# CHAPTER 6

## Singularity Analysis of generating functions

- singularities more general than poles
- subexponential factors more general than polynomials

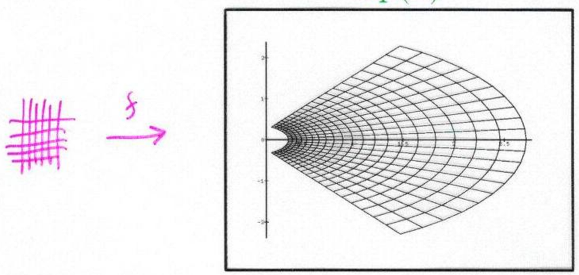
PREAMBLE: We may assume freely that functions have radius of convergence/singularity $\rho = 1$, since

$$\operatorname{coeff}[z^n]\, f(\lambda z) = \lambda^n \times \operatorname{coeff}[z^n]\, f(z).$$

(with $\lambda = \frac{1}{\rho}$) A factor of $(\rho^{-n})$ comes in ...

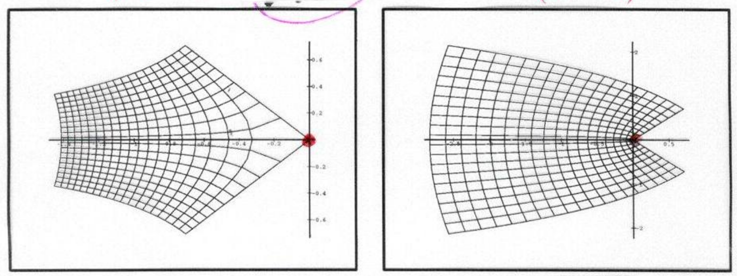
Regular point $\quad f(z) \approx f(z_0) + f'(z_0)(z - z_0)$

exp($z$)



Singular point

$-\sqrt{1-z}$ $\qquad\qquad$ $-(1-z)^{3/2}$



**(Singularity analysis)** [F+ Odlyzko, 90]

$\boxed{\text{PRINCIPLE}}$

$$\underset{[z\to 1]}{f(z) \sim} C \cdot (1-z)^{-\alpha} \log^k \frac{1}{1-z}$$

singularity, ANALYTIC CONTINUATION $\Bigg\} \implies$ $\displaystyle [z^n]f(z) \underset{[n\to\infty]}{\sim} C \cdot \frac{n^{\alpha-1}}{\Gamma(\alpha)} \log^k n$

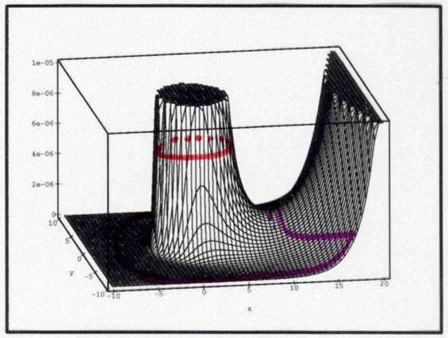
## FROM FUNCTIONS TO COEFFICIENTS

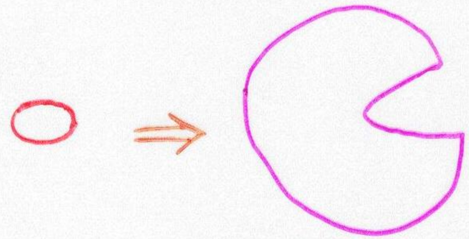
$\boxed{\text{FUNCTION}} \longrightarrow \boxed{\text{COEFFS}}$

| | |
|---|---|
| $f(z) = \sum_{n=0}^{\infty} f_n z^n$ | $f_n$? asympt$(f_n)$? |
| $\frac{1}{\sqrt{1-z}}$ | $\frac{1}{4^n}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}$ |
| $\frac{1}{1-z}$ | $1 \sim 1$ |
| $\frac{1}{1-z}\log\frac{1}{1-z}$ | $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n} \sim \log n$ |
| $\frac{1}{(1-z)^2}$ | $(n+1) \sim n$ |

GENERAL PATTERN

$$\text{Coeff}\,[z^n]\, \underbrace{f(z)}_{} = \frac{1}{2i\pi}\oint \overset{(1-z)^{-\alpha}}{f(z)}\, \frac{dz}{z^{n+1}}.$$





CAUCHY CONTOUR

HANKEL CONTOUR

THEOREM A: Coefficients of basic scale

$$[z^n]\frac{1}{(1-z)^{\alpha}} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}. \qquad \alpha \in \mathbb{C}$$

⊗ powers of $\left(\log\frac{1}{1-z}\right)$ ⊗ corresponding powers of $(\log n)$

⊗ same for log-log's etc...

