ALGEBRAIC LATTICES ARE COMPLETE SUBLATTICES OF THE CLONE LATTICE OVER AN INFINITE SET

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Abstract. The clone lattice $\mathrm{Cl}(X)$ over an infinite set X is a complete algebraic lattice with $2^{|X|}$ compact elements. We show that every algebraic lattice with at most $2^{|X|}$ compact elements is a complete sublattice of $\mathrm{Cl}(X)$.

1. How complicated is the clone lattice?

Fix a base set X and denote for all $n \geq 1$ the set X^{X^n} of all n-ary operations on X by $\mathscr{O}^{(n)}$. Then $\mathscr{O} = \bigcup_{n\geq 1} \mathscr{O}^{(n)}$ is the set of all functions on X which have finite arity. A set of finitary functions $\mathscr{C} \subseteq \mathscr{O}$ is called a *clone* iff it is closed under composition and contains all projections, i.e. for all $1 \leq i \leq n$ the function π_i^n satisfying $\pi_i^n(x_1, \ldots, x_n) = x_i$. The set of all clones over X forms a complete algebraic lattice $\mathrm{Cl}(X)$ with respect to inclusion. This lattice is countably infinite and completely known if $|X| = 2$ by a result of Post's [Pos41]; however, describing the clone lattice completely for larger X is believed impossible.

Several known results suggest this: To begin with, $\mathrm{Cl}(X)$ is large; it is of size continuum if X is finite and has at least three elements, and $|\mathrm{Cl}(X)| = 2^{2^{|X|}}$ if X is infinite. Then, the clone lattice does not satisfy any non-trivial lattice identity if $|X| \geq 3$ [Bul93]; it does not satisfy any quasi-identity if $|X| \geq 4$ [Bul94]. Also, if $|X| \geq 4$, then every countable product of finite lattices is a sublattice of $\mathrm{Cl}(X)$ [Bul94]. As for examples on infinite X, every completely distributive lattice having not more than $2^{|X|}$ compact elements is a subinterval of a monoidal interval of $\mathrm{Cl}(X)$ [Pin] (a *monoidal interval* being an interval of clones which have the same unary functions). Moreover, specific complicated parts of $\mathrm{Cl}(X)$ have been exposed, such as an interval which is isomorphic to the lattice of all filters on X in [GSa]. There exist several examples of parts of $\mathrm{Cl}(X)$ that are still "well-behaved" for finite X, but which seem to be hopelessly complicated for infinite X: The interval above $\mathscr{O}^{(1)}$ is a finite chain for finite X [Bur67] but huge and extremely complex for infinite X ([GS02] and [GSb]), and whereas $\mathrm{Cl}(X)$

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is dually atomic with a finite number of dual atoms which are all known if X is finite [Ros70], it is not dually atomic on countably infinite X if the continuum hypothesis holds [GS05], and there exist as many dual atoms as there are clones on all infinite X [Ros76]. A recent survey of clones on infinite sets is [GP].

We are interested in which lattices can be embedded into the clone lattice over an infinite set. Assume henceforth X to be infinite. The compact elements of $\mathrm{Cl}(X)$ are easily seen to be exactly the clones which are generated by a finite number of functions. Since $|\mathscr{O}| = 2^{|X|}$, this implies that $\mathrm{Cl}(X)$ has at most $2^{|X|}$ compact elements, and it is readily verified that the compact elements really amount to this number. We are going to prove that $\mathrm{Cl}(X)$ is in some sense the most complicated algebraic lattice with this property.

Theorem 1. *Let X be infinite. Then every algebraic lattice with at most $2^{|X|}$ compact elements can be completely embedded into* $\mathrm{Cl}(X)$*.*

We remark that the corresponding statement does not hold on finite X: There, $\mathrm{Cl}(X)$ has countably infinitely many compact (finitely generated) elements, but as has been proven in [Bul01], the countably infinite lattice M_ω (consisting of a countably infinite antichain plus a smallest and a largest element) does not embed into the clone lattice over any finite set. Observe also that our result implies that the clone lattice on infinite X does not satisfy any non-trivial properties such as the infinite quasi-identity exposed in [Bul01] which holds for $\mathrm{Cl}(X)$ if X is finite.

1.1. **Notation.** We denote the unary projection π_1^1 by the somewhat simpler symbol id, and use $\mathscr{J}$ for the set of projections on X. If $\mathscr{F} \subseteq \mathscr{O}$, then we write $\langle \mathscr{F} \rangle$ for the clone generated by $\mathscr{F}$. Three lattices will appear in the proof, the clone lattice $\mathrm{Cl}(X)$, the lattice $\mathfrak{L}$ to be embedded into the clone lattice, and the lattice of join-semilattice ideals of compact elements of $\mathfrak{L}$: For all of them, we use the symbols $\wedge, \vee, \bigwedge, \bigvee$ with their standard meanings, and confusion shall be carefully avoided. If $\Phi \subseteq \mathscr{O}^{(1)}$ is a set of unary operations, then Φ^* will stand for all those functions which arise from functions of Φ by the addition of any finite number of dummy variables. Such functions will remain *essentially unary*, i.e. although possibly non-unary they depend on only one variable, as opposed to *essentially at least binary* functions, which are functions that depend on at least two of their variables.

