Bivariate Linear Recurrences in Enumeration – Asymptotics and Application

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TU Graz, Austria

Enumerative combinatorics and effective aspects of differential equations
Combinatoire énumérative et aspects effectifs des équations différentielles

February 24–28, 2025

Outline

1. Part I: Bivariate Recurrences
2. Part II: The Stretched Exponential Method
3. Part III: Applications in Computer Science and Mathematical Biology

Part I
Bivariate Recurrences

A counting problem

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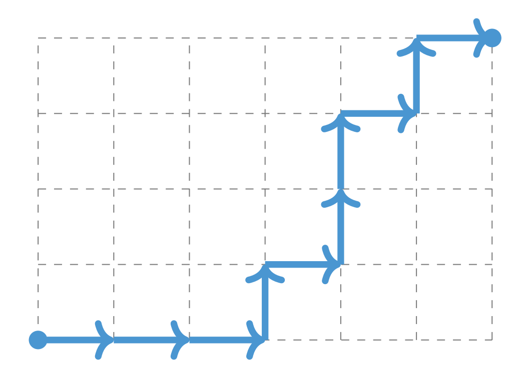

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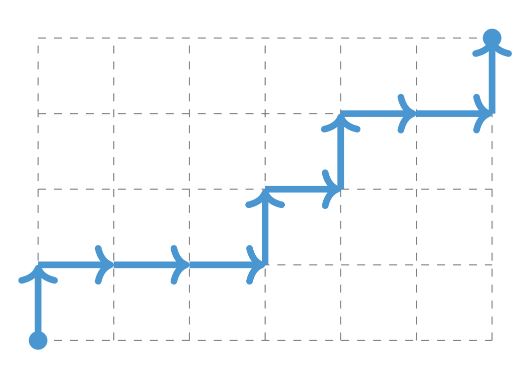

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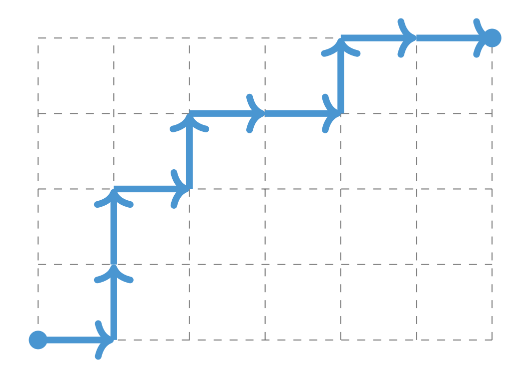

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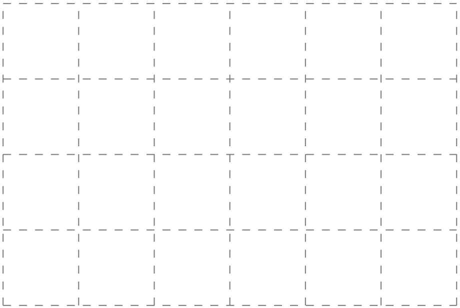

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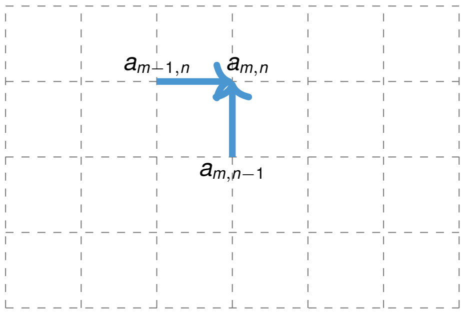

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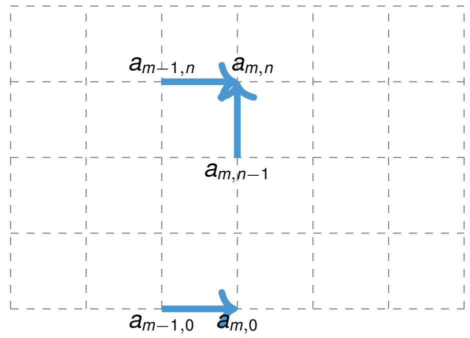

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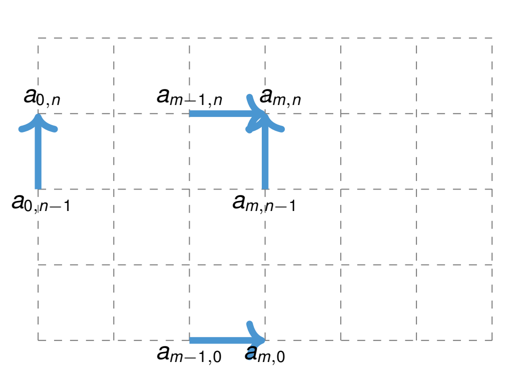

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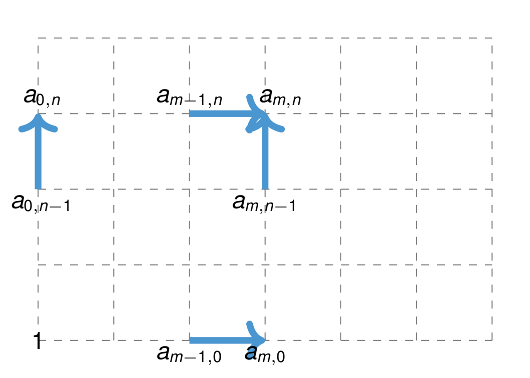

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1	5	15	35	70	126	210
1	4	10	20	35	56	84
1	3	6	10	15	21	28
1	2	3	4	5	6	7
1	1	1	1	1	1	1

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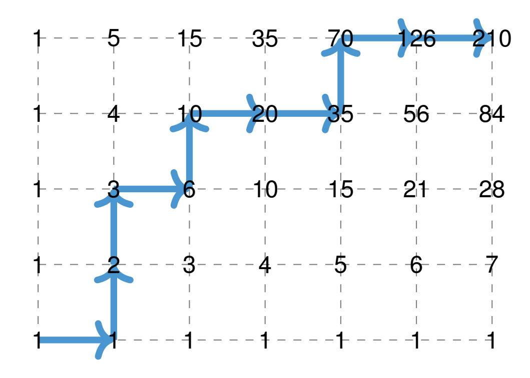

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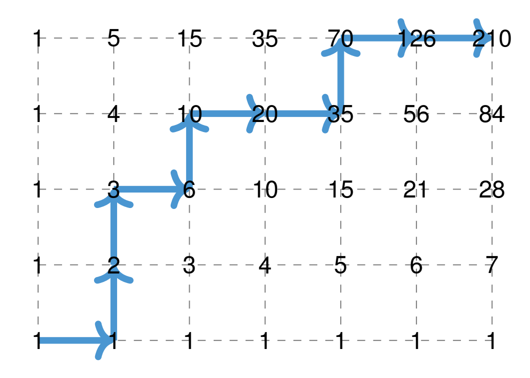

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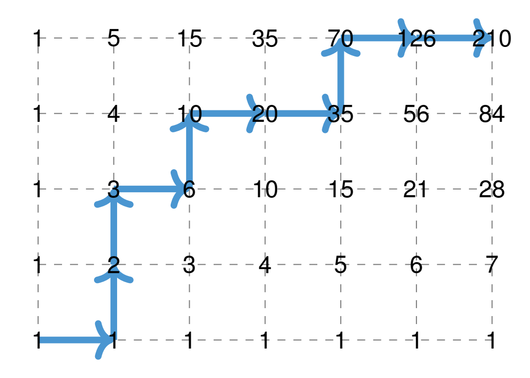

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- But what happens if we change the domain and add polynomial weights?

$$a_{m,n} = (n + 1)a_{m-1,n} + a_{m,n-1} \qquad \text{for } m \geq n > 0$$

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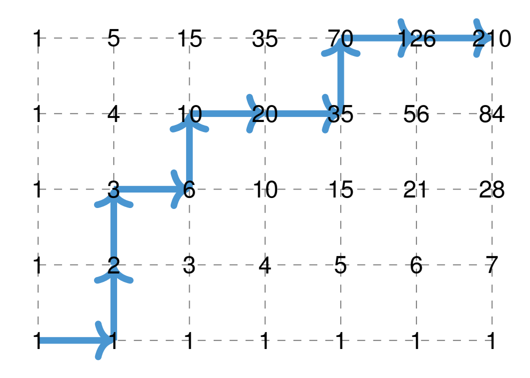

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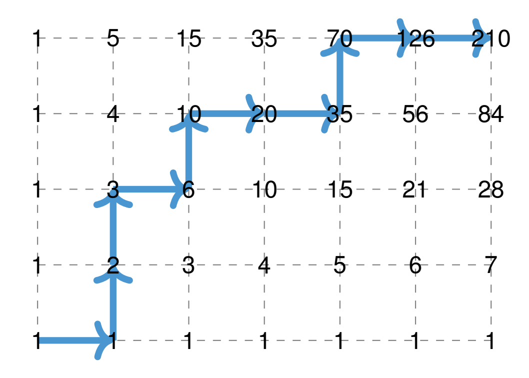

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$\rightarrow$ In this course you will learn what *asymptotic* information we can deduce!

Asymptotic counting

Landau notation

Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, $b_n > 0$ be two sequences.

- $a_n = \mathcal{O}(b_n)$ if $\limsup_{n\to\infty} \frac{|a_n|}{b_n} < \infty$
- $a_n = \Theta(b_n)$ if $0 < \liminf_{n\to\infty} \frac{|a_n|}{b_n}$ and $\limsup_{n\to\infty} \frac{|a_n|}{b_n} < \infty$
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Examples:

Stirling's formula

- $n! = \mathcal{O}(n^n)$
- $n! = \Theta\left(n^{n+1/2}\, e^{-n}\right)$
- $n! \sim \sqrt{2\pi n} n^n e^{-n}$

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- Approximations
- Universality like $n^{-1/2}$
- Large-scale behavior:
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  - phase transitions
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Allows to prove

- transcendence (i.e., non-algebraic, non-D-finite) [Bostan, Raschel, Salvy 2014]
- ambiguity of context-free languages [Flajolet 1987]
- transience of drunkard walk in 3D and higher [Pólya 1921]
- capacity of a channel/needed bits for encoding [MacKay 2003]

Types of recurrences

Linear recurrences

In this course we will only consider finite order linear recurrences

$$a_{m,n} = c_1 a_{m+i_1,n+j_1} + c_2 a_{m+i_2,n+j_2} + \cdots + c_d a_{m+i_d,n+j_d} \qquad \text{for } (m,n) \in \mathcal{C} \tag{1}$$

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Let $H = \{(i_1, j_1), \ldots, (i_d, j_d)\}$ and $\mathcal{C} = \mathbb{Z}^2_{\geq 0}$. Then (1) has a unique solution if $\mathbb{R}^2_{\geq 0} \cap \operatorname{conv} H = \emptyset$.

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- The recurrence $a_{m,n} = a_{m-1,n} + a_{m,n-1}$ has a unique solution in the following two cones:
  1. For $m, n > 0$ we have $H = \{(-1,0),(0,-1)\}$

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- But **not** the recurrence $b_{m,n} = b_{m-1,n} + b_{m,n-1} + b_{m+1,n} + b_{m,n+1}$ for $m, n > 0$.
  Here $H = \{(\pm 1, 0), (0, \pm 1)\}$

Interpretation as paths

General shape

$$a_{m,n} = c_1 a_{m+i_1,n+j_1} + c_2 a_{m+i_2,n+j_2} + \cdots + c_d a_{m+i_d,n+j_d}$$

How can we reach (m, n)?

- From $(m + i_1, n + j_1)$ with step $(-i_1, -j_1)$, or
- from $(m + i_2, n + j_2)$ with step $(-i_2, -j_2)$, or
- ...
- from $(m + i_d, n + j_d)$ with step $(-i_d, -j_d)$.

Knight variation

Let $a_{0,0} = 1$ and for $m, n \geq 0$:

$$a_{m,n} = a_{m+1,n-2} + 2a_{m-2,n+1} + 3a_{m-1,n} + 4a_{m,n-1}$$

The four steps are

$$(-1, 2), (2, -1), (1, 0), (0, 1)$$

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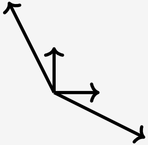

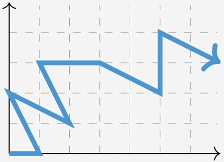

Interpretation as paths

General shape

$$a_{m,n} = c_1 a_{m+i_1,n+j_1} + c_2 a_{m+i_2,n+j_2} + \cdots + c_d a_{m+i_d,n+j_d}$$

How can we reach (m,n)?