⊗ full asymptotic expansion is available

The GAMMA function (cultural note)

$$\Gamma(s) := \int_0^\infty e^{-t}\, t^{s-1}\, dt \qquad \mathrm{Re}(s) > 0$$

**generalizes** the factorial function $\Gamma(1+s) = s!$ if $s \in \mathbb{Z}_{\geq 0}$

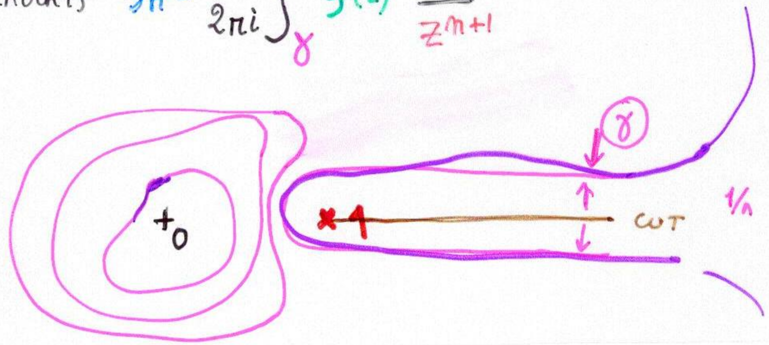
Since

$$\Gamma(s+1) = s\,\Gamma(s) \;;\quad \Gamma(1) = 1.$$

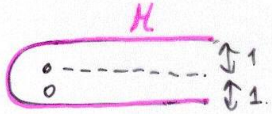
(extends to the whole of $\mathbb{C}$ with poles at $0, -1, -2, \dots$).

PROOF: $f(z) = (1-z)^{-\alpha}$, $\alpha \in \mathbb{C}$

[CAUCHY] $$f_n = \frac{1}{2\pi i}\int_\gamma f(z)\,\frac{dz}{z^{n+1}}$$




Set $\boxed{z = 1 + \frac{t}{n}}$ $t \in$




$$f_n = \frac{1}{2\pi i}\int_H \left(-\frac{t}{n}\right)^{-\alpha} \frac{\frac{1}{n}\,dt}{\left(1+\frac{t}{n}\right)^{n+1}}$$

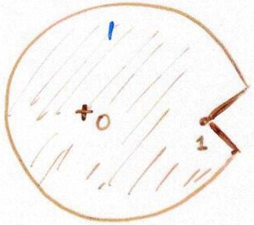
$$\sim n^{\alpha-1} \times \underbrace{\left(\frac{1}{2i\pi}\int_H (-t)^{-\alpha} e^{-t}\, dt\right)}_{\frac{1}{\Gamma(\alpha)}}$$

**Theorem B** Transfer of asymptotic properties

If $f(z) = O((1-z)^{-\alpha})$ as $z \to 1$ in a Camembert region

Then

$$\boxed{\text{coeff } [z^n]\, f(z) = O(n^{\alpha-1})}$$




* same for $\Theta(-)$ ; * same for log's, etc..

"ALGORITHM": ⊛ Make sure function exists in larger region than disc.

⊛ Expand as $z \to 1$

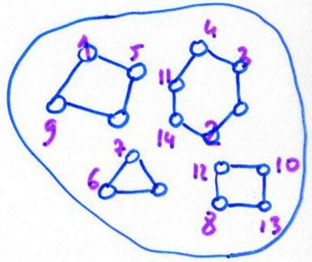
⊛ Translate formally (and safely) by theorems A+B.

**2-regular graphs** [Each node has exact degree = 2]

$$\mathcal{R} = \text{Set}(\text{UnorderedCycle}(\mathcal{Z}, \text{card} \geq 3))$$

$$R(z) = \exp\left(\frac{1}{2}\log\frac{1}{1-z} - \frac{z}{2} - \frac{z^2}{4}\right)$$

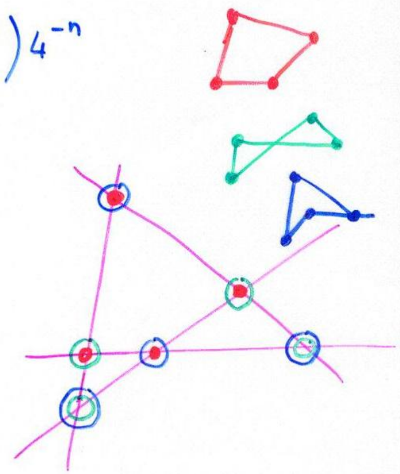
$$R(z) = \frac{e^{-\frac{z}{2}-\frac{z^2}{4}}}{\sqrt{1-z}}$$




By singularity analysis,

$$R(z) \sim \frac{e^{-3/4}}{\sqrt{1-z}}$$

$$\frac{R_n}{n!} \sim e^{-3/4}\binom{2n}{n}4^{-n} \sim \frac{e^{-3/4}}{\sqrt{\pi n}}$$