2. Proof of the main theorem

Let $\mathfrak{L}$ be the lattice to be embedded into $\mathrm{Cl}(X)$ and denote by $\mathfrak{P}$ the set of all compact elements of $\mathfrak{L}$. Then $\mathfrak{P}$ is a join-semilattice (cf. the textbook [Grä78]). By an *ideal* $I \subseteq \mathfrak{P}$ we mean a lower subset of $\mathfrak{P}$ closed under (finite) joins. The set of all ideals of $\mathfrak{P}$ is a complete algebraic lattice, and in fact

Fact 2. *$\mathfrak{L}$ is isomorphic to the lattice of ideals of $\mathfrak{P}$.*

We are going to assign a clone $\mathscr{C}_I$ to every ideal $I \subseteq \mathfrak{P}$ in such a way that the resulting mapping is a complete embedding of $\mathfrak{L}$ into $\mathrm{Cl}(X)$. Fix four elements $0, 1, 2, 4 \in X$ and set $A = X \setminus \{0, 1, 2, 4\}$. Let $\mathscr{A} = (A_p)_{p\in\mathfrak{P}}$ be a family of subsets of A indexed by the elements of $\mathfrak{P}$ with the following property: Whenever $A_p, A_{q_1}, \ldots, A_{q_k} \in \mathscr{A}$ and $p \neq q_i$ for all $1 \leq i \leq k$, then $A_p \nsubseteq A_{q_1} \cup \ldots \cup A_{q_k}$. Such a family exists: For example, there exist *independent* families of size $2^{|X|}$, where a family $\mathscr{F}$ of subsets of A is called independent iff for all finite disjoint $\mathscr{F}_1, \mathscr{F}_2 \subseteq \mathscr{F}$

$$\bigcap\{F : F \in \mathscr{F}_1\} \cap \bigcap\{A \setminus F : F \in \mathscr{F}_2\} \neq \emptyset.$$

See the textbook [Jec03, Lemma 7.7]. If $|X| = \aleph_0$, then one could also take $\mathscr{A}$ to be *almost disjoint*, meaning that all members of $\mathscr{A}$ are infinite and the intersection of any two distinct sets from $\mathscr{A}$ is finite (cf. [Jec03, Lemma 9.21]).

Define for all $p \in \mathfrak{P}$ a unary function $\phi_p \in \mathscr{O}^{(1)}$ by

$$\phi_p(x) = \begin{cases} 0 & , x \in A \setminus A_p \\ 1 & , x \in A_p \\ 2 & , x = 2 \\ 4 & , x \in \{0, 1, 4\}; \end{cases}$$

so on A, ϕ_p is the characteristic function of A_p. Set $\Phi = \{\phi_p : p \in \mathfrak{P}\}$. Now define for all $p, q_1, q_2 \in \mathfrak{P}$ with $p \leq q_1 \vee q_2$ a ternary function $m_p^{q_1,q_2}$ by

$$m_p^{q_1,q_2}(x, y, z) = \begin{cases} \phi_p(x) & , y = \phi_{q_1}(x) \wedge z = \phi_{q_2}(x) \\ 2 & , (x = 2 \vee y = 2 \vee z = 2) \wedge (y \notin \{1, 4\}) \wedge (z \notin \{1, 4\}) \\ 4 & , \text{otherwise.} \end{cases}$$

The function is well-defined: We only have to check that there is no conflict between the conditions for $m_p^{q_1,q_2}(x, y, z)$ to yield $\phi_p(x)$ and 2, respectively. If both conditions are satisfied, then one of the components of the tuple (x, y, z) equals 2; since $y = \phi_{q_1}(x)$ and $z = \phi_{q_2}(x)$, this implies $x = y = z = 2$, making the function value $m_p^{q_1,q_2}(x, y, z) = 2 = \phi_p(x)$ unique.