- From $(m+i_1, n+j_1)$ with step $(-i_1, -j_1)$, or
- from $(m+i_2, n+j_2)$ with step $(-i_2, -j_2)$, or
- ...
- from $(m+i_d, n+j_d)$ with step $(-i_d, -j_d)$.

What is the weight of a path ending at (m,n)?

1. Each step has a weight:
   - Step $(-i_1, -j_1)$ has weight c_1
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   - ...
   - Step $(-i_d, -j_d)$ has weight c_d
2. The weight of a path is the product of the weights of its steps.

Knight variation

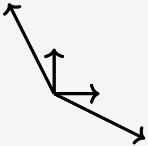
Let $a_{0,0} = 1$ and for $m, n \geq 0$:

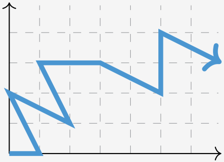
$$a_{m,n} = a_{m+1,n-2} + 2a_{m-2,n+1} + 3a_{m-1,n} + 4a_{m,n-1}$$

The four steps are

$$(-1,2), (2,-1), (1,0), (0,1)$$

with the weights $1, 2, 3, 4$, resp.





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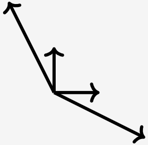

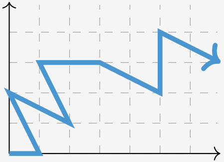

- All weights to 1: OEIS A356692
  Pascal-like triangle; family of permutations?
- Asymptotics not known! (Similar models: [Bostan, Bousquet-Mélou, Melczer 2021])
- Knight only: [Bousquet-Mélou, Petkovšek 2000]

Two-dimensional paths with a time dimension

Consider the recurrence

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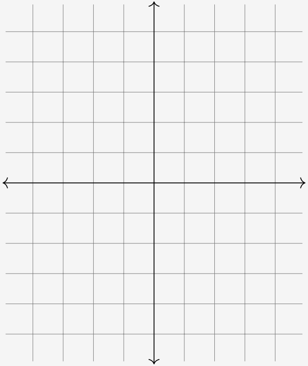

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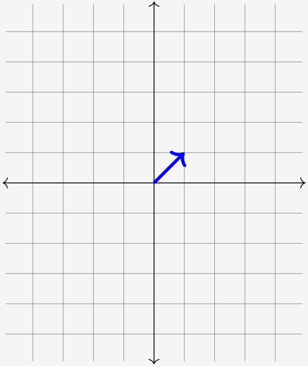

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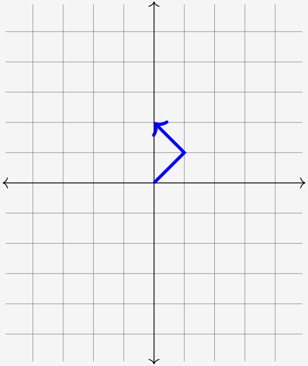

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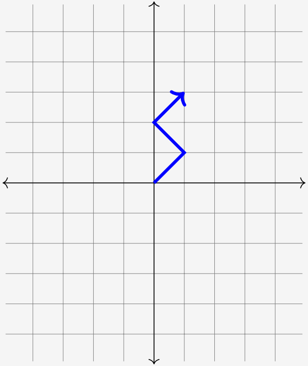

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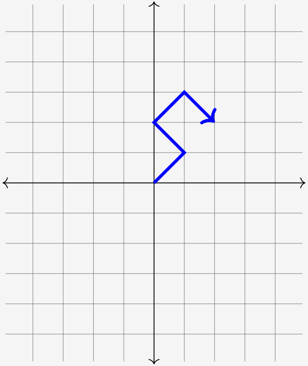

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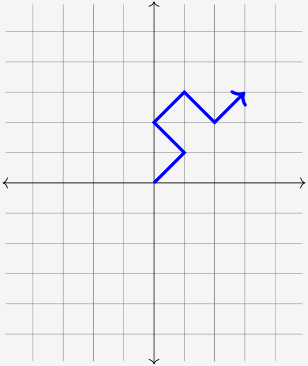

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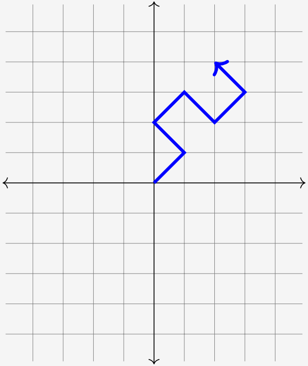

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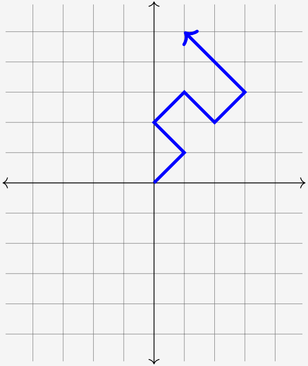

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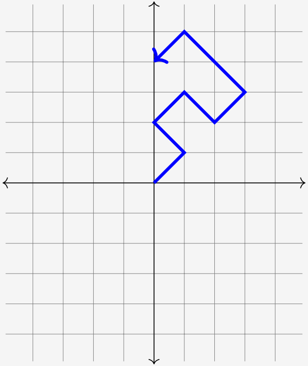

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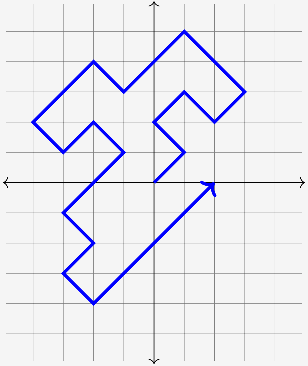

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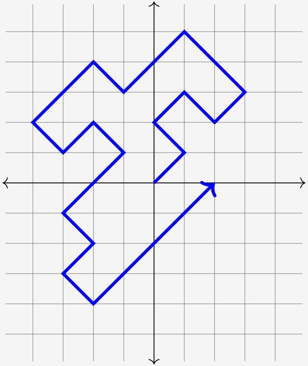

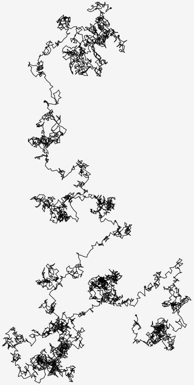

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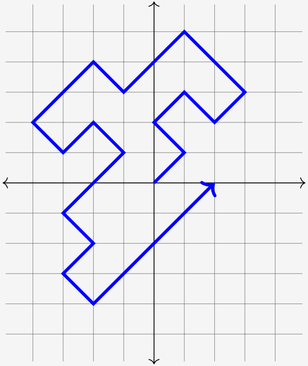

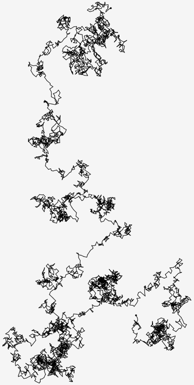

Popular models:

- Starting point: $(0,0)$
- Small steps: $\mathcal{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$

Current research: 2D lattice paths in convex and nonconvex cones

Example: King walks

$$a_{m,n;k+1} = a_{m-1,n-1;k} + a_{m-1,n;k} + a_{m-1,n+1;k} + a_{m,n-1;k} + a_{m,n+1;k} + a_{m+1,n-1;k} + a_{m+1,n;k} + a_{m+1,n+1;k}$$

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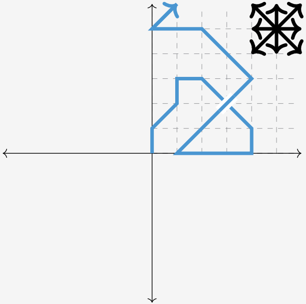
Quarter plane

$\mathcal{Q} = \{(m, n) : m \geq 0 \text{ and } n \geq 0\}$

[Bousquet-Mélou, Mishna 2010]



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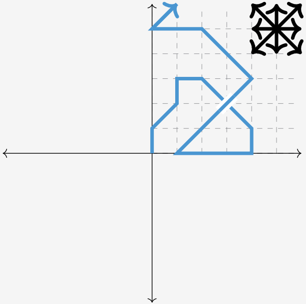

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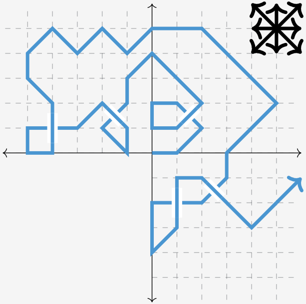
[Bousquet-Mélou 2016]



$$a_{0,0;k} \sim \alpha_9 \frac{\Gamma(2/3)}{\pi} \frac{8^k}{k^{5/3}}, \text{ where } \alpha_9 \approx 1.419$$

[Bousquet-Mélou, W 2024]

More families of multivariate recurrences

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where $p_{w,0,k} = \mathbb{1}_{w=k}$ and $p_{0,b,k} = \mathbb{1}_{k=0}$ for $w, b, k \geq 0$.

We will focus on bivariate recurrences

General assumptions on initial and boundary conditions

Let $(a_{m,n})_{(m,n)\in\mathcal{C}}$ be a recursively defined sequence on a cone $\mathcal{C} \subseteq \mathbb{Z}^2$. Throughout this course we assume

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2. The same recurrence on the triangular cone $\mathcal{C} = \{(m,n) \ : \ m \geq n \geq 0\} =$ :

$$b_{m,n} = b_{m-1,n} + b_{m,n-1} \qquad \text{for } m \geq n \geq 0$$

What we will study in this course: the diagonal entry $a_{n,n}$

Recurrences we will study

$$a_{m,n} = E(m,n)a_{m-1,n} + N(m,n)a_{m,n-1}$$

Main goal

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	$E(m,n)$	$N(m,n)$	Domain		$a_{n,n}$	Description
(1)	1	1	$m, n \geq 0$		$\binom{2n}{n}$	Binomial coefficients
(2)	1	1	$m \geq n \geq 0$		$\frac{1}{n+1}\binom{2n}{n}$	Catalan numbers
(3)	$n+1$	1	$m, n \geq 0$		$S(2n+1, n+1)$	Stirling numbers 2nd kind
(4)	$n+1$	1	$m \geq n \geq 0$		$\Theta\left(n! 4^n e^{3a_1 n^{1/3}} n\right)$	Compacted binary trees

(In the last case, $a_1 \approx -2.338$ is the largest root of the Airy function $\mathrm{Ai}(x)$ that is the unique function satisfying $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ and $\lim_{x\to\infty} \mathrm{Ai}(x) = 0$.)

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Outline of the course:

- Today: Solve Examples (1)–(3)
- Wednesday: Stretched exponential method to solve Example (4)
- Friday: Applications to computer science and phylogenetics solving open counting problems

Examples of different weights in a triangular cone

The recurrence includes many known sequences already for $a_{n,n}$ in 

$$a_{m,n} = E(m,n)a_{m-1,n} + N(m,n)a_{m,n-1} \qquad m \geq n > 0$$

$E(m,n)$	$N(m,n)$	Description	$a_{n,n}$	OEIS
1	1	Dyck paths	$(1,1,2,5,14,42,132,\dots)$	A000108
$n+1$	1	Automata/Compacted trees	$(1,1,3,16,127,\dots)$	A082161
$2m+n-1$	1	Phylogenetic networks	$(1,1,7,106,2575,\dots)$	A213863
$2n+1$	1	Matrix recursion	$(1,1,4,33,436,\dots)$	A102321

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$2(m-n)+1$	1	Class of four-regular maps	$(1,3,24,297,\dots)$	A292186
$n+1$	$m+2$	Polytope volumes	$(1,3,40,1225,\dots)$	A012250
$n+1$	$8(m-n+1)$	Evaluated Riemann ζ fct.	$(1,8,256,17408,\dots)$	A253165
$2n+1$	$m-n+1$	Secant numbers	$(1,1,5,61,1385,\dots)$	A000364
$2n+2$	$m-n+1$	Tangent numbers	$(1,2,16,272,\dots)$	A000182
$m-n+1$	$2n$	Connected Feynman diag.	$(1,4,80,3552,\dots)$	A214298

Classical Methods

Solving Examples (1)–(3)

Overview of methods

1. Generating functions
2. Recurrence relations
3. Context free grammars
4. Bijections
5. Determinants
6. Continued fractions
7. Kernel method
8. Integral transforms
9. Saddle point method
10. Singularity analysis
11. Analytic Combinatorics
12. Analytic Combinatorics in Several Variables
13. Probability Theory
14. Guess-and-check
15. Stretched exponential method
16. Random walk method
17. . . .

