WELCOME TO THE SUMMER SCHOOL

Probabilistic Methods in Combinatorics*

Graz Maria Trost, July 17-19

Course A Michael Drmota

The probabilistic method, random graphs and Stein's method

Course B Philippe Flajolet

Singularities and Random Combinatorial Structures

Course C Ralph Neininger

Distributional analysis of recursive algorithms and random trees

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Analytic Combinatorics and Probabilistic Number Theory.

	Monday, July 17	Tuesday, July 18	Wednesday, July 19
08:45 – 08:50	Opening		
08:50 – 09:35	A 1	B 2	C 3
09:35 – 09:45	Short Break		
09:45 – 10:30	A 1	B 2, Exercises	C 3
10:30 – 10:50	Break		
10:50 – 11:35	B 1	C 2	A 4
11:35 – 11:45	Short Break		
11:45 – 12:30	B 1	C 2, Exercises	A 4 Exercises
12:30	Lunch		
14:50 – 15:35	C 1	A 3	B 4
15:35 – 15:45	Short Break		
15:45 – 16:30	C 1	A 3	B 4 Exercises
16:30 – 16:50	Break		
16:50 – 17:35	A 2	B 3	C 4
17:35 – 17:45	Short Break		
17:45 – 18:30	A 2 Exercises	B 3	C 4 Exercises
18:30	Dinner		

SUMMER SCHOOL ON PROBABILISTIC METHODS IN COMBINATORICS

The Probabilistic Method, Random Graphs and Stein's Method

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Contents

Introduction

The **Probabilistic Method** has been initiated by Paul Erdős (1947) in order to prove the existence of certain combinatorial objects. The principle idea is to define a proper probability distribution on a class of (discrete) objects and to show that the probability of a certain property is positive. Of course this also proves that there exists such an object with this property. We will apply this approach to various problems on **random graphs**.

However, the main goal of this course is to give an introduction to **Stein's method** that proves asymptotic normality for sums of (in some sense) weakly dependent random variables. This method has turned out to be very successful, in particular in random graph problems.

Books

Noga Alon and Joel H. Spencer. *The probabilistic method.* Second edition. Wiley-Interscience, New York, 2000

Béla Bollobás, *Random graphs*. Second edition. Cambridge Studies in Advanced Mathematics, 73. Cambridge University Press, Cambridge, 2001.

Svante Janson, Tomasz Łuczak, and Andrzej Rucinski, *Random graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000

Valentin F. Kolchin, *Random graphs*. Encyclopedia of Mathematics and its Applications, 53. Cambridge University Press, Cambridge, 1999.

Books

Charles Stein, *Approximate computation of expectations.* Institute of Mathematical Statistics Lecture Notes—Monograph Series, 7. Institute of Mathematical Statistics, Hayward, CA, 1986.

Andrew D. Barbour, Lars Holst, and Svante Janson. *Poisson approximation*, Oxford Studies in Probability, 2. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.

Lower Bound for the Ramsey Number

Definition *The* **Ramsey number** $R(k,l)$ *is the smallest number* n *such that any 2-coloring of the edges on the conplete graph* K_n *on* n *vertices contains either a monochromatic* K_k *(in* K_n*) of the first color or a monochromatic* K_l *(in* K_n*) of the second color.*

Ramsey's theorem: $R(k,l)$ exists for all positive integers k and l.

Example: $R(3,3) = 6$.

Remark: $R(k,k) \leq (4 + o(1))^k$.

Lower Bound for the Ramsey Number

Theorem

$$\boxed{R(k,k) > 2^{k/2}}$$

for all $k \geq 3$.

Proof

K_n ... complete graph with vertex set $\{1, 2, \ldots\}$

Take a random 2-coloring of the $\binom{n}{2}$ edges
(Each edge is colored independently and with equal probability $\frac{1}{2}$.)

Lower Bound for the Ramsey Number

$R \subseteq \{1, 2, \ldots\}$, $|R| = k$

$$A_R := \{\text{the induced subgraph of } R \text{ is monochromatic}\}$$

$$\Longrightarrow \quad \mathbb{P}(A_R) = 2\,\frac{1}{2^{\binom{k}{2}}} = 2^{1-\binom{k}{2}}$$

$$\Longrightarrow \quad \mathbb{P}\{\exists\, R \subseteq \{1, 2, \ldots\} : |R| = k,\ A_R \text{ occurs}\} \le \binom{n}{k} 2^{1-\binom{k}{2}}.$$

Lower Bound for the Ramsey Number

$n = \lfloor 2^{k/2} \rfloor$ (and $k \geq 3$)

$$\implies \quad \binom{n}{k} 2^{1-\binom{k}{2}} < 2 \frac{n^k}{k!} \frac{1}{2^{k^2/2-k/2}} \leq 2 \frac{2^{k/2}}{k!} < 1$$

$$\implies \quad \mathbb{P}\{\forall\, R \subseteq \{1, 2, \ldots\} : |R| = k,\ R \text{ is not monochromatic}\} > 0$$

$$\implies \quad \boxed{R(k, k) > n}.$$

Lower Bound for the Ramsey Number

Notation: We use the notion *almost always* as an abbreviation for the property that the probability that a certain condition holds converges to 1 as the *size* of the problem goes to the infinity.

Remark. $n = \lfloor 2^{k/2} \rfloor \longleftrightarrow k = \lceil 2 \log_2 n \rceil$,

$$\lim_{k \to \infty} 2 \frac{2^{k/2}}{k!} = 0$$

$\Longrightarrow$ Almost always there exists no monochromatic $K_{\lceil 2 \log_2 n \rceil}$ in a randomly edge colored K_n.

First Moment Method

Linearity of the expectation:

$$\boxed{X = \sum_{i \in I} Y_i \quad \Longrightarrow \quad \mathbb{E}\, X = \sum_{i \in I} \mathbb{E}\, Y_i}$$

- The expected value is usually easy to compute.
- The dependence structure between the Y_i is irrelevant.

First Moment Method

Theorem Suppose that $\mathbb{E}\, X$ ist finite.

$$\Longrightarrow \quad \boxed{\mathbb{P}\{X \le \mathbb{E}\, X\} > 0} \quad \text{and} \quad \boxed{\mathbb{P}\{X \ge \mathbb{E}\, X\} > 0}.$$

Proof (indirect)

Suppose that $\mathbb{P}\{X \le \mathbb{E}\, X\} = 0$

$$\Longrightarrow \quad \mathbb{P}\{X > \mathbb{E}\, X\} = 1$$

$$\Longrightarrow \quad \mathbb{E}\, X = \mathbb{E}\left(\mathbb{I}_{[X > \mathbb{E}\, X]} \cdot X\right) = \mathbb{E}\, X + \mathbb{E}\underbrace{\left(\mathbb{I}_{[X > \mathbb{E}\, X]} \cdot (X - \mathbb{E}\, X)\right)}_{>0} > \mathbb{E}\, X$$

which is a contradiction!

First Moment Method

Theorem

X ... discrete random variable on **non-negative integers**.

$$\Longrightarrow \quad \boxed{\mathbb{P}\{X > 0\} \le \mathbb{E}\, X}.$$

Proof

$$\mathbb{E}\, X = \sum_{k \ge 0} k\, \mathbb{P}\{X = k\} \le \sum_{k \ge 1} \mathbb{P}\{X = k\} = \mathbb{P}\{X > 0\}.$$

First Moment Method

As an **first application** we prove $R(k,k) > 2^{k/2}$ a second time:

K_n ... complete graph with vertex set $\{1, 2, \ldots\}$

Take a random 2-coloring of the $\binom{n}{2}$ edges

$\mathcal{S}_{n,k}$ ... set of all subgraphs of K_n with k nodes

$$\Longrightarrow \quad \boxed{X_n := \sum_{R \in \mathcal{S}_{n,k}} \mathbb{I}_{[R \text{ is monochromatic}]}}$$

is the **(random) number of monochromatic subgraphs of K_n that are isomorphic to K_k.**