We write $\mathscr{M} = \{m_p^{q_1,q_2} : p, q_1, q_2 \in \mathfrak{P} \wedge p \leq q_1 \vee q_2\}$ and $\mathscr{C} = \langle \Phi \cup \mathscr{M} \rangle$. The clones $\mathscr{C}_I$ will all be subclones of $\mathscr{C}$ and will all contain $\mathscr{M}$. They will essentially consist of those ϕ_p for which $p \in I$, plus the functions from $\mathscr{M}$; the exact definition can only be given later. One can think of the ϕ_p as functions that represent the elements of $\mathfrak{P}$ in such a way that they are in some sense "independent" of each other, and of the $m_p^{q_1,q_2}$ as functions representing the order of $\mathfrak{P}$, since $m_p^{q_1,q_2}(\mathrm{id}, \phi_{q_1}, \phi_{q_2}) = \phi_p$ and since $m_p^{q_1,q_2}$ is defined only if $p \leq q_1 \vee q_2$. The following lemma follows easily by induction over terms in $\mathscr{C}$.

Lemma 3. *The only functions in $\mathscr{C}$ which take values in A are the projections.*

Definition 4. We call a function $f \in \mathscr{O}^{(1)}$ *distracted* iff there exists $a \in A$ such that $f(a) \in \{2, 4\}$.

Lemma 5. *Let $t \in \mathscr{C}^{(n)}$ and $t_1, \ldots, t_n \in \mathscr{O}^{(1)}$. If t depends on its i-th variable, where $1 \leq i \leq n$, and if t_i is distracted, then $t(t_1, \ldots, t_n)$ is distracted.*

Proof. We use induction over terms in $\mathscr{C}$. To start with, let $t \in \mathscr{J} \cup \Phi \cup \mathscr{M}$. There is nothing to show if t is a projection. If $t \in \Phi$ and $t_1 \in \mathscr{O}^{(1)}$ is distracted, then there exists $a \in A$ such that $t_1(a) \in \{2, 4\}$, so $t(t_1(a)) \in \{2, 4\}$ and $t(t_1)$ is distracted. If $t = m_p^{q_1,q_2} \in \mathscr{M}$ and t_i is distracted for some $i \in \{1, 2, 3\}$, then $t_i(a) \in \{2, 4\}$ for some $a \in A$ implies that $m_p^{q_1,q_2}(t_1, t_2, t_3)(a) \in \{2, 4\}$: Indeed, if $m_p^{q_1,q_2}(t_1, t_2, t_3)(a) \in \{0, 1\}$, then the definition of $m_p^{q_1,q_2}$ would allow us to conclude $t_1(a) \in A$ and $t_2(a) = \phi_{q_1}(t_1(a)) \in \{0, 1\}$ and $t_3(a) = \phi_{q_2}(t_1(a)) \in \{0, 1\}$, which is clearly impossible as $t_i(a) \in \{2, 4\}$.
For the induction step, assume that $t = f(s_1, \ldots, s_m)$, where $f \in \mathscr{J} \cup \Phi \cup \mathscr{M}$ and s_j satisfies the induction hypothesis, $1 \leq j \leq m$. Now there exists $1 \leq j \leq m$ such that f depends on its j-th variable and s_j depends on its i-th variable. By induction hypothesis $s_j(t_1, \ldots, t_n)$ is distracted and so is $f(s_1(t_1, \ldots, t_n), \ldots, s_m(t_1, \ldots, t_n))$, by the same proof as for the induction beginning. □

Lemma 6. *Let $m_p^{q_1,q_2} \in \mathscr{M}$ and $t_1, t_2, t_3 \in \Phi \cup \{\mathrm{id}\}$. Then $f = m_p^{q_1,q_2}(t_1, t_2, t_3)$ is distracted unless $t_1 = \mathrm{id}$, $t_2 = \phi_{q_1}$, and $t_3 = \phi_{q_2}$. In the latter case we have $f = \phi_p$.*

Proof. If $t_2 = \mathrm{id}$ or $t_3 = \mathrm{id}$, then $f(a) \in \{2, 4\}$ for all $a \in A$, since $m_p^{q_1,q_2}$ can yield 0 or 1 only if its second and third argument is in the range of a function in Φ; hence f is distracted in that case. Assume henceforth $t_2, t_3 \in \Phi$ and write $t_2 = \phi_r$ and $t_3 = \phi_s$, where $r, s \in \mathfrak{P}$.
If $t_1 = \mathrm{id}$, then f yields 4 on the symmetric differences $A_{q_1} \Delta A_r$ and $A_{q_2} \Delta A_s$

by the very definition of $m_p^{q_1,q_2}$. Hence f is distracted unless those sets are empty, i.e. $s = q_1$ and $r = q_2$; in the latter case we have $f = \phi_p$ as asserted. If $t_1 = \phi_l \in \Phi$, then $m_p^{q_1,q_2}(\phi_l, \phi_r, \phi_s)$ yields by definition either $2, 4$, or an element of the form $\phi_p(\phi_l(x)) \in \{2, 4\}$, so f is distracted. □

Lemma 7. *All* $t \in \mathscr{C}^{(1)} \setminus (\Phi \cup \{\mathrm{id}\})$ *are distracted.*

Proof. We prove this by induction over terms in $\mathscr{C}$. The beginning is trivial since there are no unary functions in the generating set $\mathscr{J} \cup \Phi \cup \mathscr{M}$ of $\mathscr{C}$ except those from $\Phi \cup \{\mathrm{id}\}$.