[Comtet's Clouds]
cf

TREES (Catalan model, binary variety)

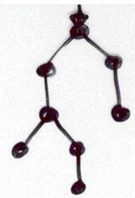
$$\mathcal{B} = \square + \begin{array}{c}\bullet\\ \mathcal{B}\quad\mathcal{B}\end{array}$$

$$B(z) = \frac{1-\sqrt{1-4z}}{2z}$$

$\mathrm{Sing}(B) = \frac{1}{4}$ ; exponent : $\alpha = -\frac{1}{2}$ in form $(1-z)^{-\alpha}$

$$B_n \sim \frac{1}{4\sqrt{\pi n^3}} \cdot 4^n .$$

Application 1: Unary-binary trees



$$T = z + zT + zT^2$$

$$\Rightarrow T = \frac{1-z-\sqrt{1-2z-3z^2}}{2z}$$

$$1-2z-3z^2 = (1-3z)(1+z)$$

$\Rightarrow \sqrt{\ }$ - singularity @ $\frac{1}{3}$ ,

$\alpha = -\frac{1}{2}$

$$\boxed{T_n \sim c \cdot 3^n n^{-3/2}}$$

In fact, universality for many classes of trees ...

Application 2: Mean # of cycles in permutation.

$$[z^n]?\ M(z) = \left(\frac{\partial}{\partial u} \exp\left(u \log \frac{1}{1-z}\right)\right)_{u \to 1}$$

$$= \frac{1}{1-z} \log \frac{1}{1-z}$$

$$\boxed{\mu_n \sim \log n}$$

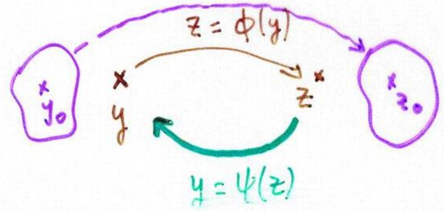
In fact universality for "exp-log" structures

# Chapter 7 (Applications of singularity analysis.)

## Recursive structures

— TREES

- Universality of the $\sqrt{\ }$- law for functions
- Universality of $\rho^{-n} n^{-3/2}$ estimate for counts




**Inversion Theorem:** $\phi$ is analytically invertible iff $\phi'(y_0) \neq 0$.

If not invertible $\phi''(y_0) \neq 0$ : $y \xrightarrow{} z \approx y^2$, $y \approx \sqrt{z}$

Cayley trees : $T = ze^T$ or $z = Te^{-T}$

not invertible if $\frac{d}{dT}(Te^{-T}) = (1-T)e^{-T} = 0$,

that is $T = 1$; $z = e^{-1}$

**square-root singularity**

$$T(z) \underset{z \to e^{-1}}{\sim} 1 - \sqrt{2}\sqrt{1-ez} + O((1-ez))$$

$$[z^n]\, T(z) \sim \frac{e^n}{\sqrt{2\pi n^3}}$$

$$\left(= \frac{n^{n-1}}{n!}\right)$$

**THEOREM** Let $\phi$ have positive coefficients and be entire (e.g. a polynomial, $\phi(y) = e^y$, etc.) with $\phi(0) \neq 0$. Then the function that solves

$$Y(z) = z\,\phi(Y(z))$$

has a $\sqrt{\ }$-singularity, so that

$$[z^n]\, Y(z) \sim C\rho^{-n} n^{-3/2}$$

- This is the framework of simple families of trees [Meir & Moon, 1978]. It also applies to unlabelled trees (e.g. Cayley) as well.

$\mathcal{U} = \mathcal{Z} \times \mathfrak{M}\mathrm{Set}(\mathcal{U})$ $\quad U(z) = z\exp\left(U(z) + \frac{1}{2}U(z^2) + \cdots\right)$

Say that $n^{-3/2}$ behaviour is <u>universal</u>.

- Using BGF's and singularities
  - Meir & Moon: Path length is on average $\sim Cn^{3/2}$
  - F. Odlyzko: Height is on average $\sim C'n^{1/2}$
  - Marckert et al.: Width is on average $\sim C''n^{1/2}$

EXAMPLE . Cyclic Points in Random Mappings

$$\begin{cases} \text{graph:} & \mathcal{G} = \mathrm{Set}(\mathcal{K}) \\ \text{Connected:} & \mathcal{K} = \mathrm{Cyc}(\mathcal{T}) \\ \text{tree:} & \mathcal{T} = \mathcal{O} * \mathrm{Set}(\mathcal{T}) \end{cases} \qquad \begin{cases} G = e^{K} \\ K = \log \dfrac{1}{1-uT} \\ T = z e^{T} \end{cases}$$