First Moment Method

$$X_n = \sum_{R \in \mathcal{S}_{n,k}} \mathbb{I}_{[R \text{ is monochromatic}]}$$

$$\Longrightarrow \quad \mathbb{E}\, X_n = \sum_{R \in \mathcal{S}_{n,k}} \mathbb{P}\{R \text{ is monochromatic}\} = \binom{n}{k} 2 \; 2^{-\binom{k}{2}}$$

$$\Longrightarrow \quad \mathbb{P}\{X_n > 0\} \le \binom{n}{k} 2^{1-\binom{k}{2}} < 2 \frac{2^{k/2}}{k!} < 1$$

$$\Longrightarrow \quad \boxed{\mathbb{P}\{X_n = 0\} > 0}.$$

First Moment Method

Theorem

$v_1, \ldots, v_n$ ... unit vectors in $\mathbb{R}^n$

$\Longrightarrow \exists\ \varepsilon_1, \ldots, \varepsilon_n \in \{-1, +1\}$:

$$\boxed{|\varepsilon_1 v_1 + \cdots + \varepsilon_n v_n| \leq \sqrt{n}}$$

$\Longrightarrow \exists\ \varepsilon'_1, \ldots, \varepsilon'_n \in \{-1, +1\}$:

$$\boxed{|\varepsilon'_1 v_1 + \cdots + \varepsilon'_n v_n| \geq \sqrt{n}}.$$

First Moment Method

Proof

$\varepsilon_1, \ldots, \varepsilon_n \in \{-1, +1\}$ random signs (independent equal probability $\frac{1}{2}$)

$$X := \left| \sum_{i=1}^{n} \varepsilon_i v_i \right|^2$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon_i \varepsilon_j v_i \cdot v_j$$

$\mathbb{E}\left(\varepsilon_i \varepsilon_j\right) = \delta_{i,j}$

$$\Longrightarrow \quad \mathbb{E}\, X = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}\left(\varepsilon_i \varepsilon_j\right) v_i \cdot v_j = \sum_{i=1}^{n} v_i \cdot v_i = n.$$

$+$ application of first moment method.

First Moment Method

Definition *A set of nodes I in a graph G is called* **independent** *if no two nodes of I are adjacent.*

The **independence number** $\alpha(G)$ *of G is the maximal size of an independent set of nodes of G.*

Theorem

$G = (V, E)$ ... graph with $|V| = n$ nodes and $|E| = m \geq n/2$ edges.

$$\Longrightarrow \quad \boxed{\alpha(G) \geq \frac{n^2}{4m}}.$$

First Moment Method

Proof

$p = n/(2m) \Longrightarrow 0 \le p \le 1.$

S ... random subset of vertices: $\mathbb{P}\{v \in S\} = p$ (independent)

$X = |S|$ ... (random) size of S, $\boxed{\mathbb{E}\, X = np = \frac{n^2}{2m}}$

Y .. (random) number of edges in $G|_S$ (= induced subgraph of G)

$$\boxed{Y = \sum_{e \in E} \mathbb{I}_{[\text{both endpoints of } e \text{ are in } S]}}$$

First Moment Method

$$Y = \sum_{e \in E} \mathbb{I}_{[\text{both endpoints of } e \text{ are in } S]}$$

$$\implies \quad \mathbb{E}\, Y = \sum_{e \in E} p^2 = mp^2 = \frac{n^2}{4m}$$

$$\implies \quad \mathbb{E}\,(X - Y) = np - mp^2 = \frac{n^2}{2m} - \frac{n^2}{4m} = \frac{n^2}{4m}.$$

First Moment Method

• There exists some specific S for which the number of vertices of S minus the number of edges of S is at least $n^2/(4m)$.

• Select one vertex from each edge of S and delete it. This leaves a set S^* with at least $n^2/(4m)$ vertices.

• S^* is an independent set (all edges of S have been destroyed)

$$\Longrightarrow \quad \boxed{\alpha(G) \geq \frac{n^2}{4m}}$$

First Moment Method

Definition *The* **girth** $\text{girth}(G)$ *of a graph* G *is the size of the shortest cycle.*

The **chromatic number** $\chi(G)$ *of a graph* G *is the smallest number* k *such that there exists a regular* k*-coloring of the vertices of* G*, that is, a coloring of at* k *colors of the vertices such that adjacent vertices have different colors.*

Theorem [Erdős 1959]

For all (positive integers) k and ℓ there exists a graph G with

$$\boxed{\text{girth}(G) > \ell} \quad \text{and} \quad \boxed{\chi(G) > k}.$$

First Moment Method

Proof

$p = n^{\theta-1}$ for some $0 < \theta < 1/\ell$ (n be chosen sufficiently large)

$V = \{1, 2, \ldots, n\}$ ... vertex set of a random graph:

$$\mathbb{P}\{e \in E(G)\} = p \qquad \text{(independently)}$$

X ...number of cycles of size $\leq \ell$.

$\theta\ell < 1$

$$\Longrightarrow \quad \mathbb{E}\, X = \sum_{i=3}^{\ell} \frac{(n)_i}{2i} p^i \leq \sum_{i=3}^{\ell} \frac{n^i}{2i} n^{(\theta-1)i} = \sum_{i=3}^{\ell} \frac{n^{\theta i}}{2i} = o(n).$$

First Moment Method

$$\mathbb{E}\,X \geq \mathbb{E}\left(X \cdot \mathbb{I}_{[X \geq n/2]}\right) \geq \frac{n}{2}\,\mathbb{P}\{X \geq n/2\}$$

$$\mathbb{E}\,X = o(n)$$

$$\Longrightarrow \quad \boxed{\mathbb{P}\{X \geq n/2\} = o(1)}.$$

First Moment Method

$$
\begin{aligned}
\mathbb{P}\{\alpha(G) \geq m\} &= \mathbb{P}\{\exists\, S \subseteq \{1,2,\ldots,n\} : |S| = m,\ S \text{ is independent}\} \\
&\leq \mathbb{E}\left(\sum_{|S|=m} \mathbb{I}_{[S \text{ is independent}]}\right) \\
&= \sum_{|S|=m} \mathbb{P}\{S \text{ is independent}\} \\
&= \binom{n}{m}(1-p)^{\binom{m}{2}} \\
&\leq \frac{n^m}{m!} e^{-p\binom{m}{2}} \\
&\leq \left(n e^{-p(m-1)/2}\right)^m
\end{aligned}
$$

First Moment Method

$$m = m(n) = \lceil \tfrac{3}{p} \log n \rceil \sim 3n^{1-\theta} \log n$$

$$\Longrightarrow \quad ne^{-p(m-1)/2} \to 0 \qquad (n \to \infty)$$

$$\Longrightarrow \quad \boxed{\mathbb{P}\{\alpha(G) \geq m(n)\} \to 0} \qquad (n \to \infty)$$

First Moment Method

n sufficiently large that $\mathbb{P}\{X \geq n/2\} < \frac{1}{2}$ and $\mathbb{P}\{\alpha(G) \geq m(n)\} < \frac{1}{2}$.

- Take G with $X < n/2$ (less than $n/2$ cycles of length at most ℓ) and $\alpha(G) < m(n) \sim 3n^{1-\theta}\log n$.

- Remove from G a vertex from each cycle of length at most ℓ.

- New graph G^* has at least $n/2$ vertices, $\boxed{\text{girth}(G^*) > \ell}$

- $\alpha(G^*) \leq \alpha(G)$

$$\Longrightarrow \quad \chi(G^*) \geq \frac{|G^*|}{\alpha(G)} \geq \frac{n/2}{3n^{1-\theta}\log n} = \frac{n^\theta}{6\log n}.$$

- n sufficiently large that $n^\theta/(6\log n) > k \Longrightarrow \boxed{\chi(G) > k}$.

Second Moment Method

Second moment $\mathbb{E}\,(X^2)$

Variance $\boxed{\mathbb{V}\,X = \mathbb{E}\,(X^2) - (EX)^2 = \mathbb{E}\,((X - \mathbb{E}\,X)^2)}$

Theorem [Chebyshev's Inequality] Suppose that $\mathbb{E}\,(X^2)$ is finite.

$$\Longrightarrow \quad \boxed{\mathbb{P}\{|X - \mathbb{E}\,X| \geq \lambda\sqrt{\mathbb{V}\,X}\} \leq \frac{1}{\lambda^2}}.$$

Proof

$$\begin{aligned}
\mathbb{V}\,X &= \mathbb{E}\,((X - \mathbb{E}\,X)^2) \\
&\geq \mathbb{E}\,\left((X - \mathbb{E}\,X)^2 \mathbb{I}_{[|X - \mathbb{E}\,X| \geq \kappa]}\right) \\
&\geq \kappa^2 \mathbb{P}\{|X - \mathbb{E}\,X| \geq \kappa\}.
\end{aligned}$$

and use $\kappa = \lambda \cdot \sqrt{\mathbb{V}\,X}$.

Second Moment Method

Theorem

X ... discrete random variable on **non-negative integers**

$$\Longrightarrow \quad \boxed{\mathbb{P}\{X = 0\} \le \frac{\mathbb{V}\,X}{(\mathbb{E}\,X)^2}}.$$

Proof Set $\lambda = \mathbb{E}\,X/\sqrt{\mathbb{V}\,X}$ in Chebyshev's Inequality.