For the induction step, assume that $t = f(t_1, \ldots, t_n)$, where $f \in \mathscr{J} \cup \Phi \cup \mathscr{M}$ and t_i satisfies the induction hypothesis, for all $1 \le i \le n$. The case $f \in \mathscr{J}$ is trivial. If $f \in \Phi$ and $t_1 \ne \mathrm{id}$, then t_1 takes only values outside A by Lemma 3, so $f(t_1)$ takes only values in $\{2, 4\}$ and is distracted. The other possibility is that $f \in \mathscr{M}$, so write $t = m_p^{q_1,q_2}(t_1, t_2, t_3)$. If any of the t_i is distracted then so is t, by Lemma 5. We may therefore assume that the t_i are not distracted and hence are elements of $\Phi \cup \{\mathrm{id}\}$. But then Lemma 6 tells us that t, not being an element of $\Phi \cup \{\mathrm{id}\}$ by assumption, must be distracted. □

Definition 8. We say that $t \in \mathscr{C}^{(n)}$ is *unspoilt* iff there exist $t_1, \ldots, t_n \in \mathscr{C}^{(1)}$ such that $t(t_1, \ldots, t_n) \in \Phi$. Otherwise we call t *spoilt.*

Remark 9. By Lemmas 5 and 7, t_i must be in $\Phi \cup \{\mathrm{id}\}$ if t depends on its i-th variable, for all $1 \le i \le n$.

Remark 10. An easy induction using Lemmas 5 and 6 shows that t_i is uniquely determined if t depends on its i-th variable, for all $1 \le i \le n$.

Remark 11. By Lemmas 5 and 7, a unary $t \in \mathscr{C}^{(1)}$ is distracted iff it is spoilt.

Lemma 12. *Let* $t \in \mathscr{C}^{(n)}$ *be unspoilt, and assume it depends on its first variable. Then* $t(2, x_2, \ldots, x_n) \in \{2, 4\}$ *for all* $x_2, \ldots, x_n \in X$.

Proof. We use induction over the complexity of t. The lemma is trivial if $t \in \mathscr{J} \cup \Phi \cup \mathscr{M}$. For the induction step, since the range of $\phi_p(t_1)$ is contained in $\{2, 4\}$ and since therefore $\phi_p(t_1)$ is spoilt for all $\phi_p \in \Phi$ and all $t_1 \in \mathscr{C} \setminus \mathscr{J}$, we may assume $t = m_p^{q_1,q_2}(t_1, t_2, t_3)$, where t_i satisfies the induction hypothesis, $1 \le i \le 3$. Now one of the t_i must depend on its first variable, implying $t_i(2, x_2, \ldots, x_n) \in \{2, 4\}$ by induction hypothesis. Hence, $m_p^{q_1,q_2}(t_1, t_2, t_3)(2, x_2, \ldots, x_n) \in \{2, 4\}$ by the definition of $m_p^{q_1,q_2}$. □

Let $t(x, y) \in \mathscr{C}^{(2)}$, and consider a concrete representation $r = r(t)$ of t as a term over the generating set $\mathscr{J} \cup \Phi \cup \mathscr{M}$ of $\mathscr{C}$. In the following, we write such representations without the use of projections, using the variables x, y instead: For example, we write $m_p^{q_1,q_2}(x,y,y)$ instead of $m_p^{q_1,q_2}(\pi_1^2, \pi_2^2, \pi_2^2)$. This is no loss of generality and only avoids unnecessary usage of the projections, as in $\pi_1^2(\pi_2^2, \phi_p(\pi_1^2))$ (equivalently, we could demand the projections to appear only as innermost arguments in the representation). We say that a subterm s of r is a *leaf* of r iff it involves exactly one function symbol from $\Phi \cup \mathscr{M}$. For example, the leaves of

$$m_p^{q_1,q_2}(m_u^{v_1,v_2}(x, \phi_l(y), \phi_r(x)), \phi_d(y), m_g^{h_1,h_2}(x,x,x))$$

are $\phi_l(y), \phi_r(x), \phi_d(y)$, and $m_g^{h_1,h_2}(x,x,x)$. Thinking of r as a tree in which the variables are not represented by their own nodes, the leaves of r are really exactly the leaves of the tree.