Solving Example (1): Generating Functions

Unweighted model in the quarter plane

$$a_{m,n} = a_{m-1,n} + a_{m,n-1} \qquad \text{for } m, n \geq 0$$

- First, we define the generating function

$$A(x, y) = \sum_{m \geq 0} \sum_{n \geq 0} a_{m,n} x^m y^n.$$

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- Recall, $a_{0,0} = 1$ and $a_{m,n} = 0$ for $(m,n) \notin \mathbb{Z}^2_{\geq 0}$. Therefore, we get

$$\begin{cases} a_{m,n} = a_{m-1,n} + a_{m,n-1} & \text{for } m,n \geq 1, \\ a_{m,0} = a_{0,n} = 1 & \text{for } m,n \geq 0. \end{cases}$$

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- We multiply by $x^m y^n$ and sum over $m, n \geq 1$. This gives

$$A(x, y) = xA(x, y) + yA(x, y) + 1.$$

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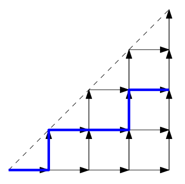
- Therefore, we get

$$A(x,y) = \frac{1}{1-x-y} = \sum_{k \geq 0} (x+y)^k = \sum_{m \geq 0} \sum_{n \geq 0} \binom{m+n}{n} x^m y^n. \quad \square$$

Solving Example (2): Generating Functions

Unweighted model below the diagonal

$$b_{m,n} = b_{m-1,n} + b_{m,n-1} \qquad \text{for } m \geq n \geq 0$$



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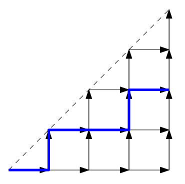

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- Here, we need to be careful at the diagonal, due to the boundary conditions. As before, we multiply by $x^m y^n$ and sum over $m \geq n \geq 0$:

$$B(x,y) = 1 + xB(x,y) + y\left(B(x,y) - D(xy)\right),$$

where $D(z) = \sum_{n\geq 0} b_{n,n} z^n$ is the diagonal of $B(x,y)$.

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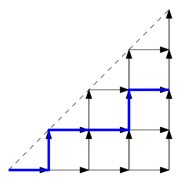

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$$(1 - x - y)B(x, y) = 1 - yD(xy).$$

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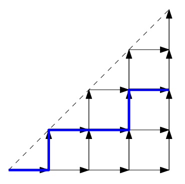

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- Two important ideas:
  1. Capture time evolution by change of coordinates
  2. Solve it using the kernel method

Solving Example (2): Kernel Method

We continue with

$$(1 - x - y)B(x, y) = 1 - yD(xy).$$

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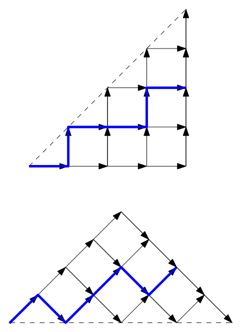

1. Capture time evolution
   - Idea: Instead of the number of $E = (1, 0)$ and $N = (0, 1)$ steps in x and y, we track the total number of steps in t and the distance to the diagonal in u:

$$x = tu \qquad \text{and} \qquad y = \frac{t}{u}.$$

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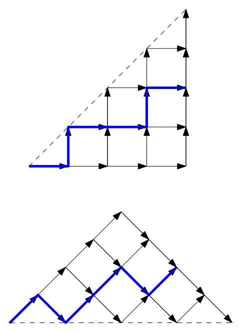

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   - This gives

$$\underbrace{\left(1 - tu - \frac{t}{u}\right)}_{=:K(t,u)} \hat{B}(t, u) = 1 - \frac{t}{u} D(t^2).$$

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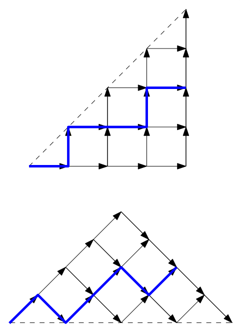

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2. Solve it using the kernel method
   - Idea: Bind u and t such that the left-hand side vanishes. Let $u_1(t)$ and $u_2(t)$ be the solutions of $K(t, u_i(t)) = 0$:

$$u_1(t) = \frac{1 - \sqrt{1 - 4t^2}}{2t} = t + \mathcal{O}(t^3) \qquad u_2(t) = \frac{1 + \sqrt{1 - 4t^2}}{2t} = \frac{1}{t} + \mathcal{O}(t).$$

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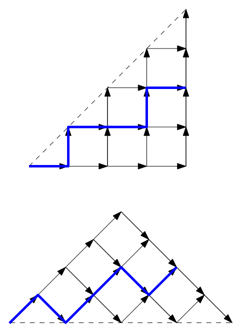

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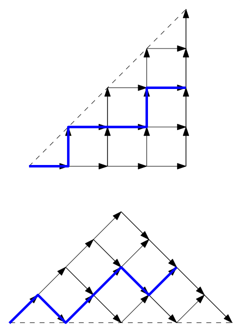

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$$D(t^2) = \frac{u_1(t)}{t} = \frac{1-\sqrt{1-4t^2}}{2t^2} = 1 + t^2 + 2\,t^4 + 5\,t^6 + 14\,t^8 + 42\,t^{10} + 132\,t^{12} + 429\,t^{14} + \dots$$

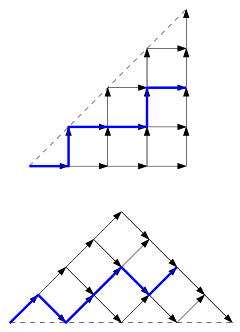
Solving Example (2): Final result

Final result for (prefixes) of Dyck paths

$$\hat{B}(t,u) = \frac{1-2ut-\sqrt{1-4t^2}}{2t(u^2t-u+t)}$$

or equivalently

$$B(x,y) = -\frac{1-2x-\sqrt{1-4xy}}{2x(1-x-y)}.$$



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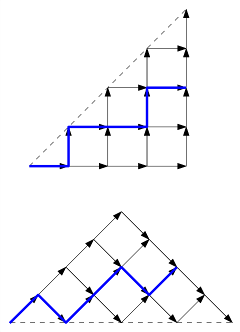

Direct corollaries:

- Paths with a fixed number of $E=(1,0)$ steps and an arbitrary number of $N=(0,1)$ steps:

$$B(x,1) = \frac{1-2x-\sqrt{1-4x}}{2x^2} = \sum_{n\geq 0} \frac{1}{n+2}\binom{2(n+1)}{n+1} x^n$$

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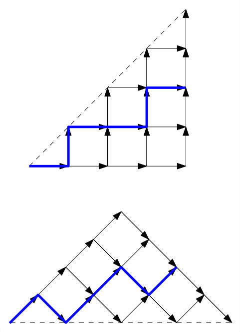

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- The total number of paths of length n:

$$\hat{B}(t,1) = \frac{1-2t-\sqrt{1-4t^2}}{2t(2t-1)} = \sum_{n\geq 0}\binom{2n}{n} t^{2n} + \sum_{n\geq 1}\frac{1}{2}\binom{2n}{n} t^{2n-1}$$

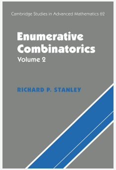
Sidenote: hierarchy of formal power series

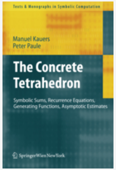
The formal power series $C(t)$ is

- rational if it can be written as

$$C(t) = \frac{P(t)}{Q(t)},$$

  where $P(t)$ and $Q(t)$ are polynomials in t.





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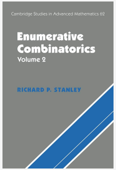

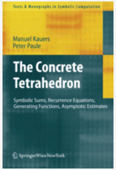

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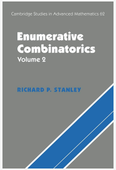
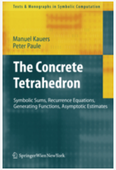
- D-finite if it satisfies a (non-trivial) linear differential equation with polynomial coefficients:

$$p_k(t)C^{(k)}(t) + \cdots + p_0(t)C(t) = 0.$$





Sidenote: hierarchy of formal power series

The formal power series $C(t)$ is

- rational if it can be written as

$$C(t) = \frac{P(t)}{Q(t)},$$

where $P(t)$ and $Q(t)$ are polynomials in t.

- algebraic (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation

$$P(t, C(t)) = 0.$$

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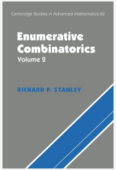

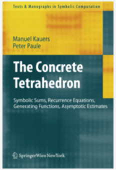

Why is it important to be D-finite?

- Nice and effective closure properties (sum, product, differentiation, . . .)
- Fast algorithms to compute coefficients
- Asymptotics of coefficients

Solving Example (3): Bijection

Weighted model in the quarter plane

$$c_{m,n} = (n+1)c_{m-1,n} + c_{m,n-1} \qquad \text{for } m, n \geq 0$$

Stirling numbers $S(n, k)$ of the second kind

- Number of set partitions of $\{1, 2, \ldots, n\}$ into k nonemtpy sets
- For example, $S(3, 2) = 3$ due to $\{\{1\}, \{2, 3\}\}$, $\{\{2\}, \{1, 3\}\}$, and $\{\{3\}, \{1, 2\}\}$

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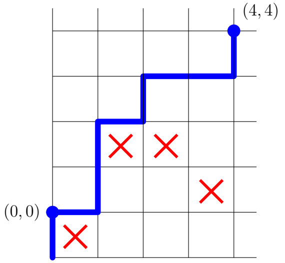

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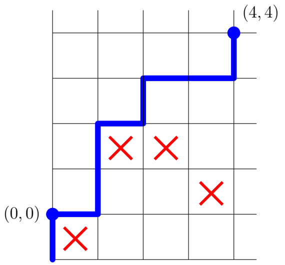

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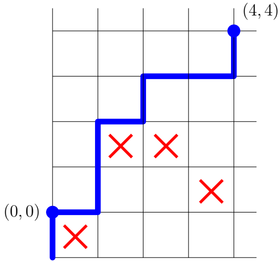

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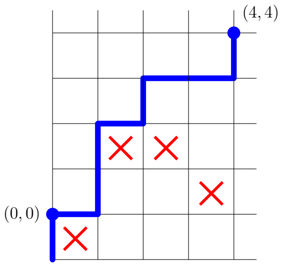

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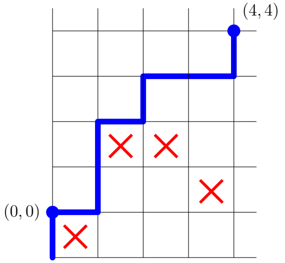

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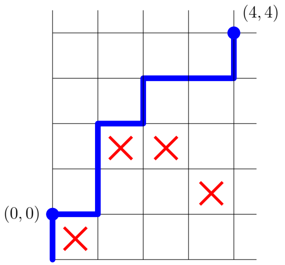

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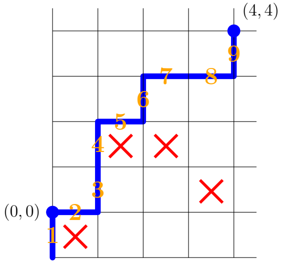

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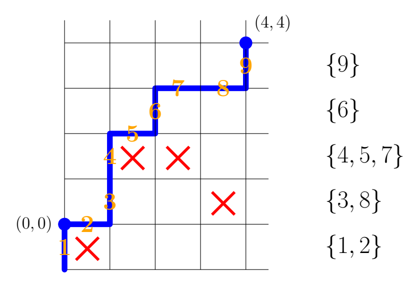

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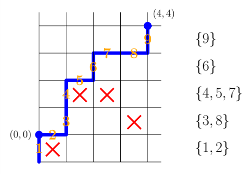

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$$c_{m,n} = S(m+n+1, n+1)$$

- Known exponential generating function for Stirling numbers of the second kind:

$$\sum_{n\geq 0}\sum_{k\geq 0} S(n,k)\frac{z^n u^k}{n!} = e^{u\left(e^z-1\right)}$$

- This allows us to conclude

$$C(x,y) = \sum_{m\geq 0}\sum_{n\geq 0} \frac{c_{m,n}x^m y^n}{(m+n+1)!} = \frac{e^{y\left(\frac{e^x-1}{x}\right)}-1}{y}$$

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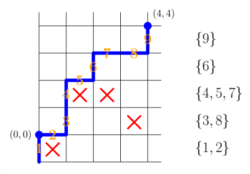

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- $C(x,y)$ is not D-finite (but it satisfies an algebraic differential equation!)
- Follows from, e.g., the following asymptotics (see saddle point method [Flajolet, Sedgewick 2009]):

$$S_n = \sum_{k=0}^{n} S(n,k) \sim n!\frac{e^{e^r-1}}{r^n\sqrt{2\pi r(r+1)e^r}},$$

where $re^r = n+1$, so that $r = \log n - \log\log n + o(1)$.