Then $\lambda\sqrt{\mathbb{V}\,X} = \mathbb{E}\,X$ and consequently

$$\mathbb{P}\{X = 0\} \le \mathbb{P}\{|X - \mathbb{E}\,X| \ge \mathbb{E}\,X\} \le \frac{1}{\lambda^2} = \frac{\mathbb{V}\,X}{(\mathbb{E}\,X)^2}.$$

Second Moment Method

Remark

Sharpened Version: $\mathbb{E}\,X = \mathbb{E}\,(X \cdot \mathbb{I}_{[X>0]}) \le \sqrt{\mathbb{E}\,X^2} \cdot \sqrt{\mathbb{P}\{X > 0\}}$.

$$\Longrightarrow \quad \boxed{\mathbb{P}\{X > 0\} \ge \frac{(\mathbb{E}\,X)^2}{\mathbb{E}\,X^2}}.$$

$$\Longrightarrow \quad \boxed{\mathbb{P}\{X = 0\} \le \frac{\mathbb{V}\,X}{\mathbb{E}\,X^2}}.$$

This complements the inequality $\mathbb{P}\{X > 0\} \le \mathbb{E}\,X$:

$$\boxed{\frac{(\mathbb{E}\,X)^2}{\mathbb{E}\,X^2} \le \mathbb{P}\{X > 0\} \le \mathbb{E}\,X}$$

Second Moment Method

Theorem

X_n ... sequence of random variables with

$$\mathbb{E}\, X_n \to \infty \quad \text{and} \quad \mathbb{E}\,(X_n)^2 \sim (\mathbb{E}\, X_n)^2$$

as $n \to \infty$.

$$\Longrightarrow \quad \boxed{X_n > 0} \quad \text{and} \quad \boxed{\frac{X_n}{\mathbb{E}\, X_n} \to 1}$$

almost always.

Second Moment Method

Proof

- $\mathbb{E}\,(X_n)^2 \sim (\mathbb{E}\,X_n)^2 \quad \Longrightarrow \quad \mathbb{V}\,X_n = o((\mathbb{E}\,X_n)^2)$.

- $\mathbb{P}\{|X_n - \mathbb{E}\,X_n| \geq \varepsilon\,\mathbb{E}\,X_n\} \leq \dfrac{\mathbb{V}\,X_n}{\varepsilon^2(\mathbb{E}\,X_n)^2}$

  (Take $\lambda = \varepsilon\mathbb{E}\,X_n/\sqrt{\mathbb{V}\,X_n}$ in Chebyshev's inequality.)

$$\Longrightarrow \quad \boxed{\mathbb{P}\{|X_n - \mathbb{E}\,X_n| \geq \varepsilon\,\mathbb{E}\,X_n\} \to 0}.$$

Remark. A relation of the kind $\mathbb{P}\{|X_n - \mathbb{E}\,X_n| \geq \varepsilon\,\mathbb{E}\,X_n\} \to 0$ is a so-called **concentration property** of X_n.

Second Moment Method

Application

$X = X_n$ ... number of **triangles** in random graph $G(n,p)$.

$$\mathbb{P}\{e \in E(G)\} = p \qquad \text{(independently)}$$

$\mathcal{T}$ ... (random) set of triangles in $G(n,p)$:

$$X = \sum_{1 \le i_1 < i_2 < i_3 \le n} \mathbb{I}_{[(i_1,i_2,i_3)\in\mathcal{T}]}$$

$$\mathbb{E}\, X = \sum_{1 \le i_1 < i_2 < i_3 \le n} \mathbb{P}\{(i_1,i_2,i_3) \in \mathcal{T}\} = \binom{n}{3} p^3.$$

Second Moment Method

$$\mathbb{E}\left(X^2\right) = \mathbb{E}\left(\sum_{1\le i_1<i_2<i_3\le n}\ \sum_{1\le j_1<j_2<j_3\le n} \mathbb{I}_{[(i_1,i_2,i_3)\in\mathcal{T}]}\cdot \mathbb{I}_{[(j_1,j_2,j_3)\in\mathcal{T}]}\right)$$

$$= \mathbb{E}\left(\sum_{1\le i_1<i_2<i_3\le n}\ \sum_{1\le j_1<j_2<j_3\le n} \mathbb{I}_{[(i_1,i_2,i_3),(j_1,j_2,j_3)\in\mathcal{T}]}\right)$$

$$= \sum_{1\le i_1<i_2<i_3\le n}\ \sum_{1\le j_1<j_2<j_3\le n} \mathbb{P}\{(i_1,i_2,i_3),(j_1,j_2,j_3)\in\mathcal{T}\}$$

Second Moment Method

1. If $|\{i_1, i_2, i_3\} \cap \{j_1, j_2, j_3\}| = 3$, that is, $i_1 = j_1$, $i_2 = j_2$, and $i_3 = j_3$ then

$$\mathbb{P}\{(i_1, i_2, i_3), (j_1, j_2, j_3) \in \mathcal{T}\} = p^3$$

and there are $\binom{n}{3}$ cases of that kind.

2. If $|\{i_1, i_2, i_3\} \cap \{j_1, j_2, j_3\}| = 2$ then

$$\mathbb{P}\{(i_1, i_2, i_3), (j_1, j_2, j_3) \in \mathcal{T}\} = p^5$$

and there are $12\binom{n}{4}$ cases of that kind.

3. If $|\{i_1, i_2, i_3\} \cap \{j_1, j_2, j_3\}| \leq 1$ then the events $\{(i_1, j_1, k_1) \in \mathcal{T}\}$ and $\{(i_2, j_2, k_2) \in \mathcal{T}\}$ are independent and consequently

$$\mathbb{P}\{(i_1, j_1, k_1), (i_2, j_2, k_2) \in \mathcal{T}\} = p^6.$$

Second Moment Method

$$\begin{aligned}\mathbb{E}\,(X^2) &= \binom{n}{3}p^3 + 12\binom{n}{4}p^5 + \left(\binom{n}{3}^2 - \binom{n}{3} - 12\binom{n}{4}\right)p^6 \\ &= (\mathbb{E}\,X)^2 + \binom{n}{3}p^3(1-p^3) + 12\binom{n}{4}p^5(1-p).\end{aligned}$$

$$np \to \infty \quad \Longleftrightarrow \quad \mathbb{E}\,X^2 \sim (\mathbb{E}\,X)^2$$

Proposition

If $np \to \infty$ then almost always the number of triangles in $G(n,p)$ is approximated by the their expected number $\binom{n}{3}p^3$.

Random Graphs

Definition *Let n be a positive integer and p a real number with $0 \leq p \leq 1$. The* **random graph** *$G(n,p)$ is a probability space over the set of graphs on the vertex set $\{1,2,\ldots,n\}$ determined by*

$$\mathbb{P}\{(i,j) \in G\} = p$$

for all possible (undirected) edges (i,j) with $1 \leq i,j \leq n$ and $i \neq j$ with these events mutually independent.

Similarly one also considers random graphs $G(n,m)$, where m is also a given integer with $0 \leq m \leq \binom{n}{2}$. Here one considers the set of all graphs on the set of vertices $\{1,2,\ldots,n\}$ with exactly m (undirected) edges where each of these graphs is equally likely. Due to the law of large numbers $G(n,m)$ will have very similar properties as $G(n,p)$ with $p = m/\binom{n}{2}$.