We call the representation $r(t)$ *reduced* iff it has no subterms of the form $m_p^{q_1,q_2}(x, \phi_{q_1}(x), \phi_{q_2}(x))$. Such subterms can be replaced by $\phi_p(x)$ by virtue of Lemma 6, so every term t has a reduced representation. We are only interested in representations of unspoilt functions that depend on both variables, so all unary subterms of any representation correspond to elements of Φ, by Lemmas 5 and 7; working with reduced terms means that we demand those unary subterms to be represented by only one function symbol.

Let $r(t)$ be reduced. We set $Leaf(r)$ to consist of all leaves of $r(t)$. Note that $Leaf(r)$ depends on the representation of the function t.

Lemma 13. *Let $r(x, y)$ be a reduced representation of a binary function in $\mathscr{C}$ that is unspoilt and depends on both of its variables. Let $a \in A$. Then $r(2, a) = 4$ iff $a \in \bigcup\{A_v : \phi_v(y) \in Leaf(r)\}$.*

Proof. We use induction over the complexity of r. The beginning is trivial as there are no binary functions depending on both variables in the generating set of $\mathscr{C}$. For the induction step, write $r = f(r_1, \ldots, r_n)$, where $f \in \Phi \cup \mathscr{M}$, and where r_i satisfies the induction hypothesis, $1 \leq i \leq n$. If $f \in \Phi$, then using Lemma 3 it is readily verified that $f(r_1)$ is spoilt unless r_1 is a projection, in which case $r \in \Phi^*$, contradicting that r depends on both variables. Assume henceforth that $f = m_p^{q_1,q_2} \in \mathscr{M}$.

Observe that all r_i must be unspoilt, for otherwise r would be spoilt as well by Lemmas 5 and 7. Since r is unspoilt, there exist $s_1, s_2 \in \mathscr{C}^{(1)}$ such that $m_p^{q_1,q_2}(r_1(s_1,s_2), \ldots, r_3(s_1,s_2)) \in \Phi$. By Lemmas 5, 6 and 7, this is only possible if $r_1(s_1, s_2)$ is the identity, which together with Lemma 3 implies that r_1 is a projection. Suppose that $r_2 = r_1 = \pi_i^2$, where $i \in \{1,2\}$. Then $r(s_1,s_2) = m_p^{q_1,q_2}(s_i, s_i, r_3(s_1,s_2)) \in \Phi$ and Lemma 6 implies that the first argument in $m_p^{q_1,q_2}$ must be the identity, while the second must equal ϕ_{q_1}, an obvious contradiction. The same contradiction occurs assuming $r_3 = r_1$, and hence we have $r_i \neq r_1$, $i = 2, 3$. We now distinguish six cases.

Assume first that $r_2, r_3 \in \mathscr{J}$. Then $r = m_p^{q_1,q_2}(x,y,y)$ or $r = m_p^{q_1,q_2}(y,x,x)$. In either case we have $r(2,a) = 2 \neq 4$, in accordance with our assertion as r does not have any leaves of the form $\phi_v(y)$.

Consider the case where $r_2 \in \mathscr{J}$ and $r_3 \in \Phi^*$ (by symmetry, this also treats the case $r_3 \in \mathscr{J}$ and $r_2 \in \Phi^*$). Keeping Lemma 6 in mind we conclude that $r = m_p^{q_1,q_2}(x,y,\phi_{q_2}(x))$ or $r = m_p^{q_1,q_2}(y,x,\phi_{q_2}(y))$ or $r = m_p^{q_1,q_2}(x,y,\phi_{q_2}(y))$ or $r = m_p^{q_1,q_2}(y,x,\phi_{q_2}(x))$. The latter two possibilities, however, are spoilt as substitution of ϕ_{q_1} for y and x, respectively, yields a distracted third argument $\phi_{q_2}(\phi_{q_1})$ of $m_p^{q_1,q_2}$. The first possibility gives us $r(2,a) = m_p^{q_1,q_2}(2,a,2) = 2 \neq 4$, in accordance with our assertion. Finally, for the second term we have $r(2,a) = m_p^{q_1,q_2}(a,2,\phi_{q_2}(a))$, which equals 4 iff $\phi_{q_2}(a) \in \{1,4\}$ iff $a \in A_{q_2}$.

Now assume that $r_2 \in \mathscr{J}$ and $r_3 \notin \mathscr{J} \cup \Phi^*$. Then r_3 depends on both of its variables by Lemma 7, and therefore satisfies the assertion of this lemma by induction hypothesis. By Lemma 12 we have that $r(2,a) = 4$ iff $r(2,a) \neq 2$; the definition of $m_p^{q_1,q_2}$ tells us that this is the case iff $2 \notin \{r_1(2,a), r_2(2,a), r_3(2,a)\}$ or $r_2(2,a) \in \{1,4\}$ or $r_3(2,a) \in \{1,4\}$. Now $r_3(2,a) \in \{2,4\}$ by Lemma 12, and $r_2(2,a) \in \{2,a\}$ since r_2 is a projection. Thus, $r(2,a) = 4$ iff $r_3(2,a) = 4$, which by induction hypothesis is the case iff $a \in \bigcup\{A_v : \phi_v(y) \in \mathit{Leaf}(r_3)\}$. Since the leaves of r_3 are the just the leaves of r we are done.

