x_2 and y_1, y_2 are equivalent with respect to R componentwise; we must show that $G(x_1, y_1)$ and $G(x_2, y_2)$ are equivalent with respect to R. Assume that $G(x_1, y_1) = g(x_1)$ as in Case (a1), and $G(x_2, y_2) = g(x_2)$ as in Case (a2). So for all i we have $f(c_i(x_1)) \; R \; f(c_i(x_2))$, hence $x_1 \; \theta \; x_2$. Together with $x_1 \; R^k \; x_2 \; R^k \; x_2'$ we get $G(x_1, y_1) = g(x_1) \; R \; g(x_2) = G(x_2, y_2)$. The other cases are left to the reader. □

We turn to the proof of Proposition 8.

Definition 14. Let Q be a binary relation on a set Y, and θ be an equivalence relation on the same set. We say that θ is *canonical* for Q iff

For all θ-classes $A \neq B$ we have
either $\forall a \in A \, \forall b \in B \, (a,b) \in Q$, or $\forall a \in A \, \forall b \in B \, (a,b) \notin Q$.

Definition 15. For a binary relation R on X, $\mathrm{Pol}^c(R)$ is the set of all functions $f \in \mathscr{O}$ such that there is an R^{n_f}-*canonical* equivalence relation $\theta = \theta_f$ on X^{n_f} with finitely many classes such that

$\forall x, y \in X^{n_f}$: If $x \;\theta\; y$ and $x \; R^{n_f} \; y$, then $f(x) \; R \; f(y)$.

Lemma 16. $\mathrm{Pol}(R) \subseteq \mathrm{Pol}^c(R) \subseteq \mathrm{Pol}^*(R)$.

Proof. Trivial. □

The following is a counterexample to the "cover conjecture".

Proof of Proposition 8. If R is a linear order, then each Dedekind cut determines a canonical equivalence relation on X. We know from [RS84] that $\mathrm{Pol}(R)$ is locally maximal. We claim that $\mathrm{Pol}(\leq) \subsetneq \mathrm{Pol}^c(\leq) \subsetneq \mathrm{Pol}^*(\leq)$:

To see that $\mathrm{Pol}(R) \subsetneq \mathrm{Pol}^c(R)$, let (D_1, D_2) be any Dedekind cut of X, and pick $d_i \in D_i$, for $i = 1, 2$. The mapping $h \in \mathscr{O}^{(1)}$ which sends all elements of D_1 to d_2 and all elements of D_2 to d_1 is in $\mathrm{Pol}^c(R)$ but does not preserve R.

We prove that $\mathrm{Pol}^c(R) \subsetneq \mathrm{Pol}^*(R)$: By Ramsey's theorem, X contains a copy of ω or of its inverse order ω^*; say wlog that $C \subseteq X$ has order type ω. Divide C into two alternating sets C_1, C_2. Now the mapping $g \in \mathscr{O}^{(1)}$ sending all elements of C_1 to the smallest element of C_1, and all other elements of X to the smallest element of C_2, is clearly an element of $\mathrm{Pol}^*(R)$ but not of $\mathrm{Pol}^c(R)$. □

Another counterexample is the following:

Another proof of Proposition 8. Let R be the partial order of finite subsets of $\mathbb{N}$ (including the empty set). Then $\mathrm{Pol}(R)$ is locally maximal by results of [RS84] (since R is a locally bounded partial order). Because of the equivalence separating the empty set from the rest, we get $\mathrm{Pol}(R) \subsetneq \mathrm{Pol}^c(R)$, and any other equivalence relation proves $\mathrm{Pol}^c(R) \subsetneq \mathrm{Pol}^*(R)$. □

We remark that one cannot simply replace Pol^* by Pol^c in order to "repair" the cover conjecture, as $\mathrm{Pol}^c(R) = \mathrm{Pol}(R)$ for many relations R. We include here the following

Example. Let X consist of the elements of the lattice freely generated by a countably infinite set $\{x_0, x_1, \ldots\}$, and let R be the order of the lattice. By [RS84], $\mathrm{Pol}(R)$ is locally maximal. We claim that for all $n \geq 1$, there are no non-trivial R^n-canonical equivalence relations with finitely many classes, hence $\mathrm{Pol}(R) = \mathrm{Pol}^c(R)$:

Consider the set D of those tuples of X^n all of whose components are equal to some x_i. If there were an R^n-canonical equivalence relation θ, then one of its classes would contain infinitely many elements of D; call this class A. Consider any $a \in A \cap D$, and any $x \in X^n$. If $(a, x) \in R$, $x \notin A$ would imply $(a', x) \in R$ for all $a' \in A \cap D$, which is impossible; hence A contains all tuples x with $(a, x) \in R$ for some $a \in A \cap D$. Thus, if θ is to be non-trivial, A cannot contain D. Pick a class B and $b \in B \cap D$, and pick any $a \in A \cap D$. The vector c which is obtained by taking the join of a and b componentwise satisfies $(a, c) \in R$, hence $c \in A$. From $(b, c) \in R$ and $b \notin A$ we infer $(b, a) \in R$, a contradiction.