Random Graphs

Definition

A **martingale** *is a sequence* $X_0, X_1, \ldots, X_m$ *of random variables with*

$$\mathbb{E}\left(X_{i+1} | X_i, X_{i-1}, \ldots, X_0\right) = X_i \qquad (0 \leq i < m)$$

"Fair Game"

Random Graphs

Edge Exposure Martingale

$V = \{1, 2, \dots, n\}$, $E = \{e_1, e_2, \dots, e_m\}$ with $m = \binom{n}{2}$.

f ... graph theoretic function (e.g. chromatic number), $G \sim G(n, p)$

$X_0(H) := \mathbb{E}\, f(G)$

$X_1(H) := \mathbb{E}\,(f(G) | e_1 \in G \Longleftrightarrow e_1 \in H)$

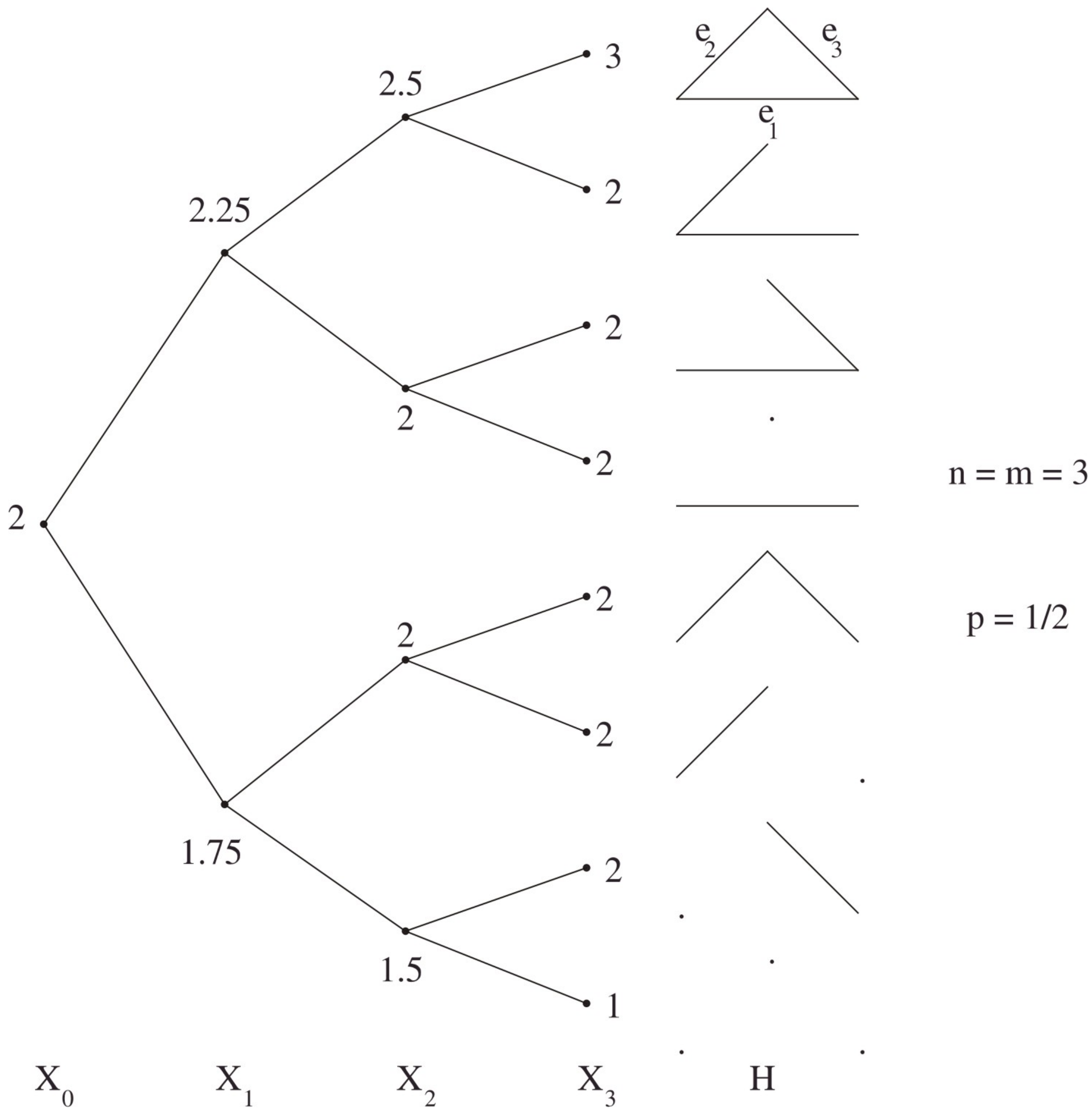
$X_2(H) := \mathbb{E}\,(f(G) | e_1 \in G \Longleftrightarrow e_1 \in H,\; e_2 \in G \Longleftrightarrow e_2 \in H)$

...

$X_m(H) := f(H)$

Random Graphs

Edge exposure martingale for the chromatic number χ



Random Graphs

Lemma

f ... graph theoretic function with the property that

- if H, H' differ in one edge then $|f(H) - f(H')| \leq 1$.

$X_0, X_1, \ldots X_m$ edge exposure martingale on $G(n,p)$

$$\Longrightarrow \quad \boxed{|X_{i+1} - X_i| \leq 1}.$$

Proof (Idea)

Pairing H, H' that differ exactly by edge e_{i+1}.

Random Graphs

Theorem [Azuma's Inequality]

Suppose that $0 = X_0, X_1, \ldots, X_m$ is a martingale with $|X_{i+1} - X_i| \leq 1$.

$$\Longrightarrow \quad \boxed{\mathbb{P}\{X_m > \lambda\sqrt{m}\} < e^{-\frac{1}{2}\lambda^2}}.$$

Proof

$$x \in [-1, 1] \quad \Longrightarrow \quad e^{\lambda x} \leq \cosh(\lambda) + x \sinh(\lambda)$$

$$Y_i := X_i - X_{i-1} \quad \Longrightarrow \quad \mathbb{E}\,(Y_i | X_{i-1}, \ldots, X_0) = 0.$$

$$\Longrightarrow \quad \mathbb{E}\left(e^{\alpha Y_i} | X_{i-1}, \ldots, X_0\right) \leq \cosh(\alpha) + 0 \cdot \sinh(\alpha) \leq e^{\frac{1}{2}\alpha^2}$$

Random Graphs

$$
\begin{aligned}
\mathbb{E}\left(e^{\alpha X_m}\right) &= \mathbb{E}\left(\prod_{i=1}^{m} e^{\alpha Y_i}\right) \\
&= \mathbb{E}\left(\prod_{i=1}^{m-1} e^{\alpha Y_i} \cdot \mathbb{E}\left(e^{\alpha Y_m} | X_{m-1}, \ldots, X_0\right)\right) \\
&\leq \mathbb{E}\left(\prod_{i=1}^{m-1} e^{\alpha Y_i}\right) \cdot e^{\frac{1}{2}\alpha^2} \\
&\leq e^{\frac{1}{2}\alpha^2 m}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}\{X_m > \lambda\sqrt{m}\} &= \mathbb{P}\{e^{\alpha X_m} > e^{\alpha\lambda\sqrt{m}}\} \\
&< \mathbb{E}\left(e^{\alpha X_m}\right) \cdot e^{-\alpha\lambda\sqrt{m}} \\
&\leq e^{\frac{1}{2}\alpha^2 m - \alpha\lambda\sqrt{m}} \\
&= e^{-\frac{1}{2}\lambda^2} \qquad (\alpha = \lambda/\sqrt{m})
\end{aligned}
$$

Random Graphs

$k = k(n) = k_0(n) - 4$, where $k_0 = k_0(n)$ is defined by

$$\binom{n}{k_0 - 1} 2^{-\binom{k_0-1}{2}} > 1 > \binom{n}{k_0} 2^{-\binom{k_0}{2}}$$

$$k = k(n) \sim 2 \log_2 n, \quad \binom{n}{k(n)} 2^{-\binom{k(n)}{2}} > n^{3+(1)}$$

Random Graphs

Lemma

Y ... maximal size of a family of edge disjoint cliques (= complete subgraph) of size k.

$$\Longrightarrow \quad \boxed{\mathbb{E}\, Y \geq \frac{n^2}{2k^4}(1+o(1))}.$$

Proof

$\mathcal{K}$ ... (random) set of k-cliques of G, $\mu := \mathbb{E}\,(|\mathcal{K}|) = \binom{n}{k} 2^{-\binom{k}{2}}$

W ... (unordered) pairs $\{A, B\}$ of k-cliques of G with $2 \leq |A \cap B| < k$.

$$\mathbb{E}\, W = \frac{\Delta}{2} \sim \frac{\mu^2 k^4}{2n^2}$$

with $\Delta = \sum_{i=2}^{k-1} \binom{k}{i}\binom{n-k}{k-i} 2^{\binom{i}{2}-\binom{k}{2}}$.

Random Graphs

$q := \mu/\Delta$.

$\mathcal{C}$ ... random subfamily of $\mathcal{K}$ with $\mathbb{P}\{A \in \mathcal{C}\} = q$.

W' ... (random) number of (unordered) pairs $\{A, B\}$, $A, B \in \mathcal{C}$ with $2 \le |A \cap B| < k$.

$$\mathbb{E}\, W' = q^2 \mathbb{E}\, W = q^2 \Delta/2.$$

Delete from $\mathcal{C}$ one set from each such pair. This gives a set $\mathcal{C}^*$ of edge disjoint k-cliques of G and

$$\mathbb{E}\, Y \ge \mathbb{E}\,(|\mathcal{C}^*|) \ge \mathbb{E}\,(|\mathcal{C}|) - \mathbb{E}\, W' = \mu q - q^2\Delta/2 = \frac{\mu^2}{2\Delta} \sim \frac{n^2}{2k^4}.$$

Random Graphs

Lemma

$\omega(G)$ ... size of the maximum clique of G

$$\Longrightarrow \quad \boxed{\mathbb{P}\{\omega(G) < k\} < e^{-(c+o(1))\frac{n^2}{(\log n)^8}}}.$$

Proof

$Y_0, Y_1, \ldots, Y_m$ ... edge exposure martingale on $G(n, \frac{1}{2})$ with Y from above.