Advanced generating function methods

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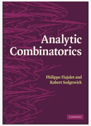
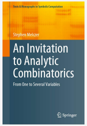
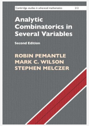
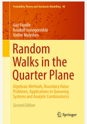
  Main tools: Saddle point method, singularity analysis, integral transforms, etc.









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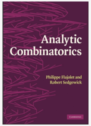

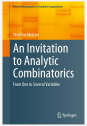

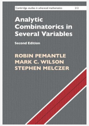

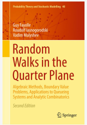

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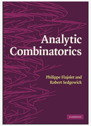

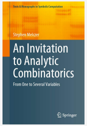

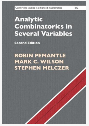

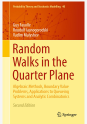

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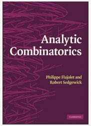

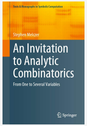

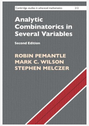

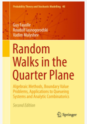

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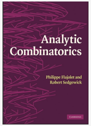

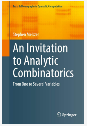

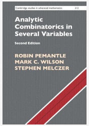

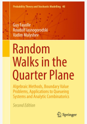

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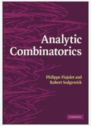

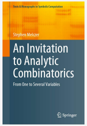

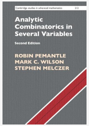

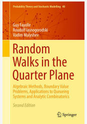

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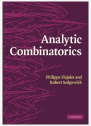
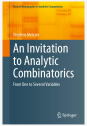
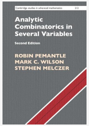
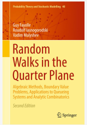
- Analytic Combinatorics in Several Variables [Pemantle, Wilson, Melczer 2024], [Melczer 2021]
  Works well with detailed information on the multivariate generating function
- Galois theory [Dreyfus, Hardouin, Roques, Singer 2018]
- Complex analysis [Bostan, Raschel, Salvy 2014]
- Probability theory [Denisov, Wachtel 2015]
- Computer algebra: Guess-and-check [Kauers, Paule 2011]
- Different extensions of the kernel method:
  - Iterated kernel method [Bousquet-Mélou, Petkovšek 2003]
  - Obstinate kernel method [Bousquet-Mélou 2002]
  - Vectorial kernel method [Asinowski, Bacher, Banderier, Gittenberger 2020]
  - Similar approaches developed in, e.g., statistical mechanics (algebraic Bethe ansatz [Gaudin 2014]), probability theory and queuing theory [Fayolle, Iasnogorodski, Malyshev 1999]



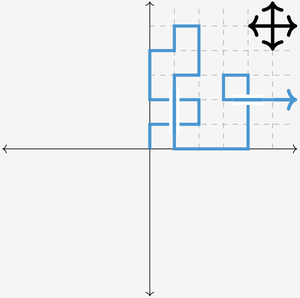






Highlight: The quarter plane

Great interdisciplinary success: combinatorics, algebra, computer algebra, complex analysis, probability theory, and Galois theory.



- Quarter plane

$$\mathcal{Q} = \{(m, n) : m, n \geq 0\}.$$

- Generating function

$$Q(x, y; t) = \sum_{m,n \geq 0} \sum_{k \geq 0} q_{m,n;k} t^k.$$

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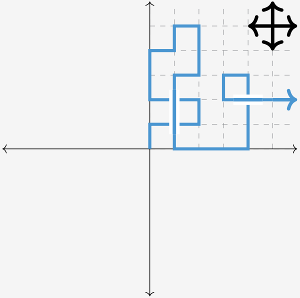

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- The chosen step set is associated with a group G of birational transformations of $\mathbb{Z}^2$.
  - Here, $\phi(x,y) = (\frac{1}{x}, y)$ and $\psi(x,y) = (x, \frac{1}{y})$
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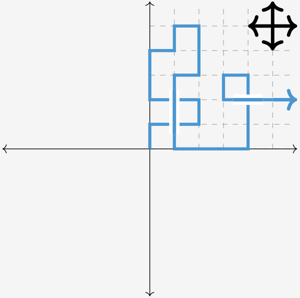

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Theorem [Bousquet-Mélou, Mishna 10], [Bostan, Kauers 10], [Kurkova, Raschel 12], [Mishna, Rechnitzer 07], [Melczer, Mishna 13], [and more!]

The series $Q(x, y; t)$ is D-finite if and only if G is finite.

This is the case for 23 out of 79 non-equivalent small step models $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$.

What about Example (4)?

What about Example (4)? The core of this course!

Weighted model below the diagonal

$$a_{m,n} = (n+1)a_{m-1,n} + a_{m,n-1} \qquad \text{for } m \geq n \geq 0$$

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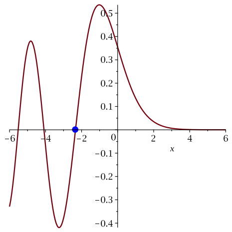
Theorem [Elvey Price, Fang, W 2021]

For $n \to \infty$ it holds that

$$a_{n,n} = \Theta\left(n!\, 4^n e^{3a_1 n^{1/3}} n\right)$$

where $a_1 \approx -2.338$ is the largest root of the Airy function $\mathrm{Ai}(x)$ characterized by $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ and $\lim_{x\to\infty} \mathrm{Ai}(x) = 0$.



$$\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x)$$

What is a stretched exponential?

General question

How does a sequence $(a_n)_{n\geq 0}$ behave for large n?

- Often we observe

$$C \cdot R^n \cdot n^\alpha,$$

for constants $C, R, \alpha \in \mathbb{R}$.

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- Much more seldom we observe (or are able to prove)

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Some deeper reasons why they are "seldom"

- Generating function cannot be algebraic
- It can be D-finite (satisfy a linear differential equation with polynomial coefficients), but only only with an *irregular singularity*, e.g., $\exp(\frac{z}{1-z})$

Appearances of stretched exponentials

Known exactly:

- Number theory (integer partitions):

$$\sim (4\sqrt{3})^{-1} e^{\pi(2n/3)^{1/2}} n^{-1}$$

- Theoretical physics (pushed Dyck paths [Beaton, McKay 14], [Guttmann 15]):

$$\sim C_1 4^n e^{-3\left(\frac{\pi \log 2}{2}\right)^{2/3} n^{1/3}} n^{-5/6}$$

- Phylogenetics (phylogenetic tree-child networks [Fuchs, Yu, Zhang 20]):

$$\Theta\left(n^{2n}(12e^{-2})^n e^{a_1 (3n)^{1/3}} n^{-2/3}\right)$$

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Conjectured:

- Permutations avoiding 1324 [Conway, Guttmann, Zinn-Justin 18]:

$$\approx \mu^n e^{-cn^{1/2}}$$

- Pushed self avoiding walks [Beaton, Guttmann, Jensen, Lawler 15]:

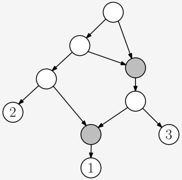
$$\approx \mu^n e^{-cn^{3/7}}$$

- and recently more and more appear in group theory, queuing theory, . . .

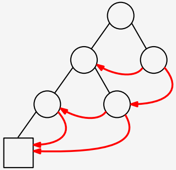
Stretched exponential method applies to many more objects

6	10	14	15	17	18
3	5	9	12	13	16
2	1	7	4	11	8

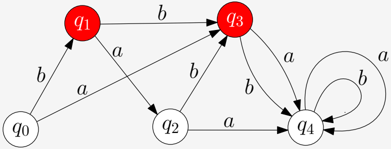
Young tableaux with walls
[Banderier, Marchal, W 2018], [Banderier, W 2021]



Phylogenetic networks
[McDiarmid, Semple, Welsh 2015]



Compacted trees
[Aho, Sethi, Ullman 1986]



Minimal automata
[Hopcroft, Ullman 1979]

BAADBACFCBEDECDFEF
Constrained words [Pons, Batle 2021]

Many new natural appearances of stretched exponentials

Theorem

The number c_n of compacted binary trees,

satisfy for $n \to \infty$

$$c_n = \Theta\left(n!\, 4^n e^{3a_1 n^{1/3}} n^{3/4}\right),$$

[Elvey Price, Fang, W 2021]

where $a_1 \approx -2.338$ is the largest root of the Airy function $\mathrm{Ai}(x)$ characterized by $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ and $\lim_{x\to\infty} \mathrm{Ai}(x) = 0$.

Many new natural appearances of stretched exponentials

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The number c_n of compacted binary trees, t_n of bicombining phylogenetic tree-child networks, satisfy for $n \to \infty$

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The number c_n of compacted binary trees, t_n of bicombining phylogenetic tree-child networks, b_n of minimal DFAs recognizing a finite binary language, and y_n of $3 \times n$ Young tableaux with walls satisfy for $n \to \infty$

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Associated recurrence relations ($m \geq n \geq 0$):

$c_n = c_{n,n}$, where $c_{m,n} = c_{m,n-1} + (n+1)c_{m-1,n} - (n-1)c_{m-2,n-1}$

$t_n = (n-1)!\,t_{m,m}$, where $t_{m,n} = \dfrac{2m+n-2}{2m+n-3} t_{m,n-1} + (2m+n-2)t_{m-1,n}$

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Part II

Asymptotics along the boundary

Recap of Part I

Recurrences we study

$$a_{m,n} = E(m,n)a_{m-1,n} + N(m,n)a_{m,n-1}$$

Main goal

- Determine $a_{n,n}$

$\rightarrow$ We focus on asymptotics for $n \to \infty$

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	$E(m,n)$	$N(m,n)$	Domain	$a_{n,n}$	Description
(1)	1	1	$m, n \geq 0$	$\binom{2n}{n}$	Binomial coefficients
(2)	1	1	$m \geq n \geq 0$	$\frac{1}{n+1}\binom{2n}{n}$	Catalan numbers
(3)	$n+1$	1	$m, n \geq 0$	$S(2n+1, n+1)$	Stirling numbers 2nd kind
(4)	$n+1$	1	$m \geq n \geq 0$	$\Theta\left(n! 4^n e^{3a_1 n^{1/3}} n\right)$	Compacted binary trees

(In the last case, $a_1 \approx -2.338$ is the largest root of the Airy function $\mathrm{Ai}(x)$ that is the unique function satisfying $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ and $\lim_{x\to\infty} \mathrm{Ai}(x) = 0$.)