Next say that $r_2 \in \Phi^*$ and $r_3 \notin \mathscr{J} \cup \Phi^*$. We have $r(2,a) = 4$ iff $r(2,a) \neq 2$, which happens iff $2 \notin \{r_1(2,a), r_2(2,a), r_3(2,a)\}$ or $r_2(2,a) \in \{1,4\}$ or $r_3(2,a) \in \{1,4\}$. Again, $r_3(2,a) \in \{2,4\}$ by Lemma 12, and $r_2(2,a) \in \{0,1,2\}$ as $r_2 \in \Phi^*$, implying $r(2,a) = 4$ iff $r_2(2,a) = 1$ or $r_3(2,a) = 4$. Now if $r_2(x,y) = \phi_{q_1}(x)$, then $r_2(2,a) = 2$ and so $r(2,a) = 4$ iff $r_3(2,a) = 4$ iff $a \in \bigcup\{A_v : \phi_v(y) \in \mathit{Leaf}(r_3)\}$ by induction hypothesis. This is in accordance with our assertion since then $\phi_v(y) \in \mathit{Leaf}(r_3)$ iff $\phi_v(y) \in \mathit{Leaf}(r)$. If on the other hand $r_2(x,y) = \phi_{q_1}(y)$, then $r_2(2,a) = 1$ iff $a \in A_{q_1}$, and hence $r(2,a) = 4$ iff $a \in A_{q_1} \cup \bigcup\{A_v : \phi_v(y) \in \mathit{Leaf}(r_3)\}$; this is the case iff $a \in \bigcup\{A_v : \phi_v(y) \in \mathit{Leaf}(r)\}$.

If $r_2, r_3 \in \Phi^*$, then up to symmetry $r = m_p^{q_1,q_2}(x,\phi_{q_1}(x),\phi_{q_2}(y))$ or $r = m_p^{q_1,q_2}(x,\phi_{q_1}(y),\phi_{q_2}(y))$ or $r = m_p^{q_1,q_2}(y,\phi_{q_1}(x),\phi_{q_2}(x))$ or $r = m_p^{q_1,q_2}(y,\phi_{q_1}(x),\phi_{q_2}(y))$. Therefore $r(2,a) = 4$ iff $a \in A_{q_2}$ in the first case, iff $a \in A_{q_1} \cup A_{q_2}$ in the second case, and iff $a \in A_{q_2}$ in the fourth case; in the third case, $r(2,a) = 2 \neq 4$.

Finally, consider $r_2, r_3 \notin \mathscr{J} \cup \Phi^*$. By Lemma 12, $\{r_2(2,a), r_3(2,a)\} \subseteq \{2,4\}$; thus, $r(2,a) = 4$ iff $r(2,a) \neq 2$ iff $r_2(2,a) = 4$ or $r_3(2,a) = 4$. Using the induction hypothesis, we get that $r(2,a)$ yields 4 iff $a \in \bigcup\{A_v : \phi_v(y) \in \mathit{Leaf}(r_2)\}$ or $a \in \bigcup\{A_v : \phi_v(y) \in \mathit{Leaf}(r_3)\}$; hence, $r(2,a) = 4$ iff $a \in \bigcup\{A_v : \phi_v(y) \in \mathit{Leaf}(r)\}$. □

Set $\mathscr{S} = \{t \in \mathscr{C} : t \text{ spoilt}\}$. Define for all $I \subseteq \mathfrak{P}$ sets of functions $\Phi_I = \{\phi_p \in \Phi : p \in I\}$ and $\mathscr{G}_I = \Phi_I \cup \mathscr{M} \cup \mathscr{S}$, and a clone $\mathscr{C}_I = \langle \mathscr{G}_I \rangle$. Write $\langle I \rangle$ for the ideal of $\mathfrak{P}$ generated by I.