- $|Y_i - Y_{i-1}| \leq 1$ (a single edge can add at most one clique to a family of edge disjoint cliques)

- G has no k-clique $\iff Y = 0$.

Random Graphs

Azuma's inequality: $m = \binom{n}{2} \sim \frac{1}{2}n^2$, $\mathbb{E}\, Y \geq \frac{n^2}{2k^4}(1 + o(1))$.

$$\begin{aligned}
\mathbb{P}\{\omega(G) < k\} &= \mathbb{P}\{Y = 0\} \leq \mathbb{P}\{|Y - \mathbb{E}\, Y| \leq \mathbb{E}\, Y\} \\
&\leq e^{-(\mathbb{E}\, Y)^2/2\binom{n}{2}} \leq e^{-(c'+o(1))n^2/k^8} \\
&= e^{-(c+o(1))\frac{n^2}{(\log n)^8}}.
\end{aligned}$$

Random Graphs

Theorem [Bollobas] We have, almost always in $G(n, \frac{1}{2})$,

$$\boxed{\chi(G) \sim \frac{n}{2\log_2 n}}.$$

Proof (Lower bound)

Almost always there exists no complete subgraph $K_{\lfloor 2\log_2 n \rfloor}$ in $G(n, \frac{1}{2})$.

The same holds for the complement. Consequently almost always there is no independent set of size $\lfloor 2\log_2 n \rfloor$.

$$\Longrightarrow \quad \chi(G) \geq \frac{n}{\alpha(G)} \geq \frac{n}{2\log_2 n}.$$

($\alpha(G)$ ... independece number of G.)

Random Graphs

Proof (Upper bound)

$m = \lfloor n/(\log n)^4 \rfloor$.

S .. set of m vertices

$G|_S$ ... restriction of to G to S. $G|_S$ has the distribution $G(m, \frac{1}{2})$.

$k = k(m) = k_0(m) - 4 \sim 2 \log_2$ as above.

$$\boxed{\mathbb{P}\{\alpha(G|_S) < k\} < e^{-m^{2+o(1)}}}.$$

($\alpha(G)$ has the same distribution as $\omega(G)$ for $p = \frac{1}{2}$.)

Random Graphs

There are now $\binom{n}{m} < 2^n = 2^{m^{1+o(1)}}$ such sets S. Hence

$$\mathbb{P}\{\alpha(G|_S) < k \text{ for some } m\text{-set } S\} < 2^{m^{1+o(1)}} e^{-m^{2+o(1)}} = o(1).$$

Always always *every* m vertices contain a k-element independent set.

Take G with this property.

Pull out k-element independent sets and give each a distint color until there are less than m vertices left.

Give each remaining point a distinct color.

$$\begin{aligned}\implies \chi(G) &\leq \left\lceil \frac{n-m}{k} \right\rceil + m \leq \frac{n}{k} + m \\ &= \frac{n}{2\log_2 n}(1+o(1)) + o\left(\frac{n}{\log_2 n}\right) \\ &= \frac{n}{2\log_2 n}(1+o(1)),\end{aligned}$$

This proves the upper bound (almost always).

Central Limit Theorem

Definition

A random variable Z *is said to be* **normally distributed** *(or* **Gaussian***) with mean* μ *and variance* σ^2 *if its distribution function* $F_Z(x) = \mathbb{P}\{Z \le x\}$ *is given by*

$$\boxed{F_Z(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)},$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt.$$

Notation. $\mathcal{L}(Z) = N(\mu, \sigma^2)$.

Central Limit Theorem

Density of Z:

$$f_Z(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

Characteristic function of Z:

$$\varphi_Z(t) = \mathbb{E}\, e^{itZ} = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}.$$

Central Limit Theorem

Definition

Weak convergence:

$$X_n \xrightarrow{\text{d}} X \iff \boxed{\mathbb{E}\, h(X_n) \to \mathbb{E}\, h(X)}$$

for all continuous and bounded $h : \mathbb{R} \to \mathbb{R}$

Equivalently:

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad \text{for all points of continuity of } F_X(x)$$

If X is a continuous then convergence is uniform:

$$\|F_{X_n} - F_X\|_\infty = \sup_{x\in\mathbb{R}} |F_{X_n}(x) - F_X(x)| \to 0.$$

Central Limit Theorem

Levy's Criterion

$$\boxed{X_n \xrightarrow{\mathsf{d}} X \quad\iff\quad \mathbb{E}\,e^{itX_n} \to \mathbb{E}\,e^{itX} \quad (t \in \mathbb{R})}$$

Moreover, if for all $t \in \mathbb{R}$

$$\psi(t) := \lim_{n\to\infty} \mathbb{E}\,e^{itX_n}$$

exists and $\psi(t)$ is continous at $t = 0$ then $\psi(t)$ is the characteristic function of a random variable X for which we have $X_n \xrightarrow{\mathsf{d}} X$.

Central Limit Theorem

Notation. "iid" ... independently and identically distributed

Theorem

$Y_1, Y_2, \ldots$ iid, $\mathbb{E}\, Y_i^2 < \infty$, $S_n = Y_1 + Y_2 + \ldots + Y_n$

$$\Longrightarrow \quad \boxed{\tilde{S}_n := \frac{S_n - \mathbb{E}\, S_n}{\sqrt{\mathbb{V}\, S_n}} \xrightarrow{\text{d}} N(0,1)}$$

Remark. $\mathbb{P}\{S_n \leq \mathbb{E}\, S_n + x\sqrt{\mathbb{V}\, S_n}\} \to \Phi(x)$.

Proof

$\mu = \mathbb{E}\, Y_i$, $\sigma^2 = \mathbb{V}\, Y_i = \mathbb{E}\,(Y_i^2) - (\mathbb{E}\, Y_i)^2 \quad \Longrightarrow \mathbb{E}\, S_n = n\mu$, $\mathbb{V}\, S_n = n\sigma^2$.

Central Limit Theorem

$$\varphi_{Y_i}(t) = \mathbb{E}\, e^{itY_i} = e^{it\mu - \frac{1}{2}\sigma^2 t^2\,(1+o(1))} \qquad (t \to 0)$$

$$\begin{aligned}
\Longrightarrow \boxed{\varphi_{\tilde{S}_n}(t)} &= \mathbb{E}\, e^{it\tilde{S}_n} \\
&= e^{-it\sqrt{n}\mu/\sigma} \cdot \mathbb{E}\left(e^{(it/(\sqrt{n}\sigma))(Y_1+\cdots+Y_n)}\right) \\
&= e^{-it\sqrt{n}\mu/\sigma} \cdot \left(\mathbb{E}\, e^{(it/(\sqrt{n}\sigma)Y_1}\right)^n \\
&= e^{-it\sqrt{n}\mu/\sigma} \cdot e^{it\sqrt{n}\mu/\sigma - \frac{1}{2}t^2\,(1+o(1))} \\
&= e^{-\frac{1}{2}t^2\,(1+o(1))} \to \boxed{e^{-\frac{1}{2}t^2}}.
\end{aligned}$$

+ Levy's criterion.

Central Limit Theorem

Quantified version for finite third moments $\mathbb{E}\,|Y_i|^3$:

$$\mathbb{P}\{S_n \leq n\mu + x\sqrt{n}\,\sigma\} = \Phi(x) + O\left(\frac{\mathbb{E}\,|Y_i - \mu|^3}{\sigma^3\sqrt{n}}\right).$$

uniformly for $x \in \mathbb{R}$.

Stein's Method

Lemma

$$\mathcal{L}(Z) = N(\mu, \sigma^2) \quad \Longleftrightarrow \quad \mathbb{E}\,(Z - \mu) f(Z) = \sigma^2 \mathbb{E}\, f'(Z)$$

for all smooth functions f

with $f(x) e^{-\frac{1}{2}x^2} \to 0$ as $|x| \to \infty$

and $\int_{-\infty}^{\infty} |x f(x)| e^{-\frac{1}{2}x^2}\, dx < \infty$.

Stein's Method

Proof

Wlog $\mu = 0$ and $\sigma^2 = 1$.