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Today we solve Example (4): weighted model below the diagonal

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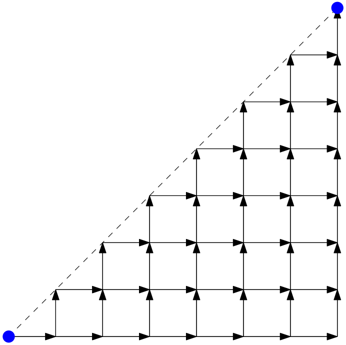
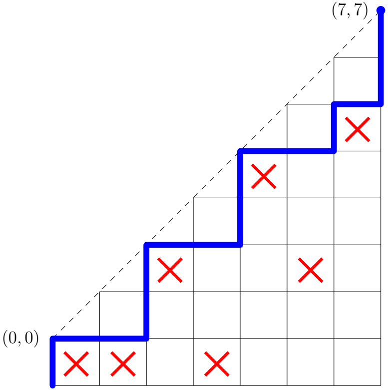
Step 1: Transformation of the recurrence

Step 1: Transform recurrence into a Dyck-like recurrence





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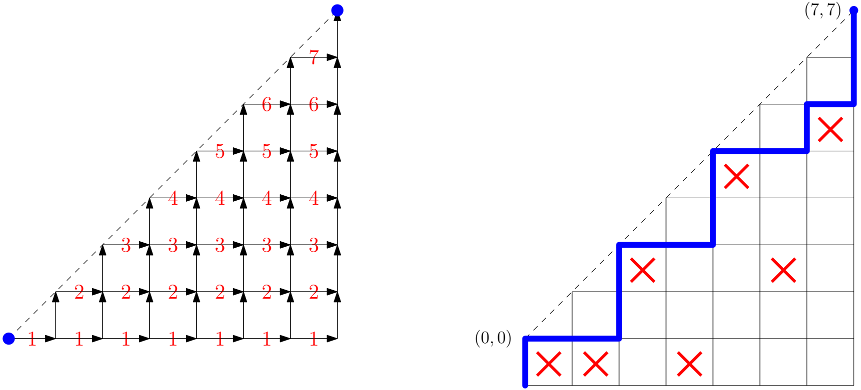

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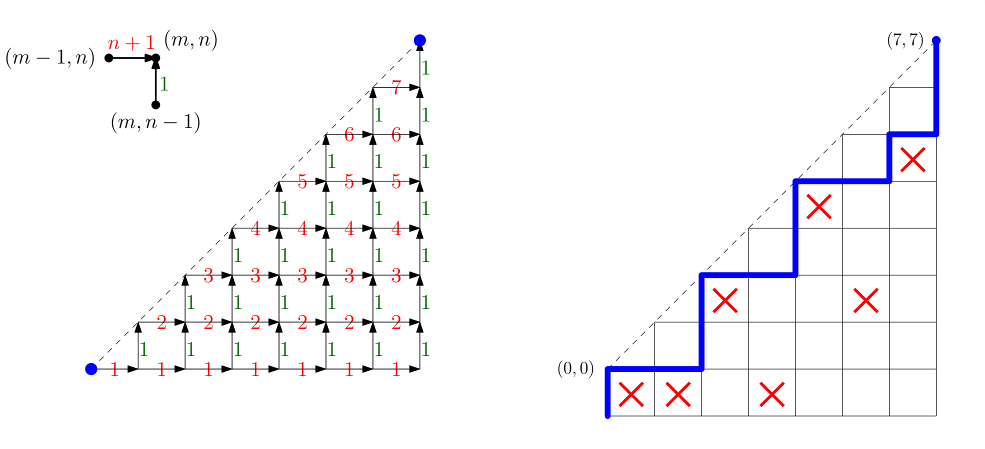

- Path starts at $(0,-1)$ and ends at (n,n)
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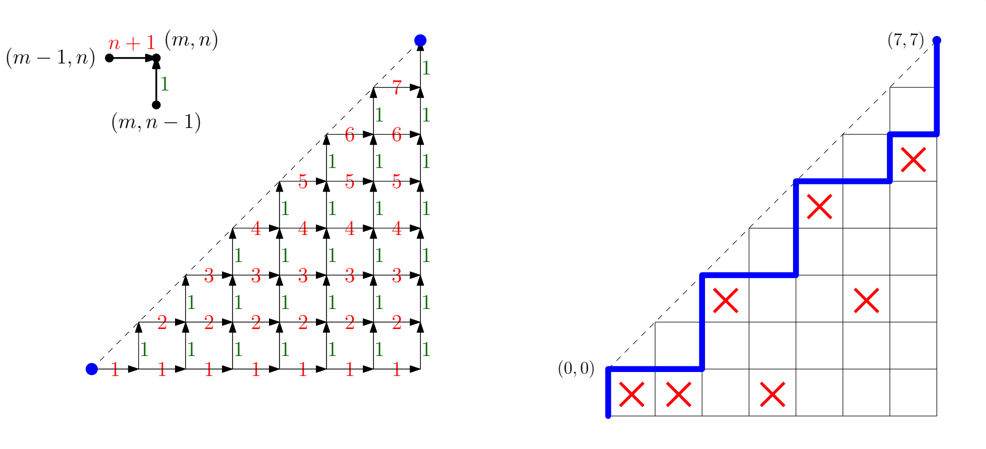
- One box is marked below each horizontal step
- Each vertical step has weight 1

Recurrence for decorated paths



Recurrence: Let $a_{m,n}$ be the number of paths ending at (m,n)

$$a_{m,n} = a_{m,n-1} + (n+1)a_{m-1,n}, \qquad \text{for } m \geq n$$
$$a_{0,0} = 1.$$

Number of relaxed compacted trees is $a_{n,n}$

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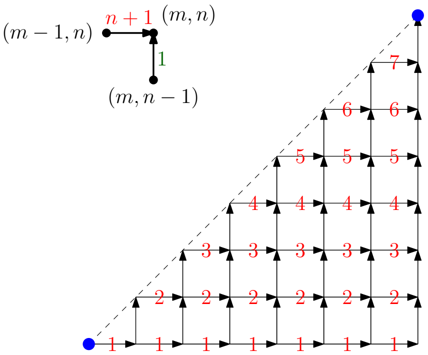

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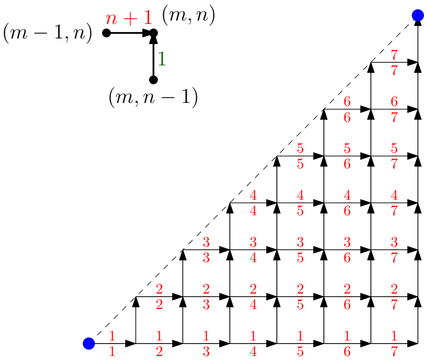


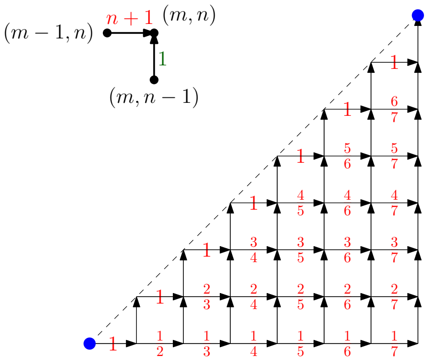
Recurrence: Let $\tilde{a}_{m,n}$ be the number of paths ending at (m, n) with weights divided by column number

$$\tilde{a}_{m,n} = \tilde{a}_{m,n-1} + \frac{n+1}{m}\, \tilde{a}_{m-1,n}, \qquad \text{for } m \geq n$$

$$\tilde{a}_{0,0} = 1.$$

Number of relaxed compacted trees is $n!\, \tilde{a}_{n,n}$

Recurrence for decorated paths



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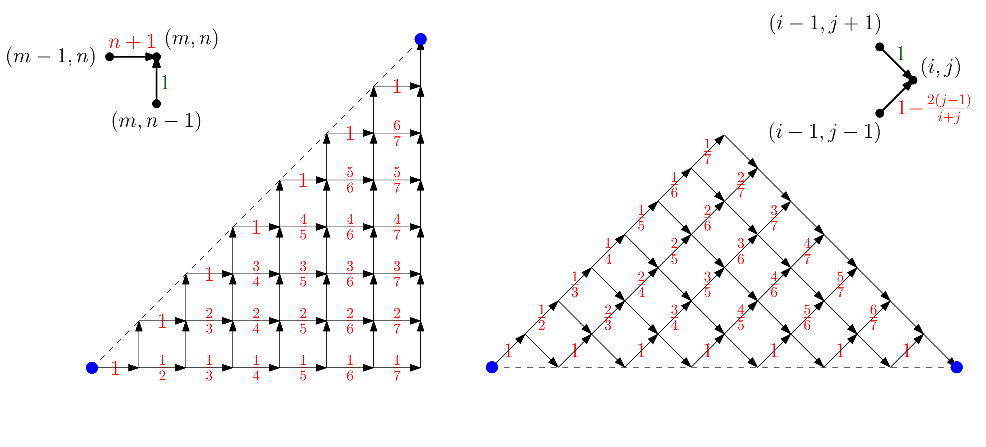

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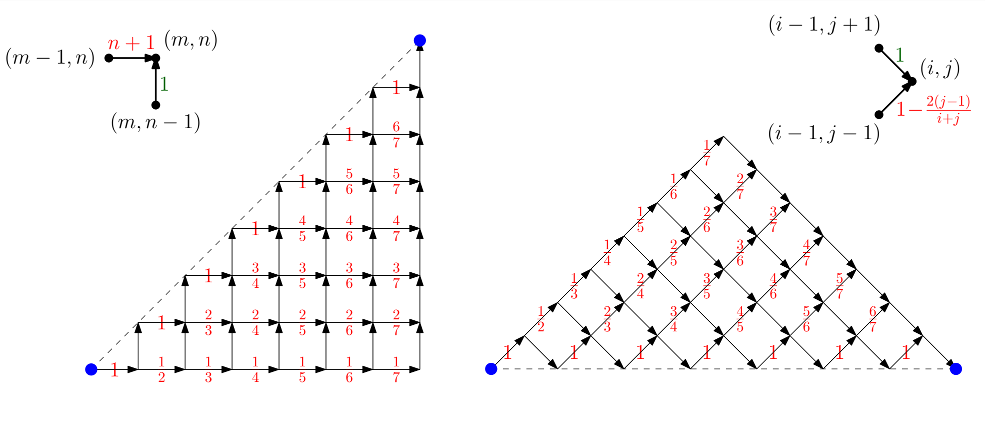


Recurrence: Let $d_{i,j}$ be the number of decorated paths ending at (i, j) shown on the right

$$d_{i,j} = d_{i-1,j+1} + \left(1 - \frac{2(j-1)}{i+j}\right) d_{i-1,j-1}, \qquad \text{for } i > 0,\ j \geq 0$$

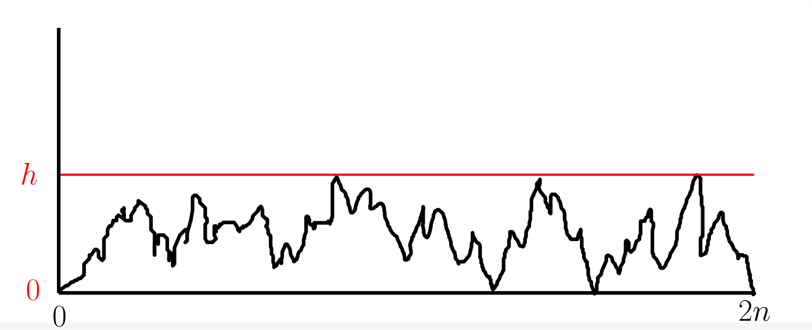
$$d_{0,0} = 1.$$

$\Rightarrow a_{n,n} = n!\, d_{2n,0}$

Step 2: Heuristic analysis

Intuition stretched exponential: Pushed Dyck paths [Beaton, McKay 14], [Guttmann 15]

Dyck paths of length $2n$ where paths of height h get weight 2^{-h}



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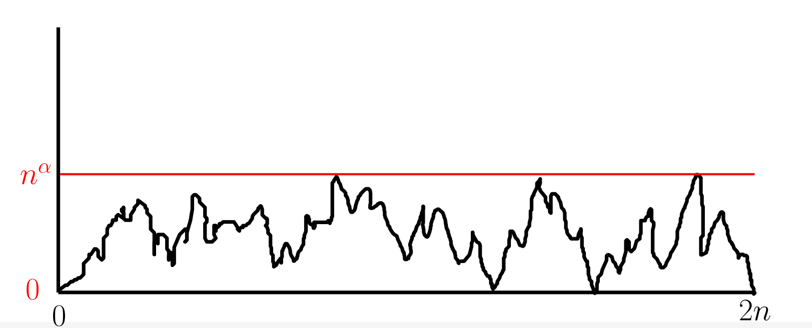

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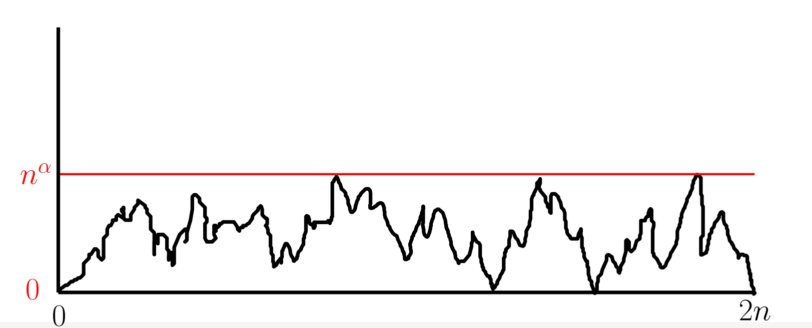

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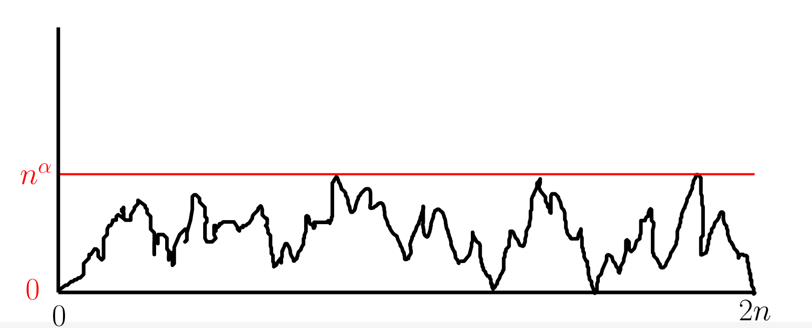

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Our case: weights decrease similarly with height so we expect similar behavior

Step 2: Heuristics analysis of recurrence: What happens for large (fixed) n?