Lemma 14. *Let $p \in \mathfrak{P}$ and $I \subseteq \mathfrak{P}$. Then $\phi_p \in \mathscr{C}_I$ iff $p \in \langle I \rangle$.*

Proof. Let $t \in \mathscr{C}_I$; using induction over the complexity of t as a term over the generating set $\mathscr{G}_I$, we show that $t = \phi_p$ implies $p \in \langle I \rangle$. The beginning is trivial, since if $t \in \mathscr{G}_I$, then $t \in \Phi_I$ and so $p \in I$. For the induction step, write $t = f(t_1, \ldots, t_n)$, with $f \in \mathscr{G}_I$ and $t_i \in \mathscr{C}_I$ satisfying the induction hypothesis, $1 \leq i \leq n$. Clearly, $f \in \mathscr{S}$ is impossible. $f \in \Phi_I$ implies that t_1 is the identity and so $f = \phi_p$; hence $p \in I$. Assume therefore that $f = m_u^{q_1,q_2} \in \mathscr{M}$. Then $u = p$, $t_1 = \mathrm{id}, t_2 = \phi_{q_1}$ and $t_3 = \phi_{q_2}$ by Lemmas 5, 6 and 7. By induction hypothesis, $q_1, q_2 \in \langle I \rangle$. Hence, $p \leq q_1 \vee q_2 \in \langle I \rangle$.
For the other direction, it is enough to show that if $\phi_{q_1}, \phi_{q_2} \in \mathscr{C}_I$, then $\phi_u \in \mathscr{C}_I$ for all $u \leq q_1 \vee q_2$. But this is clear since $\phi_u = m_u^{q_1,q_2}(\mathrm{id}, \phi_{q_1}, \phi_{q_2}) \in \mathscr{C}_I$. □

Lemma 15. *Let $\mathscr{I}$ be a family of ideals of $\mathfrak{P}$. Then $\bigvee\{\mathscr{C}_I : I \in \mathscr{I}\} = \mathscr{C}_{\bigvee \mathscr{I}}$.*

Proof. Trivially, $\mathscr{C}_{\bigvee \mathscr{I}}$ contains all $\mathscr{C}_I$, where $I \in \mathscr{I}$, hence it contains $\bigvee\{\mathscr{C}_I : I \in \mathscr{I}\}$. For the other inclusion we have to show that $\mathscr{C}_{\bigvee \mathscr{I}}$ is contained in $\bigvee\{\mathscr{C}_I : I \in \mathscr{I}\}$; clearly, it is enough to show that $\Phi_{\bigvee \mathscr{I}} \subseteq \bigvee\{\mathscr{C}_I : I \in \mathscr{I}\}$. Indeed, if $\phi_p \in \Phi_{\bigvee \mathscr{I}}$, then $p \in \bigvee \mathscr{I}$. Since $\bigvee \mathscr{I} = \langle \bigcup \mathscr{I} \rangle$, the preceding lemma implies $\phi_p \in \mathscr{C}_{\bigcup \mathscr{I}}$. Now it is enough to observe that $\mathscr{C}_{\bigcup \mathscr{I}}$ equals $\langle \bigcup \{\mathscr{C}_I : I \in \mathscr{I}\} \rangle$, which is exactly $\bigvee\{\mathscr{C}_I : I \in \mathscr{I}\}$. □

Lemma 16. *Let $\mathscr{I}$ be a family of ideals of $\mathfrak{P}$. Then $\bigwedge\{\mathscr{C}_I : I \in \mathscr{I}\} = \mathscr{C}_{\bigwedge \mathscr{I}}$.*

Proof. $\mathscr{C}_{\bigwedge \mathscr{I}}$ is a subclone of all $\mathscr{C}_I$, where $I \in \mathscr{I}$, so trivially $\mathscr{C}_{\bigwedge \mathscr{I}} \subseteq \bigwedge\{\mathscr{C}_I : I \in \mathscr{I}\}$. For the other direction, let $t \in \bigwedge\{\mathscr{C}_I : I \in \mathscr{I}\} = \bigcap\{\mathscr{C}_I : I \in \mathscr{I}\}$. If t is spoilt, then $t \in \mathscr{C}_{\bigwedge \mathscr{I}}$ by definition, so assume that t is unspoilt. If t is essentially unary, then t is a projection or an element of Φ^*, by Lemma 7. In the latter case, $t \in \bigcap\{\Phi_I^* : I \in \mathscr{I}\}$ by Lemma 14, so $t \in \mathscr{C}_{\bigcap \mathscr{I}} = \mathscr{C}_{\bigwedge \mathscr{I}}$. So let t be essentially at least binary, and assume without loss of generality that it depends on all of its variables. Because t is unspoilt, there exist $t_1, \ldots, t_n \in \Phi \cup \{\mathrm{id}\}$ such that $t(t_1, \ldots, t_n) \in \Phi$. Set $s_i(x,y) = t(t_1(x), \ldots, t_{i-1}(x), y, t_{i+1}(x), \ldots, t_n(x))$, for all $1 \leq i \leq n$. Obviously, all s_i are unspoilt. They also depend on both variables: Indeed, let without loss of generality $i = 1$. Then $s_1(2, t_1(a)) = t(t_1(a), 2, \ldots, 2) \in \{2, 4\}$ by Lemma 12 but $s_1(a, t_1(a)) = t(t_1, \ldots, t_n)(a) \in \{0, 1\}$ for all $a \in A$, so s_1 depends on the first variable. For the second variable, observe that