$\mathcal{L}(Z) = N(0,1)$

$$\begin{aligned}\implies \mathbb{E}\, f'(Z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-\frac{1}{2}x^2}\, dx \\ &= \frac{1}{\sqrt{2\pi}} f(x) e^{-\frac{1}{2}x^2} \Big|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) e^{-\frac{1}{2}x^2}\, dx \\ &= 0 + \mathbb{E}\, Z f(Z).\end{aligned}$$

Stein's Method

$\mathbb{E}\, Z f(Z) = \mathbb{E}\, f'(Z)$

$g(x)$ bounded with $\int_{-\infty}^{\infty} g(x) e^{-\frac{1}{2}x^2}\, dx = 0$

$$\Longrightarrow f(x) := e^{\frac{1}{2}x^2} \int_{-\infty}^{x} g(y) e^{-\frac{1}{2}y^2}\, dy$$
$$= -e^{\frac{1}{2}x^2} \int_{x}^{\infty} g(y) e^{-\frac{1}{2}y^2}\, dy$$

satisfies

$$\boxed{f'(x) - x f(x) = g(x)},$$

$f(x) e^{-\frac{1}{2}x^2} \to 0$ as $|x| \to \infty$ and $\int_{-\infty}^{\infty} |x f(x)| e^{-\frac{1}{2}x^2}\, dx < \infty$.

Stein's Method

$$g(x) := \mathbb{I}_{[x \le x_0]} - \Phi(x_0)$$

$$f(x) := e^{\frac{1}{2}x^2} \int_{-\infty}^{x} \left(\mathbb{I}_{[x \le x_0]} - \Phi(x_0)\right) e^{-\frac{1}{2}y^2}\, dy$$

$$f'(x) - xf(x) = \mathbb{I}_{[x \le x_0]} - \Phi(x_0)$$

$$\mathbb{E}\, f'(Z) - \mathbb{E}\, Zf(Z) = \mathbb{P}\{Z \le x_0\} - \Phi(x_0)$$

$$\Longrightarrow 0 = \mathbb{P}\{Z \le x_0\} - \Phi(x_0)$$

$$\Longrightarrow \mathcal{L}(Z) = N(0,1).$$

Stein's Method

Notation. h bounded, absolutely integrable:

$$Nh = \mathbb{E}\, h(Z/\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x/\sigma) e^{-\frac{1}{2}x^2}\, dx.$$

Lemma h ... bounded with bounded first derivative.

Then there exists f with bounded second derivative with

$$\boxed{\sigma^2 f'(w) - w f(w) = h(w/\sigma) - Nh} \quad (\textbf{Stein's equation})$$

and

$$\boxed{\|f''\|_\infty \leq K_{\mathsf{univ}} \cdot \Big(\|h\|_\infty + \|h'\|_\infty \Big)}$$

for a universal constant $K_{\mathsf{univ}} > 0$.

Stein's Method

Proof

The solution of Stein's equation has been already determined (see the previous lemma).

$$f(x) = e^{\frac{1}{2}x^2} \int_{-\infty}^{x} (h(y/\sigma) - Nh)\, e^{-\frac{1}{2}y^2}\, dy.$$

Wlog $\sigma^2 = 1$

$\overline{h}(x) := h(x) - Nh$. ($\Longrightarrow \quad N\overline{h} = 0,\ \|\overline{h}\|_\infty \leq 2\|h\|$.)

Stein's Method

Abbreviations:

$$H_0 = \|\overline{h}\|_\infty,$$

$$H_1 = \|\overline{h}'\|_\infty = \|h'\|_\infty$$

$$F_0 = \|f\|_\infty, \qquad F_1\|f'\|_\infty,$$

$$F_{11} = \|(xf)'\|_\infty,$$

$$F_2 = \|f''\|_\infty,$$

$$c_1 = \sup_{x\geq 0}\left|x\left(1 - xe^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}u^2}\,du\right)\right|,$$

$$c_2 = \sup_{x\geq 0}\, e^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}u^2}\,du,$$

$$c_3 = \sup_{x\geq 0}\, x\,e^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}u^2}\,du = 1$$

Stein's Method

1. $F_0 \leq c_2 H_0$,

2. $F_1 \leq 2H_0$,

3. $F_{11} \leq (c_1 + c_2)H_0 + H_1$,

4. $F_2 \leq (c_1 + c_2)H_0 + 2H_1$.

4. implies upper bound for $\|f''\|_\infty$ and proves the lemma.

Stein's Method

1.

$$f(x) = -e^{\frac{1}{2}x^2} \int_x^\infty \left(\overline{h}(y)\right) e^{-\frac{1}{2}y^2}\, dy \quad (x > 0)$$

$$\implies \quad F_0 \leq c_2 H_0.$$

Recall: $c_2 = \sup_{x \geq 0} e^{\frac{1}{2}x^2} \int_x^\infty e^{-\frac{1}{2}u^2}\, du$

Stein's Method

2.

$$f'(x) = xf(x) + \overline{h}(x) \quad \Longrightarrow \quad F_1 \leq \|xf(x)\|_\infty + H_0.$$

$$xf(x) = -xe^{\frac{1}{2}x^2} \int_x^\infty \left(\overline{h}(y)\right) e^{-\frac{1}{2}y^2}\, dy \quad (x > 0)$$

$$\Longrightarrow \quad \|xf(x)\|_\infty \leq c_3 H_0 = H_0.$$

$$\Longrightarrow \quad F_1 \leq 2H_0$$

Stein's Method

3.

$$(xf(x))' = f(x) + x^2 f(x) + x\overline{h}(x), \quad F_0 \le c_2 H_0.$$

$$\begin{aligned}
x^2 f(x) + x\overline{h}(x) &= -x^2 e^{\frac{1}{2}x^2} \int_x^\infty \overline{h}(y) e^{-\frac{1}{2}y^2}\, dy + x\overline{h}(x) \\
&= -x^2 e^{\frac{1}{2}x^2} \int_x^\infty \overline{h}(y) e^{-\frac{1}{2}y^2}\, dy + x e^{\frac{1}{2}x^2} \int_x^\infty y\overline{h}(y) e^{-\frac{1}{2}y^2}\, dy \\
&- x e^{\frac{1}{2}x^2} \int_x^\infty y\overline{h}(y) e^{-\frac{1}{2}y^2}\, dy + xh(x) \\
&= x^2 e^{\frac{1}{2}x^2} \int_x^\infty \overline{h}(y) \left(\frac{y}{x} - 1\right) e^{-\frac{1}{2}y^2}\, dy \\
&- x e^{\frac{1}{2}x^2} \int_x^\infty \overline{h}'(y) e^{-\frac{1}{2}y^2}\, dy.
\end{aligned}$$

$$\implies \quad \|x^2 f(x) + x\overline{h}(x)\|_\infty \le c_1 H_0 + H_1$$

$$\implies \quad F_{11} \le (c_1 + c_2) H_0 + H_1.$$

Stein's Method

4.

$f'(x) = \overline{h}(x) + xf(x).$

$$\begin{aligned}
|f'(x+t) - f'(x)| &= |\overline{h}(x+t) - \overline{h}(x) + (x+t)f(x+t) - xf(x)| \\
&\le |t|H_1 + |t|F_{11} \\
&\le |t|\left((c_1 + c_2)H_0 + 2H_1\right).
\end{aligned}$$

$$\Longrightarrow \quad F_2 \le (c_1 + c_2)H_0 + 2H_1.$$

Stein's Method

Norm $\|h\|$

$$\|h\| := K_{\mathsf{univ}} \cdot \Big(\|h\|_\infty + \|h'\|_\infty \Big)\,.$$

This norm is maybe a little unusual but it perfectly fits to Stein's method.

Distance of two probability measures P and Q

$$d_1(P,Q) := \sup_{\|h\| \le 1} |\mathbb{E}\, h(X) - \mathbb{E}\, h(Y)|$$

where $\mathcal{L}(X) = P$ and $\mathcal{L}(Y) = Q$.