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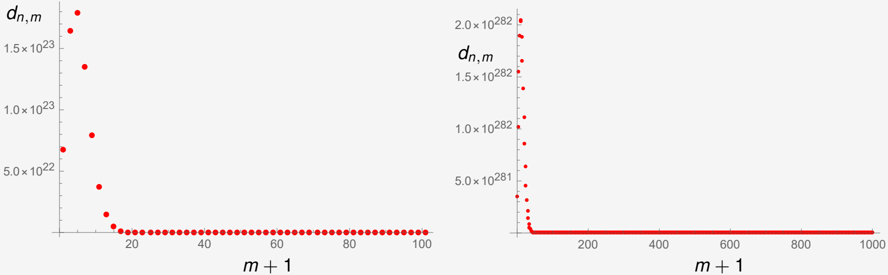

Figure: Plots of $d_{n,m}$ against $m+1$. **Left:** $n = 100$, **Right:** $n = 1000$.

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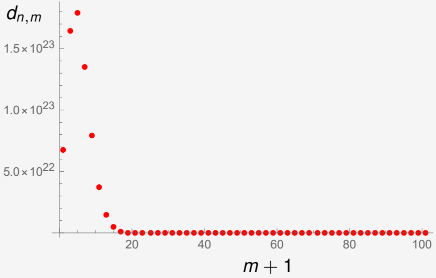

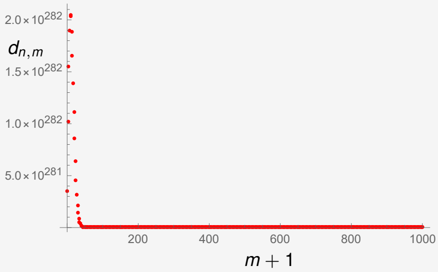

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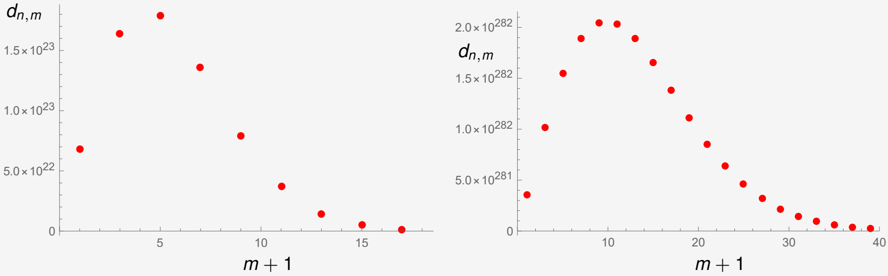

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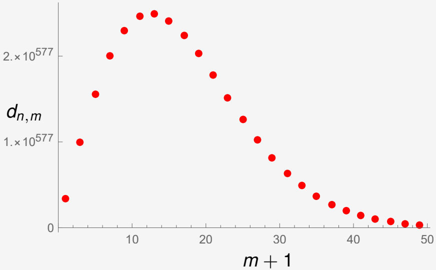

Figure: **Left:** Plot of $d_{n,m}$ against $m+1$ for $n = 2000$. **Right:** Limiting function $f(x)$.

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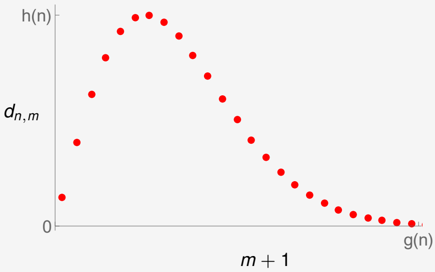

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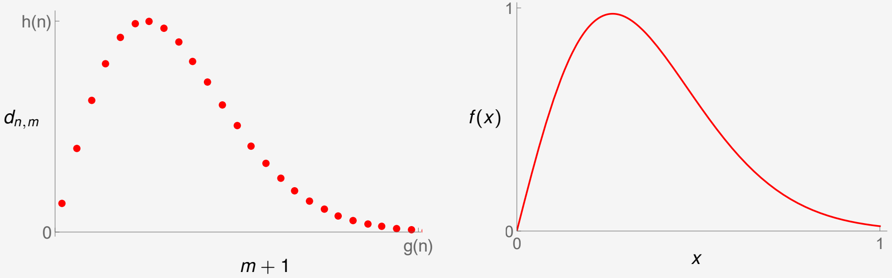

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$$\textbf{Ansatz:} \quad d_{n,m} \approx h(n) f\left(\frac{m+1}{g(n)}\right)$$

Does this ansatz work in the unweighted or unconstrained model?

$$d_{n,m} = \mu_{n,m}\, d_{n-1,m+1} + \nu_{n,m}\, d_{n-1,m-1}, \qquad m \geq 0$$

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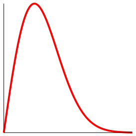

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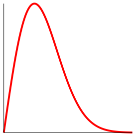

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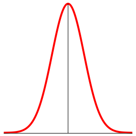

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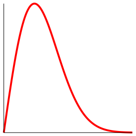

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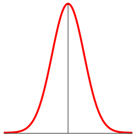

3. Relaxed binary trees $\mu_{n,m} = 1$ and $\nu_{n,m} = 1 - \frac{2(m-1)}{n+m}$ with $m \geq 0$:
   $\Rightarrow$ Based on the relation with pushed Dyck paths, we guess $g(n) = \sqrt[3]{n}$.

What are $h(n)$ and $f(x)$?

Heuristic analysis of weighted paths of relaxed binary trees

$$d_{n,m} = d_{n-1,m+1} + \left(1 - \frac{2(m+1)}{n+m}\right) d_{n-1,m-1}$$

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  Set $m = x\sqrt[3]{n} - 1$:

$$h(n) f(x) \approx h(n-1) f\left(\frac{x\sqrt[3]{n}+1}{\sqrt[3]{n-1}}\right) + \left(1 - \frac{2x\sqrt[3]{n}}{n + x\sqrt[3]{n} - 1}\right) h(n-1) f\left(\frac{x\sqrt[3]{n}-1}{\sqrt[3]{n-1}}\right)$$

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  Dividing by $h(n-1)$ and expanding the right-hand side around x for $n \to \infty$ gives

$$\frac{h(n)}{h(n-1)} \approx 2 + \frac{f''(x) - 2xf(x)}{f(x)} n^{-2/3} + O(n^{-1})$$

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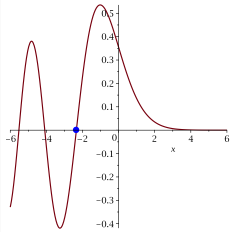
$$f''(x) = (2x + c)f(x) \qquad \Rightarrow \quad f(x) = \mathrm{Ai}\left(2^{-2/3}(2x + c)\right)$$

  where c is a constant and Ai is the Airy function.



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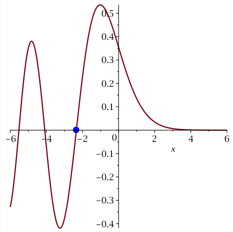

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- **Boundary condition:** $d_{n,-1} = 0$ and $d_{n,m} \geq 0$.
  Then $f(0) = 0$ implies $c = 2^{2/3} a_1$, where $a_1 \approx -2.338$ satisfies $\mathrm{Ai}(a_1) = 0$.

Refined heuristic analysis

1. **Ansatz of order 1:**

$$d_{n,m} \approx h(n) f\left(\frac{m+1}{\sqrt[3]{n}}\right) \qquad \text{and} \qquad s_n = 2 + cn^{-2/3} + O(n^{-1}).$$

yields estimates $c = 2^{2/3} a_1$ such that

$$h(n) \approx 2^n e^{3a_1 (n/2)^{1/3}} \qquad \text{and} \qquad f(x) = \mathrm{Ai}(2^{1/3} x + a_1).$$

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$$h(n) \sim cst\, 2^n e^{3a_1 (n/2)^{1/3}} n^{4/3} \qquad \text{and} \qquad \begin{aligned} f_0(x) &= \operatorname{Ai}(2^{1/3} x + a_1) = \operatorname{Ai}'(a_1) x + \dots \\ f_1(x) &= -\frac{2x^2}{3} f_0(x) \end{aligned}$$

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This way we conjecture the asymptotic form

$$a_{n,n} = n! d_{2n,0} \approx cst\, n! 4^n e^{3a_1 n^{1/3}} n.$$

Step 3: Inductive proof

Step 3: Inductive proof – Outline

Recall:

$$d_{n,m} = \left(1 - \frac{2(m+1)}{n+m}\right) d_{n-1,m-1} + d_{n-1,m+1}$$

Find **explicit sequences** $X_{n,m}$ and $Y_{n,m}$ with the **same asymptotic form**, such that

$$X_{n,m} \leq d_{n,m} \leq Y_{n,m},$$

for all m and all n large enough.

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How to find them?

1. Use heuristics
2. Adapt until $X_{n,m}$ and $Y_{n,m}$ satisfy the recurrence of $d_{n,m}$ with the equalities replaced by inequalities:

$$= \quad \longrightarrow \quad \leq \text{ and } \geq$$

3. Prove $X_{n,m} \leq d_{n,m} \leq Y_{n,m}$ by induction.

Induction (Lower bound)

$$d_{n,m} = d_{n-1,m+1} + \left(1 - \frac{2(m+1)}{n+m}\right) d_{n-1,m-1}$$

Main idea

Suppose we have found explicit sequences $(X_{n,m})_{n\geq m\geq 0}$ and $(s_n)_{n\geq 1}$ that satisfy

$$X_{n,m}s_n \leq X_{n-1,m+1} + \left(1 - \frac{2(m+1)}{n+m}\right) X_{n-1,m-1}, \tag{2}$$

for all sufficiently large n and **all** integers $m \in [0, n]$.

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Define $(h_n)_{n \geq 0}$ by $h_0 = 1$ and $h_n = s_n h_{n-1}$; then prove that

$$X_{n,m} h_n \leq b_0 d_{n,m}$$

for some constant b_0 by induction:

Induction (Lower bound)

$$d_{n,m} = d_{n-1,m+1} + \left(1 - \frac{2(m+1)}{n+m}\right) d_{n-1,m-1}$$

Main idea

Suppose we have found explicit sequences $(X_{n,m})_{n \geq m \geq 0}$ and $(s_n)_{n \geq 1}$ that satisfy

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Explicit sequences for the lower bound

Lemma (lower bound)

For all $n, m \geq 0$ let

$$\tilde{X}_{n,m} := \left(1 - \frac{2m^2}{3n} + \frac{m}{2n}\right) \mathrm{Ai}\left(a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right) \qquad \text{and}$$

$$\tilde{s}_n := 2 + \frac{2^{2/3} a_1}{n^{2/3}} + \frac{8}{3n} - \frac{1}{n^{7/6}}.$$

Then, for any $\varepsilon > 0$, there exists an $\tilde{n}_0$ such that

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Making m valid for all $[0, n]$

Define $X_{n,m} := \max\{\tilde{X}_{n,m}, 0\}$. Then,

1. $X_{n,m} \tilde{s}_n = \tilde{X}_{n,m} \tilde{s}_n$ for $m < cst\sqrt{n}$

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2. $X_{n,m}\tilde{s}_n = 0 \leq (1-\frac{2(m+1)}{n+m})X_{n-1,m-1} + X_{n-1,m+1}$ otherwise

Explicit sequences for the lower bound

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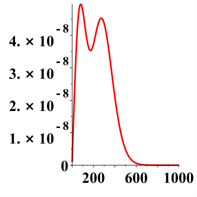
for all $n \geq \tilde{n}_0$ and for all $0 \leq m < n^{2/3-\varepsilon}$.