In fact, one will obtain other examples with any partial order which has a maximal antichain C with the property that no infinite subset of C is bounded from above or below.

4. The maximality conjecture

This section is devoted to the "maximality conjecture", i.e., the statement that $\mathrm{Pol}^*(R) = \mathscr{O}$ iff $\mathrm{Pol}(R)$ is maximal in the clone lattice, for every relation R which is not essentially unary and for which $\mathrm{Pol}(R)$ is locally maximal. Before actually disproving the conjecture, we prove Proposition 9, which is a collection of examples in favor of the conjecture.

4.1. **Examples in favor of the maximality conjecture.**

Example. According to [RS84], $\mathrm{Pol}(R)$ is locally maximal for all non-trivial equivalence relations R. And we have seen in Proposition 7 that the conjecture holds for all such R, since $\mathrm{Pol}^*(R)$ is a cover of $\mathrm{Pol}(R)$.

Example. According to [RS84], $\mathrm{Pol}(R)$ is locally maximal for all locally bounded partial orders R, where a partial order is called *locally bounded* iff for all finite $S \subseteq X$ there exist $a, b \in X$ such that $aRsRb$ for all $s \in S$. We claim that the maximality conjecture holds here as well, i.e., that $\mathrm{Pol}^*(R) = \mathscr{O}$ implies that $\mathrm{Pol}(R)$ is maximal.

Proof. If R has an infinite ascending chain C, then one readily checks that $\mathrm{Pol}^*(R) \neq \mathscr{O}$: Let $(C_i : i \in \omega)$ be a partition of this chain into infinite sets, and let $g \in \mathscr{O}^{(1)}$ map each C_i to its smallest element. Then $g \notin \mathrm{Pol}^*(R)$. Same if R has an infinite descending chain. So, since R is also locally bounded, we can assume that R is bounded, say with smallest element 0 and largest element 1. By an argument similar to the one with infinite chains, the set of cardinalities of finite chains in R is bounded. Thus every $a \in X$ has a finite level, which we define to be the length of the longest chain with smallest element 0 and largest element a. Clearly 1 has the largest level, which we denote by n. Now for the partial orders in this class, we indeed have $\mathrm{Pol}^*(R) = \mathscr{O}$: For example, if g is unary, then the partition θ of X which identifies elements of equal level witnesses this. Thus we must prove that $\mathrm{Pol}(R)$ is maximal in $\mathrm{Cl}(X)$ for such R.

Let $g \in \mathscr{O}^{(1)}$ be arbitrary. Fix a chain $(c_0, \ldots, c_n)$ starting from 0 and ending at 1. The operation h which maps every x at level i to c_i preserves R. Fix an antichain $(d_0, \ldots, d_n)$. There is an operation $t \in \mathscr{C}$ which maps every c_i to d_i. Now set $G(x, y) = g(x)$, if $y = d_i$ for some i and $y = t \circ h(x)$,

$G(x, y) = 1$, if $y > d_i$ for some i, $G(x, y) = 0$, if $y < d_i$ for some i, and $G(x, y) = x$ otherwise. Clearly, $g(x) = G(x, t \circ h(x))$. We have show $G \in \mathrm{Pol}(R)$. Let $a \leq c$ and $b \leq d$. If $b = t \circ h(a)$ and $d = t \circ h(c)$, then $G(a, b) = g(a)$ and $G(c, d) = g(c)$. However, in that case $t \circ h(a) \leq t \circ h(c)$ implies that a and c are at the same level, meaning that $a = c$. The remaining cases can be checked as well. □

Example. Let R be the graph of a fixed point free permutation of X whose cycles are all of the same prime length. According to [RS84], $\mathrm{Pol}(R)$ is locally maximal. We claim that $\mathrm{Pol}^*(R) = \mathscr{O}$ for all such R, and $\mathrm{Pol}(R)$ is maximal. Thus the maximality conjecture holds for such relations R.