Remark

$$d_1(\mathcal{L}(X_n), \mathcal{L}(X)) \to 0 \quad \Longleftrightarrow \quad X_n \xrightarrow{\mathsf{d}} X$$

Stein's Method

General situation

W can be composed in the following way

(I ... finite index set, K_i ... finite index set ($i \in I$)

$X_i, W_i, Z_i, Z_{ik}, W_{ik}, V_{ik}$ square integrable, $i \in I$ and $k \in K_i$):

1. $\boxed{W = \sum_{i \in I} X_i}$, 2. $\mathbb{E}\, X_i = 0 \quad (i \in I)$, 3. $\mathbb{V}\, W = 1$,

4. $\boxed{W = Z_i + W_i} \quad (i \in I)$, W_i is independent of X_i,

5. $\boxed{Z_i = \sum_{k \in K_i} Z_{ik}} \quad (i \in I)$,

6. $\boxed{W_i = W_{ik} + V_{ik}} \quad (i \in I,\ k \in K_i)$,

7. W_{ik} is independent of the pair $(X_i, Z_{ik}) \quad (i \in I,\ k \in K_i$.

Stein's Method

Theorem

Suppose that a random variable W decomposes as introduced above. Then

$$d_1(\mathcal{L}(W), N(0,1)) \leq \frac{1}{2}\sum_{i\in I}\mathbb{E}\left(|X_i|Z_i^2\right) + \sum_{i\in I}\sum_{k\in K_i}\Big(\mathbb{E}\,|X_i Z_{ik} V_{ik}| + \mathbb{E}\,|X_i Z_{ik}|\cdot\mathbb{E}\,|Z_i + V_{ik}|\Big).$$

Remark. If the right hand side goes to 0 then $W \xrightarrow{\text{d}} N(0,1)$.

Stein's Method

$Y_1, Y_2, \ldots$ iid, $\mathbb{E}\,|Y_i|^3 < \infty$

$\mu = \mathbb{E}\,Y_i, \;\; \sigma^2 = \mathbb{V}\,Y_i$

$$X_i := \frac{Y_i - \mu}{\sqrt{n}\,\sigma} \qquad \text{(also iid)}$$

$$W := X_1 + \cdots + X_n$$

$$\implies \quad \boxed{W = \frac{Y_1 + \cdots + Y_n - \mu n}{\sqrt{n}\,\sigma}}$$

Stein's Method

$$K_i = \{i\}$$
$$Z_i = X_i,$$
$$W_{ik} = X_k,$$
$$V_{ik} = 0.$$

$$\mathbb{E}\left(|X_i| Z_i^2\right) = \mathbb{E}\,|X_i|^3,$$
$$\mathbb{E}\,|X_i Z_{ik} V_{ik}| = 0,$$
$$\mathbb{E}\,|X_i Z_{ik}| \cdot \mathbb{E}\,|Z_i + V_{ik}| \ = \mathbb{E}\,X_i^2 \cdot \mathbb{E}\,|X_i| = \frac{1}{n}\mathbb{E}\,|X_i|.$$

$$\Longrightarrow \quad d_1\left(\mathcal{L}(W), N(0,1)\right) \leq \frac{1}{\sigma^3\sqrt{n}}\left(\frac{1}{2}\mathbb{E}\left(|Y_i - \mu|^3\right) + \mathbb{E}\,|Y_i - \mu|\right).$$

Stein's Method

Proof

Goal:

$$\begin{aligned}&\left|\mathbb{E}\, W f(W) - \mathbb{E}\, f'(W)\right| \\ &\le \|f''\|_\infty \cdot \left(\frac{1}{2}\sum_{i\in I} \mathbb{E}\, \left(|X_i| Z_i^2\right)\right. \\ &\qquad\left. + \sum_{i\in I}\sum_{k\in K_i}\Big(\mathbb{E}\, |X_i Z_{ik} V_{ik}| + \mathbb{E}\, |X_i Z_{ik}| \cdot \mathbb{E}\, |Z_i + V_{ik}|\Big)\right).\end{aligned}$$

Stein's Method

Choose h with $\|h\| \leq 1$ and use $f(x)$ with

$$f'(x) - xf(x) = h(x) - Nh = \overline{h}(x)$$

(Recall: $Nh = \mathbb{E}\, h(Z)$ with $\mathcal{L}(Z) = N(0, 1)$).

$$\begin{aligned}
\implies \quad \mathbb{E}\, h(W) - \mathbb{E}\, h(Z) &= \mathbb{E}\, f'(W) - \mathbb{E}\, W f(W) \\
\implies \quad |\mathbb{E}\, h(W) - \mathbb{E}\, h(Z)| &= |\mathbb{E}\, f'(W) - \mathbb{E}\, W f(W)| \\
&\leq \|f''\|_\infty \cdot \left(\cdots\cdots \right) \\
&\leq \left(\cdots\cdots \right)
\end{aligned}$$

for all h with $\|h\| \leq 1$. (Recall that $\|f''\|_\infty \leq \|h\| \leq 1$.)

Stein's Method

Rewrite the difference:

$$\begin{aligned}\mathbb{E}\,W f(W) - \mathbb{E}\,f'(W) &= \mathbb{E}\,W f(W) - \sum_{i\in I} \mathbb{E}\,(X_i Z_i f'(W_i)) \\ &\quad + \sum_{i\in I} \mathbb{E}\,(X_i Z_i f'(W_i) - \sum_{i\in I}\sum_{k\in K_i} \mathbb{E}\,(X_i Z_{ik})\,\mathbb{E}\,f'(W_{ik}) \\ &\quad + \sum_{i\in I}\sum_{k\in K_i} \mathbb{E}\,(X_i Z_{ik}) \left(\mathbb{E}\,f'(W_{ik}) - \mathbb{E}\,f'(W)\right),\end{aligned}$$

Here we used

$$\begin{aligned}1 = \mathbb{E}\,W^2 &= \sum_{i\in I} \mathbb{E}\,(X_i W) \\ &= \sum_{i\in I} \mathbb{E}\,(X_i)\mathbb{E}\,(W_i) + \sum_{i\in I} \mathbb{E}\,(X_i Z_i) \\ &= \sum_{i\in I} \mathbb{E}\,(X_i Z_i) \\ &= \sum_{i\in I}\sum_{k\in K_i} \mathbb{E}\,(X_i Z_{ik}).\end{aligned}$$

Stein's Method

First by Taylor's expansion:
$f(x+t) = f(x) + tf'(x) + \frac{1}{2}t^2 f''(x+\theta t)$ for some $\theta \in [0,1]$.

$$
\begin{aligned}
Wf(W) &= \sum_{i\in I} X_i f(W) \\
&= \sum_{i\in I} X_i \left(f(W_i) + Z_i f'(W_i) + \frac{1}{2} Z_i^2 f''(W_i + \theta_i Z_i) \right)
\end{aligned}
$$

X_i and W_i are independent $\Longrightarrow \mathbb{E}\,(X_i f(W_i)) = \mathbb{E}\,X_i \cdot \mathbb{E}\,f(W_i)$

$$
\Longrightarrow \left| \mathbb{E}\,Wf(W) - \sum_{i\in I} \mathbb{E}\,(X_i Z_i f'(W_i)) \right| \le \frac{\|f''\|}{2} \cdot \sum_{i\in I} \mathbb{E}\,(|X_i| Z_i^2).
$$

Stein's Method

Second:

$$X_i Z_i f'(W_i) = \sum_{k\in K_i} X_i Z_{ik} f'(W_i)$$
$$= \sum_{k\in K_i} X_i Z_{ik} \Big(f'(W_{ik} + V_{ik} f''(W_{ik} + \theta_{ik} V_{ik}) \Big)$$

$$\Longrightarrow \left| \sum_{i\in I} \mathbb{E}\,(X_i Z_i f'(W_i) - \sum_{i\in I} \sum_{k\in K_i} \mathbb{E}\,(X_i Z_{ik})\,\mathbb{E}\,f'(W_{ik}) \right|$$
$$\le \|f''\| \cdot \sum_{i\in I} \sum_{k\in K_i} \mathbb{E}\,|X_i Z_{ik} V_{ik}|$$

Stein's Method

Third:

$W_{ik} = W_i - V_{ik} = W - Z_i - V_{ik}$:

$$f'(W_{ik}) = f'(W) - (Z_i + V_{ik}) f''(W - \theta(Z_i + V_{ik}))$$

$$\Longrightarrow \left| \mathbb{E} f'(W_{ik}) - \mathbb{E} f'(W) \right| \leq \|f''\| \cdot \mathbb{E} |(Z_i + V_{ik})|.$$

Putting the three estimates together we get the proposed estimate for $|\mathbb{E} W f(W) - \mathbb{E} f'(W)|$.

Stein's Method

Simplified Version (*dissociated* composition: $Z_{ik} = X_k, i \in K_i \subseteq I$)

... more precisely:

1. $\boxed{W = \sum_{i \in I} X_i}$, 2. $\mathbb{E}\, X_i = 0 \quad (i \in I)$, 3. $\mathbb{V}\, W = 1$,

4. $\boxed{W = Z_i + W_i}$ $(i \in I)$, W_i is independent of X_i,

5. $\boxed{Z_i = \sum_{k \in K_i} X_k}$, $\boxed{W_i = \sum_{k \in I \setminus K_i} X_k}$ $(i \in I)$,

6. $\boxed{W_i = W_{ik} + V_{ik}}$ $\boxed{V_{ik} = \sum_{j \in K_k \setminus K_i} X_j}$ $(i \in I,\ k \in K_i)$,

7. $\boxed{W_{ik} = W - \sum_{j \in K_i \cup K_k} X_j}$ is independent of (X_i, X_k) $(i \in I,\ k \in K_i)$.