Approach

- Show that $P_{n,m} := -\tilde{X}_{n,m}\tilde{s}_n + \tilde{X}_{n-1,m+1} + \left(1 - \frac{2(m+1)}{n+m}\right)\tilde{X}_{n-1,m-1} \geq 0$



$P(n, m)$ for $n = 10^6$

Explicit sequences for the lower bound

Lemma (lower bound)

For all $n, m \geq 0$ let

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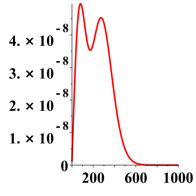
- Expand for n, m large such that $P_{n,m} = \sum a_{i,j} m^i n^j$ (converges absolutely, since Airy function is entire)



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Explicit sequences for the lower bound

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For all $n, m \geq 0$ let

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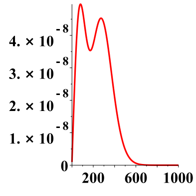
- Expand for n, m large such that $P_{n,m} = \sum a_{i,j} m^i n^j$ (converges absolutely, since Airy function is entire)
- Show that $P_{n,m} = \kappa m^{i_0} n^{j_0} + o(m^{i_0} n^{j_0})$ where $\kappa > 0$ for n large



$P(n,m)$ for $n = 10^6$

Lemma (lower bound) – Proof (1)

The following computations rely on computer algebra (Maple session available online).

1. We make the ansatz

$$X_{n,m} := \left(1 + \frac{\tau_2 m^2 + \tau_1 m}{n}\right) \mathrm{Ai}\left(a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right),$$

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2. Expand $\mathrm{Ai}(z)$ in a neighborhood of

$$\alpha = a_1 + \frac{2^{1/3} m}{n^{1/3}},$$

   using $\mathrm{Ai}''(z) = z\mathrm{Ai}(z)$. Then

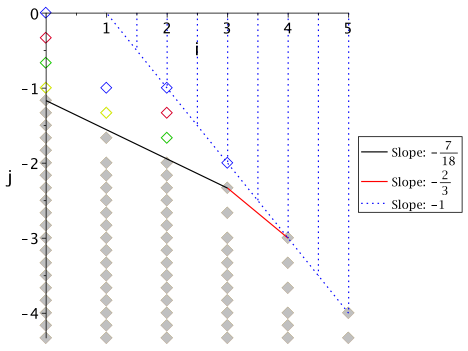
$$P_{n,m} = p_{n,m}\mathrm{Ai}(\alpha) + p'_{n,m}\mathrm{Ai}'(\alpha),$$

   where $p_{n,m}$ and $p'_{n,m}$ are power series in $n^{-1/6}$ whose coefficients are polynomials in m.

Lemma (lower bound) – Proof (2)

3 Choose σ_i and τ_i to kill lower order terms in

$$P_{n,m} = \sum a_{i,j} m^i n^j$$



$$\begin{aligned} P_{n,m} = &(\sigma_0 - 2)\mathrm{Ai}(\alpha) \\ &- \left((\sigma_1 \mathrm{Ai}(\alpha) + 2^{1/3}(\sigma_0 - 2)\right) \mathrm{Ai}'(\alpha) n^{-\frac{1}{3}} \\ &- \left(\left(\frac{a_1(\sigma_0 - 4)}{2^{1/3}} + \sigma_2\right) \mathrm{Ai}(\alpha) + 2^{1/3}\sigma_1 \mathrm{Ai}'(\alpha) \right) n^{-\frac{2}{3}} \\ &+ \dots \end{aligned}$$

- blue terms: $\sigma_0 = 2$
- red terms: $\sigma_1 = 0$
- green terms: $\sigma_2 = 2^{2/3} a_1$
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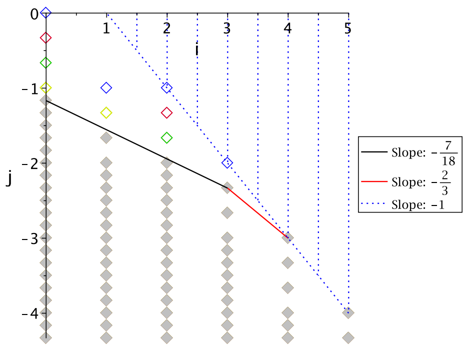

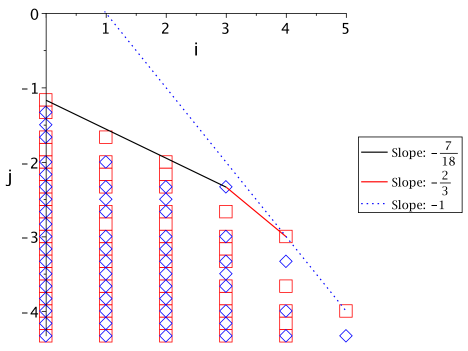

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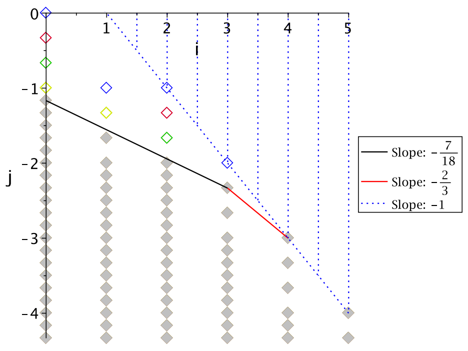

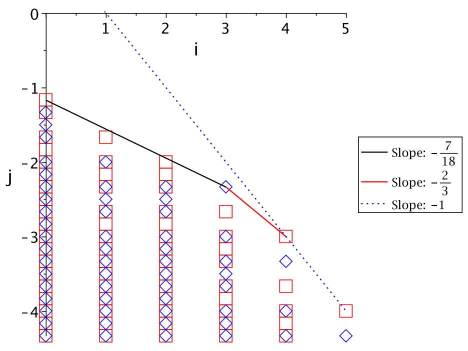

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Case analysis on non-zero coefficients:

1. $m \leq x_0 (n/2)^{1/3}$ (here $\mathrm{Ai}'(\alpha) > 0$)
2. $x_0 (n/2)^{1/3} < m \leq n^{7/18}$
3. $n^{7/18} < m \leq n^{2/3-\varepsilon}$. □

Upper bound

Lemma

Choose $\eta > 2/9$ *fixed and for all* $n, m \geq 0$ *let*

$$\hat{X}_{n,m} := \left(1 - \frac{2m^2}{3n} + \frac{m}{2n} + \eta \frac{m^4}{n^2}\right) \operatorname{Ai}\left(a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right) \qquad \text{and}$$

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Then, for any $\varepsilon > 0$*, there exists a constant* $\hat{n}_0$ *such that*

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Proof: Same idea with colorful Newton polygons works (but more complicated). □

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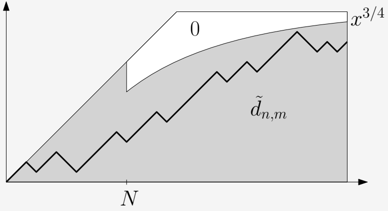
Proof: Same idea with colorful Newton polygons works (but more complicated). □

Making m valid for all $[0, n]$ (different than lower bound)

- We fix $N > 0$ and define a new sequence $\tilde{d}_{n,m}$ with the same rules as $d_{n,m}$ **except** that $\tilde{d}_{n,m} = 0$ for $m > n^{3/4}$ and $n > N$



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Then, for any $\varepsilon > 0$, *there exists a constant* $\hat{n}_0$ *such that*

$$\hat{X}_{n,m}\hat{s}_n \geq \frac{n-m+2}{n+m}\hat{X}_{n-1,m-1} + \hat{X}_{n-1,m+1},$$

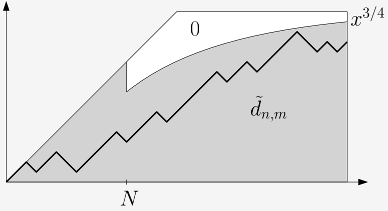
for all $n \geq \hat{n}_0$ *and all* $0 \leq m < n^{1-\varepsilon}$.

Proof: Same idea with colorful Newton polygons works (but more complicated). □

Making m valid for all $[0, n]$ (different than lower bound)

- We fix $N > 0$ and define a new sequence $\tilde{d}_{n,m}$ with the same rules as $d_{n,m}$ **except** that $\tilde{d}_{n,m} = 0$ for $m > n^{3/4}$ and $n > N$

$\Rightarrow$ Induction works and we get $\tilde{d}_{2n,m} \leq \gamma 4^n e^{3a_1 n^{1/3}} n$



Upper bound

Lemma

Choose $\eta > 2/9$ fixed and for all $n, m \geq 0$ let

$$\hat{X}_{n,m} := \left(1 - \frac{2m^2}{3n} + \frac{m}{2n} + \eta\frac{m^4}{n^2}\right) \mathrm{Ai}\left(a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right) \qquad \textit{and}$$

$$\hat{s}_n := 2 + \frac{2^{2/3}a_1}{n^{2/3}} + \frac{8}{3n} + \frac{1}{n^{7/6}}.$$

Then, for any $\varepsilon > 0$, there exists a constant $\hat{n}_0$ such that

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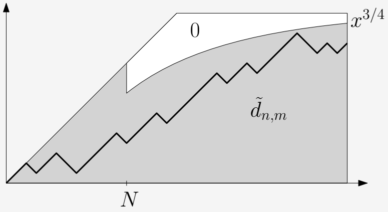
! Prove that $d_{2n,0} \leq cst\, \tilde{d}_{2n,m}$



Lattice path theory to finish the upper bound

Cropped paths

$$\begin{cases} \tilde{d}_{n,m} = d_{n-1,m+1} + \left(1 - \frac{2(m+1)}{n+m}\right) \tilde{d}_{n-1,m-1} & \text{for } m > n^{3/4} \text{ and } n > N, \\ \tilde{d}_{n,m} = 0 & \text{otherwise.} \end{cases}$$

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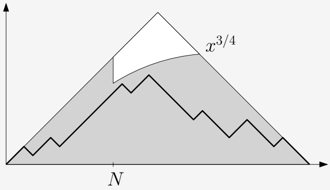
Good path



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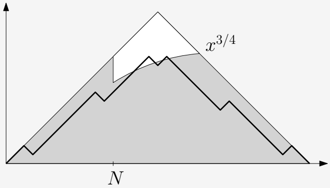

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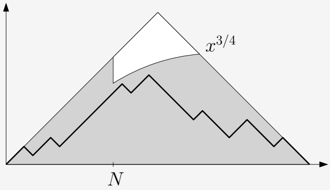

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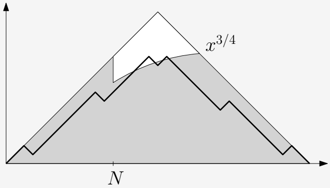

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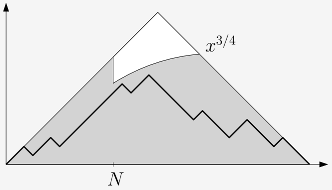

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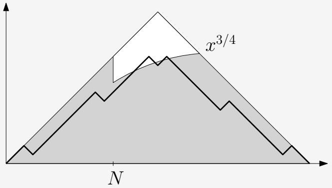

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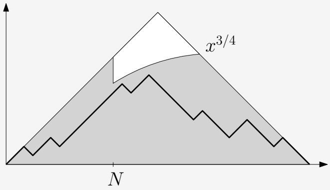

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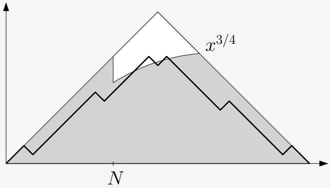

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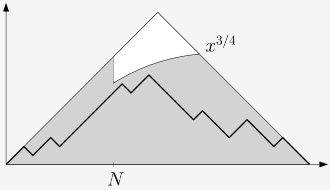

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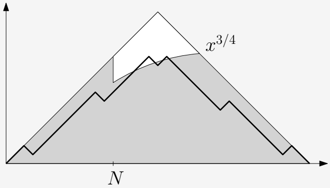