$s_1(a,2) = t(2, t_2(a), \ldots, t_n(a)) \in \{2,4\}$, so $s_1(a, t_1(a)) \neq s_1(a,2)$.
Assume that t is represented as a reduced term. The s_i might not be reduced: For example, t could have a subterm like $m_p^{q_1,q_2}(x_2, \phi_{q_1}(x_3), x_4)$, which becomes $m_p^{q_1,q_2}(x, \phi_{q_1}(x), \phi_{q_2}(x))$ when we substitute $x_2 = x_3 = x$ and $x_4 = \phi_{q_2}(x)$ upon building, say, s_1. However, such redundancies will occur only for the variable x. Thus, when simplifying s_i to a reduced term according to the equation $m_p^{q_1,q_2}(x, \phi_{q_1}(x), \phi_{q_2}(x)) = \phi_p(x)$, the leaves of the form $\phi_p(y)$, which were originally (that is, in t) leaves of the form $\phi_p(x_i)$, do not change. Therefore, $\phi_p(y)$ is a leaf of the new reduced s_i iff $\phi_p(x_i)$ is a leaf of t.
By Lemma 13, for all $1 \leq i \leq n$ and for all $a \in A$ we have that $s_i(2,a) = 4$ iff $a \in \bigcup\{A_v : \phi_v(y) \in \mathit{Leaf}(s_i)\}$. This is the case iff $a \in \bigcup\{A_v : \phi_v(x_i) \in \mathit{Leaf}(t)\}$. Therefore, there exists $1 \leq i \leq n$ with $s_i(2,a) = 4$ iff $a \in \bigcup\{A_v : \exists i\,(\phi_v(x_i) \in \mathit{Leaf}(t))\}$. Pick arbitrary $I, J \in \mathscr{I}$ and consider two reduced representations t_I, t_J of t, where t_I is a term over $\mathscr{G}_I$ and t_J one over $\mathscr{G}_J$. Then, since whether or not $s_i(2,a) = 4$ does not depend on the representation,

$$\bigcup\{A_v : \exists i\,(\phi_v(x_i) \in \mathit{Leaf}(t_I))\} = \bigcup\{A_v : \exists i\,(\phi_v(x_i) \in \mathit{Leaf}(t_J))\}.$$

Because $A_v \nsubseteq A_{q_1} \cup \ldots \cup A_{q_k}$ whenever $q_i \neq v$, $1 \leq i \leq k$, we conclude

$$\{v : \exists i\,(\phi_v(x_i) \in \mathit{Leaf}(t_I))\} = \{v : \exists i\,(\phi_v(x_i) \in \mathit{Leaf}(t_J))\}.$$

Thus, the latter set is a subset of both I and J, implying that t_I actually involves only functions from $\mathscr{G}_{I \cap J}$ as leaves. Since J was arbitrary, we may conclude that the term t_I uses only functions from $\mathscr{G}_{\bigwedge \mathscr{I}}$ as leaves. Because functions from Φ can appear only as leaves in an unspoilt term ($\phi_v(f)$ is spoilt for all $\phi_v \in \Phi$ and all $f \in \mathscr{C}$ unless f is a projection), this means that t_I contains only functions from $\mathscr{G}_{\bigwedge \mathscr{I}}$. Hence, $t \in \mathscr{C}_{\bigwedge \mathscr{I}}$. □

Proposition 17. *The mapping assigning $\mathscr{C}_I$ to every ideal $I \subseteq \mathfrak{P}$ is a complete lattice embedding of $\mathfrak{L}$ into* $\mathrm{Cl}(X)$.

Proof. The function is injective by Lemma 14 and preserves arbitrary suprema and infima by Lemmas 15 and 16. □

3. Concluding remarks and outlook

The only place where we used the infinity of the base set X is when we claim the existence of a family $\mathscr{A}$ which is as large as $\mathfrak{P}$ and has the property that whenever $A_p, A_{q_1}, \ldots, A_{q_k} \in \mathscr{A}$ and $p \neq q_i$ for all $1 \leq i \leq k$, then $A_p \nsubseteq A_{q_1} \cup \ldots \cup A_{q_k}$. Therefore surprisingly, the same proof works to show that every finite lattice $\mathfrak{L}$ is a sublattice of the clone lattice over a finite X for some X large enough ($|X| \geq |\mathfrak{L}| + 4$ suffices). However, as mentioned in the introduction, much better results already exist for finite X.

Answering the following question would be a next interesting step in answering the question of how complicated the clone lattice is.

Problem 18. *Is every algebraic lattice with at most $2^{|X|}$ compact elements an interval of* $\mathrm{Cl}(X)$*?*

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