Stein's Method

Dependency Graph $\mathcal{L}$

I ... vertices, X_i random variable $(i \in I)$

- If A, B are disjoint subsets of I that are not interconnected by an edge then two subsystems $(X_i : i \in A)$ and $(X_j : j \in B)$ are independent.

Application to Stein's Theorem

$$K_i := \{\text{neighbors of } i \text{ in } \mathcal{L}\}$$

$$W_i = \sum_{k \in I \setminus K_i} X_k \Longrightarrow X_i, W_i \text{ ind.}$$

$$W_{ik} = \sum_{j \in I \setminus (K_i \cup K_k)} X_j \Longrightarrow (X_i, X_k), W_{ik} \text{ ind.}$$

Stein's Method

Theorem

Suppose that a random variable W decomposes in a dissociated way that is induced by a dependency graph.

Then

$$\boxed{d_1\left(\mathcal{L}(W), N(0,1)\right) \leq 2 \sum_{i \in I} \sum_{j,k \in K_i} \left(\mathbb{E}\left(|X_i X_j X_k|\right) + \mathbb{E}\left(|X_i X_j|\right) \mathbb{E}\,|X_k| \right)}.$$

Stein's Method

Proof

$$Z_i = \sum_{k \in K_i} X_k$$

$$\implies \quad |X_i| Z_i^2 = |X_i| \sum_{j,k \in K_i} X_j X_k \leq \sum_{j,k \in K_i} |X_i X_j X_k|$$

$$\implies \quad \sum_{i \in I} \mathbb{E}\left(|X_i| Z_i^2 \right) \leq \sum_{i \in I} \sum_{j,k \in K_i} \mathbb{E}\left(|X_i X_j X_k| \right).$$

Stein's Method

$$Z_{ik} = X_k$$

$$V_{ik} = \sum_{j \in K_k \setminus K_i} X_j$$

$$\implies \quad |X_i Z_{ik} V_{ik}| \le \sum_{j \in K_k} |X_i X_k X_j|$$

$$\begin{aligned}
\implies \quad \sum_{i \in I} \sum_{k \in K_i} \mathbb{E}\,|X_i Z_{ik} V_{ik}| &\le \sum_{i \in I} \sum_{k \in K_i} \sum_{j \in K_k} \mathbb{E}\,|X_i X_k X_j| \\
&= \sum_{k \in I} \sum_{i \in K_k} \sum_{j \in K_k} \mathbb{E}\,|X_i X_k X_j| \\
&= \sum_{k \in I} \sum_{i,j \in K_k} \mathbb{E}\,|X_i X_k X_j|.
\end{aligned}$$

Stein's Method

$$Z_{ik} = X_k$$

$$Z_i + V_{ik} = \sum_{j \in K_k \cup K_i} X_j$$

$$\Longrightarrow \quad \mathbb{E}\,|X_i Z_{ik}| \cdot \mathbb{E}\,|Z_i + V_{ik}| \leq \sum_{j \in K_k} \mathbb{E}\,|X_i X_k| \cdot \mathbb{E}\,|X_j| + \sum_{j \in K_i} \mathbb{E}\,|X_i X_k| \cdot \mathbb{E}\,|X_j|$$

$$\begin{aligned}\Longrightarrow \quad & \sum_{i \in I} \sum_{k \in K_i} \mathbb{E}\,|X_i Z_{ik}| \cdot \mathbb{E}\,|Z_i + V_{ik}| \\ & \leq \sum_{i \in I} \sum_{k \in K_i} \sum_{j \in K_k} \mathbb{E}\,|X_i X_k| \cdot \mathbb{E}\,|X_j| + \sum_{i \in I} \sum_{k \in K_i} \sum_{j \in K_i} \mathbb{E}\,|X_i X_k| \cdot \mathbb{E}\,|X_j| \\ & = 2 \sum_{i \in I} \sum_{j,k \in K_i} \mathbb{E}\,\Big(|X_i X_j|\Big)\,\mathbb{E}\,|X_k|\end{aligned}$$

Stein's Method

$$\implies \quad \frac{1}{2}\sum_{i\in I}\mathbb{E}\left(|X_i|Z_i^2\right)+\sum_{i\in I}\sum_{k\in K_i}\Bigg(\mathbb{E}\,|X_iZ_{ik}V_{ik}|+\mathbb{E}\,|X_iZ_{ik}|\cdot\mathbb{E}\,|Z_i+V_{ik}|\Bigg)$$

$$\leq 2\sum_{i\in I}\sum_{j,k\in K_i}\Bigg(\mathbb{E}\left(|X_iX_jX_k|\right)+\mathbb{E}\left(|X_iX_j|\right)\mathbb{E}\,|X_k|\Bigg),$$

Application to Random Graphs

$G(n,p)$ ... random graph

$\mathcal{T}$ ... triangles in $G(n,p)$

$I = \{i = (i_1, i_2, i_3) : 1 \leq i_1 < i_2 < i_3 \leq n\}$

$$\boxed{X = |\mathcal{T}| = \sum_{i \in I} \mathbb{I}_{[i=(i_1,i_2,i_3)\in\mathcal{T}]}}$$ number of triangles in $G(n,p)$

$$\mathbb{E}\, X = \binom{n}{3} p^3$$
$$\sigma^2 := \mathbb{V}\, X = \binom{n}{3} p^3(1-p^3) + 12\binom{n}{4} p^5(1-p).$$

Simplification: $\boxed{p \leq \frac{1}{2},\ np \to \infty \Longrightarrow \mathbb{E}\, X \to \infty,\ \mathbb{V}\, X \to \infty}$

Application to Random Graphs

$$\boxed{X_i := \frac{1}{\sigma}\left(\mathbb{I}_{[i=(i_1,i_2,i_3)\in\mathcal{T}]} - p^3\right)}$$

$$\boxed{W = \sum_{i\in I} X_i = \frac{X - \mathbb{E}\,X}{\sqrt{\mathbb{V}\,X}}}.$$

Dependency graph $\mathcal{L}$.

$$V(\mathcal{L}) = I$$

$$E(\mathcal{L}) = \{(i,j) : |\{i_1,i_2,i_3\} \cap \{j_1,j_2,j_3\}| \geq 2\}$$

$$K_i = \{k = (k_1,k_2,k_3) \in I : |\{i_1,i_2,i_3\} \cap \{k_1,k_2,k_3\}| \geq 2\}.$$

Application to Random Graphs

$$\sum_{i\in I}\sum_{j,k\in K_i}\left(\mathbb{E}\left(|X_iX_jX_k|\right)+\mathbb{E}\left(|X_iX_j|\right)\mathbb{E}\,|X_k|\right)=???$$

(Recall: $X_i := \frac{1}{\sigma}\left(\mathbb{I}_{[i=(i_1,i_2,i_3)\in\mathcal{T}]} - p^3\right))$

- $i = j = k$:

$$\mathbb{E}\left(|X_iX_jX_k|\right)=\mathbb{E}\left(|X_i|^3\right)=\frac{1}{\sigma^3}\left(p^3(1-p^3)^3+(1-p^3)p^9\right)\le\frac{2p^3}{\sigma^3}$$

- other case are similar ...

Application to Random Graphs

$$\implies \sum_{i\in I}\sum_{j,k\in K_i}\left(\mathbb{E}\left(|X_iX_jX_k|\right)+\mathbb{E}\left(|X_iX_j|\right)\mathbb{E}\,|X_k|\right) = O\left(\frac{1}{\sigma^3}\left(n^3p^3(1+np^2)^2\right)\right)$$

$$\mathbb{V}\,X = \sigma^2 \geq c\,n^3p^3(1+np^2)$$

$$\begin{aligned}\implies \quad d_1\left(\mathcal{L}(W), N(0,1)\right) &= O\left(\frac{n^3p^3(1+np^2)^2}{n^{9/2}p^{9/2}((1+np^2)^{3/2}}\right)\\ &= O\left((np)^{-3/2}(1+np)^{1/2}\right)\\ &\to 0.\end{aligned}$$

Application to Random Graphs

Theorem

Suppose that $0 < p \leq \frac{1}{2}$ and $np \to \infty$.

Then the number of triangles in a random graph $G(n, p)$ satisfies a central limit theorem.

Remark. Similar properties hold for general subgraph counting.