Proof. To see that $\mathrm{Pol}^*(R) = \mathscr{O}$, observe that any equivalence θ with the property that each of its classes intersects each cycle at most once is a witness for every unary $g \in \mathscr{O}^{(1)}$. The n-ary case is similar.

We prove that $\mathrm{Pol}(R)$ is maximal. Let $f \notin \mathrm{Pol}(R)$, and let $g \in \mathscr{O}$ be arbitrary; we have to write g as a term of f and functions from $\mathrm{Pol}(R)$. It is not hard to see that f and $\mathrm{Pol}(R)$ together generate a unary operation that does not preserve R, so we may assume that f is itself unary. Let $C_0, C_1, \ldots$ be an enumeration of all cycles, and denote the elements of each C_j by $c_j^0, \ldots, c_j^{p-1}$ (where p is the common prime length of the cycles). The operation h which maps every c_j^i to c_0^i preserves R. Clearly, for each $1 \leq i \leq p-1$, there is a function $f^i \in \mathscr{O}^{(1)}$ generated by f and $\mathrm{Pol}(R)$ such that $(f_i(c_0^i), f_i(c_0^{i+1})) \notin R$ (where we set $(p-1)+1 := 0$). Now, writing m for the arity of g, set $G(x_1, \ldots, x_m, y_1^0, \ldots, y_1^{p-1}, \ldots, y_m^0, \ldots, y_m^{p-1})$ to equal $g(x_1, \ldots, x_m)$, if $y_j^i = f^i(h(x_j))$ for all $1 \leq j \leq m$ and $0 \leq i \leq p-1$. We have that G is a partial function of arity $m + pm = m(p+1)$. Clearly,

$$g(x_1, \ldots, x_m) = G(x_1, \ldots, x_m, f^0(h(x_1)), \ldots, f^{p-1}(h(x_1)), \ldots, f^0(h(x_m)), \ldots, f^{p-1}(h(x_m)));$$

we only have to show that we can extend G to a function in $\mathrm{Pol}(R)$. Call two tuples $u, v \in \mathscr{O}^{m(p+1)}$ *parallel* iff for the permutation α of which R is the graph there exists a number $0 \leq q \leq p-1$ such that $u = \alpha^q(v)$. It is easy to see from the definitions of h and the f^i that G is not yet defined on any two distinct parallel tuples. Now set $G(v)$ to an arbitrary value in X if v is not parallel to any tuple for which G has already been defined. Finally, set $G(\alpha^i(v)) := \alpha^i(v)$ for all v for which G has been defined so far and for all $1 \leq i \leq p-1$. It is easy to see that G preserves R. □

4.2. **Disproving the maximality conjecture.** Let R be a locally central symmetric binary relation, that is, a reflexive and symmetric relation with the additional property that for all finite $F \subseteq X$ there exists $c \in X$ with $F \times \{c\} \subseteq R$. According to [RS84], $\mathrm{Pol}(R)$ is maximal in $\mathrm{Cl}_{loc}(X)$.

We view R as a graph with vertex set X. Denote the graph obtained by deleting all loops by G_R. By Ramsey's theorem, G_R contains either an infinite complete subgraph, or an infinite induced independent set. It turns out that in the first case, $\mathrm{Pol}^*(R) = \mathscr{O}$ iff the complement of G_R (without loops) is n-colorable for some $n \geq 1$, and in the second case $\mathrm{Pol}^*(R) = \mathscr{O}$ iff G_R itself is n-colorable for some finite n (Lemma 17).

We then show that if G_R contains an infinite complete graph, then unless R is globally central, i.e., unless there exists $c \in X$ with $X \times \{c\} \subseteq R$, $\mathrm{Pol}(R)$ is not maximal.

This proves Proposition 10, since G_R can be chosen to contain an infinite complete subgraph, without globally central element, and in such a way that its complement is n-colorable, all at the same time. We then have $\mathrm{Pol}^*(R) = \mathscr{O}$, but $\mathrm{Pol}(R)$ is not maximal.