Mean number of cyclic points is

$$\mu_n = \frac{[z^n] \; \partial/\partial u \, G|_{u=1}}{[z^n] \; G|_{u=1}}, \leftarrow \quad G = \frac{1}{1-uT}$$

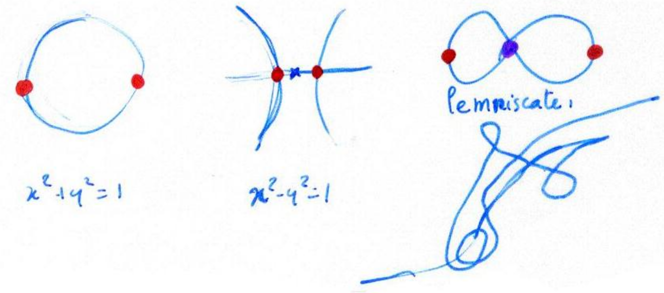
$$= \frac{[z^n] \; T/(1-T)^2}{[z^n] \; 1/(1-T)}$$

$$\sim \frac{[z^n] \, 2(1-ez)^{-1}}{[z^n] \sqrt{2}\,(1-ez)^{-1/2}} \qquad \begin{matrix} \leftarrow e^n n^0 \\ \leftarrow e^n n^{-1/2} \end{matrix}$$

$$\boxed{\mu_n \equiv \text{Mean \# cyclic points} \sim \sqrt{\frac{\pi n}{2}}}$$

Singularity analysis applies to any algebraic function (curve)

$$P(z, y(z)) = 0$$



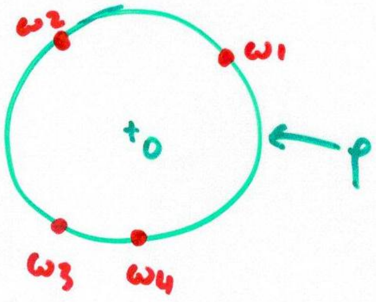
use **NEWTON-PUISEUX THEOREM**

Around any point $\xi$, $y(z)$ admits a fractional power expansion

$$y(z) = \sum_{j \geq -m} c_j (z-\xi)^{\alpha j} \qquad \alpha = p/q \in \mathbb{Q}.$$

$m=0$, $\alpha=1$: ordinary regular point

others: exceptional points $\to$ singularities.




Define an "algebraic" element to be of the form

$$\omega^{-n} \cdot \sum_{j \geq -r} d_j\, n^{-j\beta} \qquad (\beta \in \mathbb{Q})$$

**Theorem:** If $y(z)$ is an algebraic function, then there exists a finite collection of algebraic elements

$A_1$ (at $\omega_1$), $A_2$ (at $\omega_2$) ..., $A_s$ (at $\omega_s$) s.t.

$$y_n = A_1 + \cdots + A_s + O(\xi^{-n})$$

$|\omega_1| = |\omega_2| \cdots = |\omega_s| = \rho$ ; $\xi > \rho$.

- Trees with a finite ~~number~~ set of node degrees (we know already $\sqrt{\ }$-singularity ; $\rho^{-n} n^{-3/2}$).

- Excursions defined by a discrete set of steps $\Omega$ that is finite [Banderier, F. 2002]

- MAPS = graphs embedded into the plane

→ Gimenez-Noy : counting of planar graphs by gen. function + complex analysis.

**SINGULARITY ANALYSIS** applies to

- many non-linear ordinary differential equations, especially of order 1.

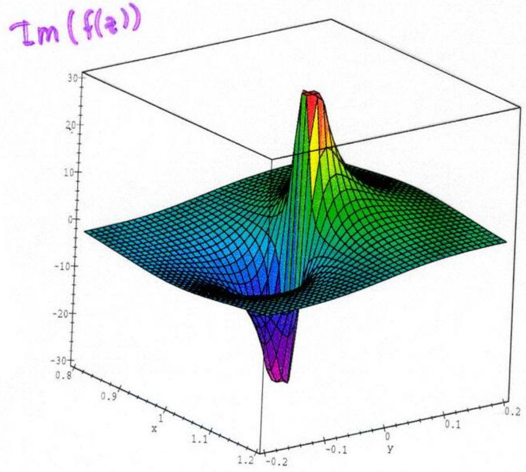
  ⇝ models of "logarithmic trees" : increasing trees, binary search trees, m-ary search, ...

- the whole class of linear ordinary diff. equations with so-called "regular singularities" [generic case].

  ⇝ the holonomic framework = functions such that coefficients of the linear ODE are in $\mathbb{C}(z)$.

Permutations: $f(z) = (1-z)^{-1}$

Im(f(z))



Trees: $f(z) = 1-\sqrt{1-z}$

Im(f(z))