Lemma 17. *Let R be a locally central symmetric binary relation.*

- *If R contains an infinite complete subgraph, then* $\mathrm{Pol}^*(R) = \mathscr{O}$ *iff the complement of G_R is n-colorable for some finite n.*
- *If R contains an infinite independent set, then* $\mathrm{Pol}^*(R) = \mathscr{O}$ *iff G_R is n-colorable for some finite n.*

Proof. We prove the first statement. Assume first that $\mathrm{Pol}^*(R) = \mathscr{O}$. Let $K \subseteq X$ be so that G_R is the complete countably infinite graph on K. Let $g \in \mathscr{O}^{(1)}$ map K bijectively onto X, and do anything outside K. Let θ be an equivalence relation on X witnessing $g \in \mathrm{Pol}^*(R)$, and let $C_1, \ldots, C_n$ be its classes. Then the sets $g[K \cap C_i]$ induce a finite coloring of X, and if $(a, b) \notin R$, then a and b have different colors.

Conversely, let $g \in \mathscr{O}^{(m)}$ be arbitrary. Let $C_1, \ldots, C_n$ be the partition corresponding to the coloring of the complement of G_R, and set $D_i = g^{-1}[C_i]$ for all $1 \leq i \leq n$. The D_i form a partition of X^m which witnesses $g \in \mathrm{Pol}^*(R)$.

For the second statement, assume first that $\mathrm{Pol}^*(R) = \mathscr{O}$. Let $E \subseteq X$ be so that G_R is the infinite graph with no edges on E. Let $g \in \mathscr{O}^{(1)}$ map X bijectively onto E. Any partition θ witnessing $g \in \mathrm{Pol}^*(R)$ is a finite coloring of G_R.

Conversely, let $g \in \mathscr{O}^{(m)}$ be arbitrary. Let θ be the finite coloring of G_R; then, via preimages, θ induces a finite coloring of X^m witnessing $g \in \mathrm{Pol}^*(R)$. □

Definition 18. We call $C \subseteq X^n$ *bounded* iff there exists $c \in X^n$ such that $\{c\} \times C \subseteq R^n$.

Thus a symmetric reflexive relation is locally central iff all finite subsets of X are bounded. The bounded subsets of X^n form an order ideal (in general no join-semilattice-ideal) in the power set of X^n, for every $n \geq 1$. A subset of X^n is bounded iff it is contained in a product $C_1 \times \ldots \times C_n$ of bounded subsets of X.

Definition 19. An operation $g \in \mathscr{O}^{(n)}$ is *tame* iff $g[C]$ is bounded for all bounded $C \subseteq X^n$.

Lemma 20. *The set $\mathscr{T}$ of tame operations is a clone.*

Proof. $\mathscr{T}$ certainly contains the projections. Let $g \in \mathscr{T}^{(m)}$ and $f_1, \ldots, f_m \in \mathscr{T}^{(n)}$, and $t = g(f_1, \ldots, f_m)$, and $C \subseteq X^n$ be bounded. We have that for all $1 \leq i \leq n$, $f_i[C_i]$ is bounded, hence the product of these sets is bounded, hence the image of this product under g is bounded, hence $t[C]$, which is contained in the latter set, is bounded. □

Lemma 21. $\mathrm{Pol}(R) \subsetneq \mathscr{T}$.

Proof. Let $h \in \mathrm{Pol}(R)^{(n)}$, and let $C \subseteq X^n$ be bounded. Pick $c \in X^n$ with $\{c\} \times C \subseteq R^n$. Then $\{h(c)\} \times h[C] \subseteq R$, proving that $h[C]$ is bounded, so $\mathrm{Pol}(R) \subseteq \mathscr{T}$. The inequality follows from the fact that $\mathscr{T}$ contains all operations with bounded range, in particular those with finite range. □

Lemma 22. *If X has an infinite bounded subset, and if X is not bounded itself, then $\mathscr{T} \neq \mathscr{O}$.*

Proof. Let S be this subset, and let $g \in \mathscr{O}^{(1)}$ map S onto X, and do anything outside S. Then $g \notin \mathscr{T}$. □

Proposition 23. *Let R be locally central, symmetric and reflexive. Assume that X contains an infinite complete subgraph, contains no globally central element, and that the complement of G_R is n-colorable for some finite $n \geq 1$. Then*

(1) $\mathrm{Pol}^*(R) = \mathscr{O}$, *and*
(2) $\mathrm{Pol}(R)$ *is locally maximal, and*
(3) $\mathrm{Pol}(R) \subsetneq \mathscr{T} \subsetneq \mathscr{O}$, *in particular is* $\mathrm{Pol}(R)$ *not globally maximal.*

Proof. (1) is Lemma 17. (2) is from [RS84]. By the preceding lemma we have $\mathscr{T} \subsetneq \mathscr{O}$, and by Lemma 21 we have $\mathrm{Pol}(R) \subsetneq \mathscr{T}$. □

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