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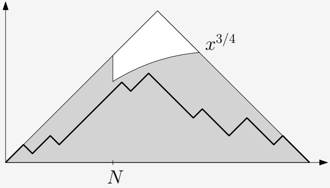

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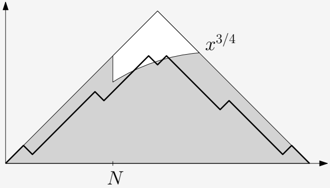

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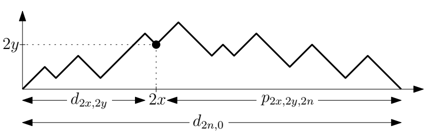

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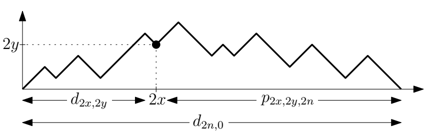

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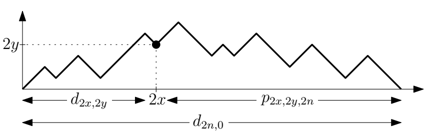

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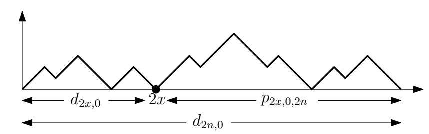

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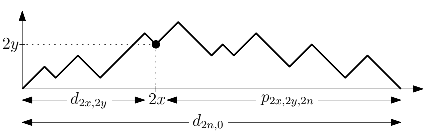

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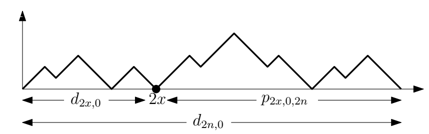

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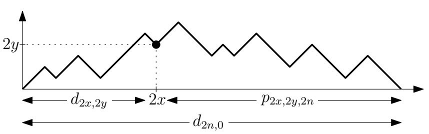

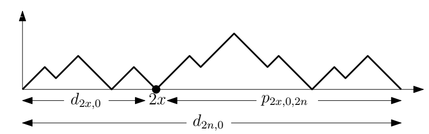

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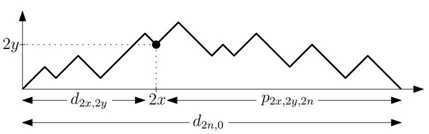

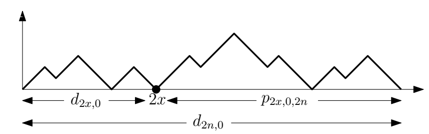

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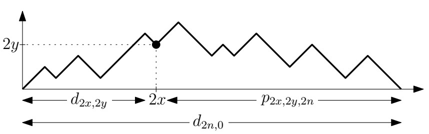

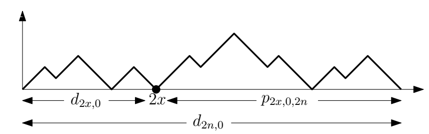

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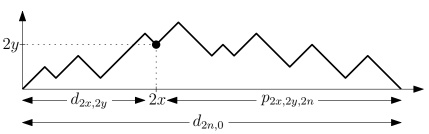

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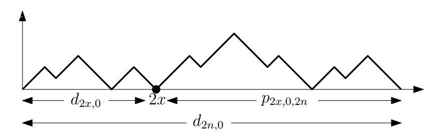

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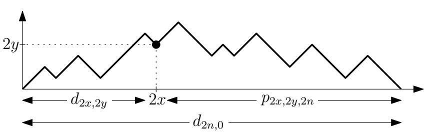

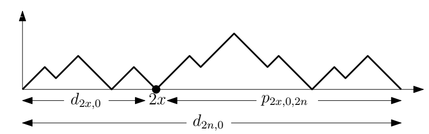

Lemma

$$p_{2x,2y,n} \leq (2y+1) p_{2x,0,n} \qquad \text{and} \qquad p_{2x,0,n} \leq \frac{d_{2n,0}}{d_{2x,0}}.$$

Therefore, we get for $x \geq y > x^{3/4}$ and x large

$$s_{2x,2y,n} = \frac{d_{2x,2y} \cdot p_{2x,2y,n}}{d_{2n,0}} \overset{\text{(Lemma)}}{\leq} \frac{(2y+1)d_{2x,2y}}{d_{2x,0}} \overset{\text{(Unweighted paths)}}{\leq} C' \frac{2y+1}{d_{2x,0}} \binom{2x}{x+y}$$

$$\overset{\text{(Lower bound)}}{\leq} C' \frac{2y+1}{4^x e^{3a_1 x^{1/3}} x} \binom{2x}{x+y} \leq C'' \frac{1}{4^x e^{3a_1 x^{1/3}}} \binom{2x}{x+x^{3/4}} \leq C''' 2^{-x^{1/2}} \qquad \square$$

Summary

1. Two-parameter recurrence relation

$$a_{m,n} = (n+1)a_{m-1,n} + a_{m,n-1}, \quad n \geq m > 0$$

$$d_{n,m} = \left(1 - \frac{2(m-1)}{n+m}\right) d_{n-1,m-1} + d_{n-1,m+1}, \quad m \geq 0$$

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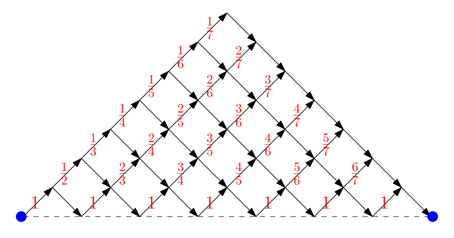

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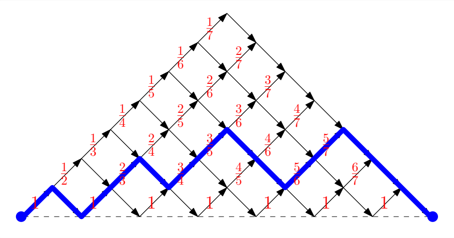

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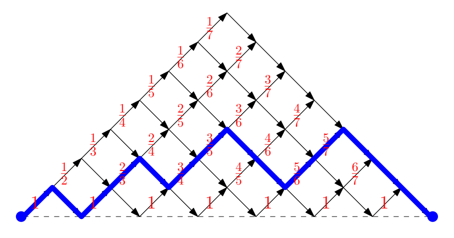

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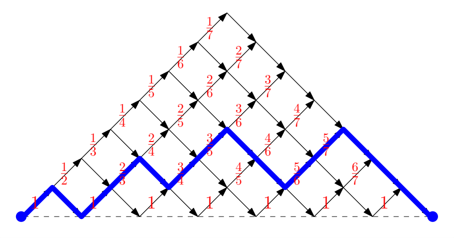

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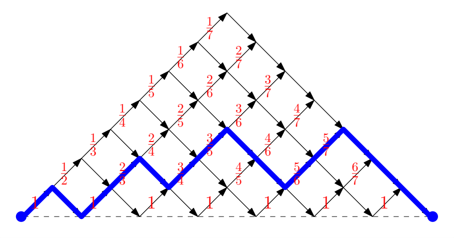

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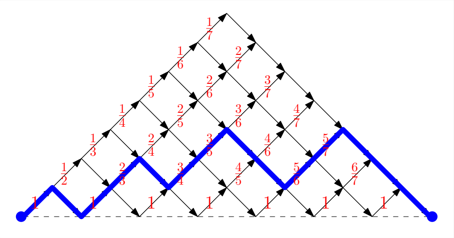

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Upper bound (similar proof, more technical)

$$a_{n,n} \leq \gamma_2 \, n! 4^n e^{3a_1 n^{1/3}} n,$$

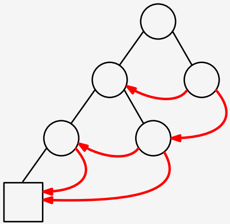
for some constant $\gamma_2 > 0$.

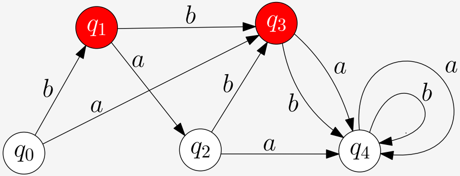
Part III

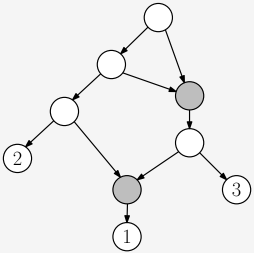
Applications in Computer Science and Biology

Stretched exponentials appear in open asymptotic counting problems

1. Compacted trees [Flajolet, Sipala, Steyaert 1990]
2. Minimal deterministic finite automata accepting a finite language [Domaratzki, Kisman, Shallit 2002]
3. Phylogenetic tree-child networks [McDiarmid, Semple, Welsh 2015]

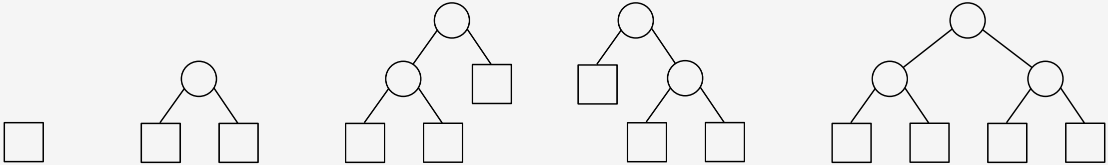






Compacted trees

Let’s start simple: binary trees



- *Internal node*: Node of out-degree 2 (circle)
- *Leave*: Node of out-degree 0 (square)
- *Root*: Distinguished node (top node)
- *Left-Right Order* of children

A recursive construction

- A binary tree is either a leaf,
- or it consists of a root and a left and right binary tree.

Motivation: Efficiently store redundant information

Example

Consider the labeled tree necessary to store the arithmetic expression

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(* (- (* x x) (* y y)) (+ (* x x) (* y y))),
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which represents $(x^2 - y^2)(x^2 + y^2)$.

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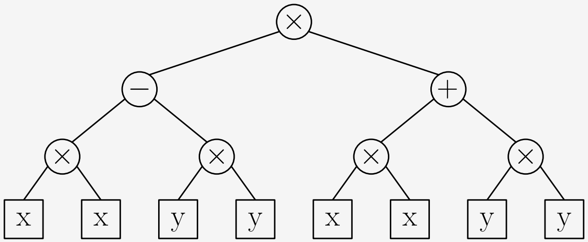

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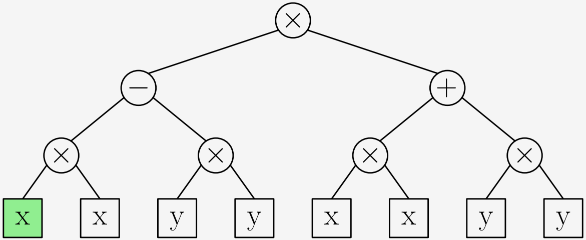

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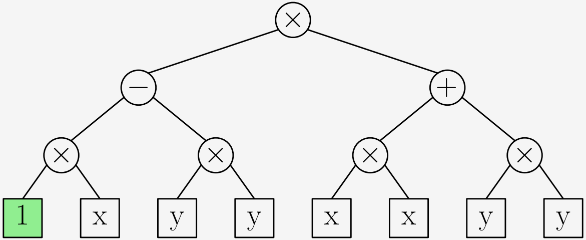

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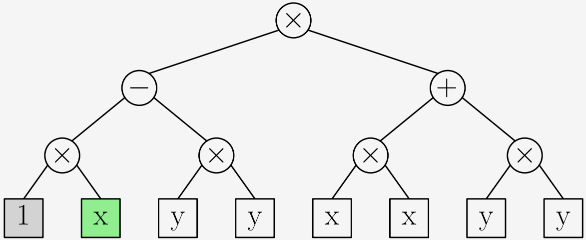

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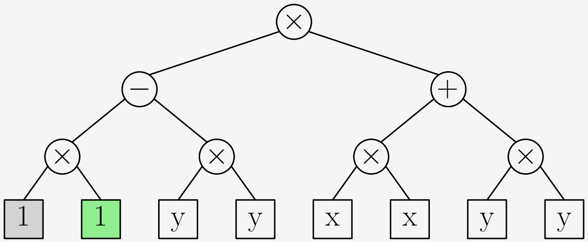

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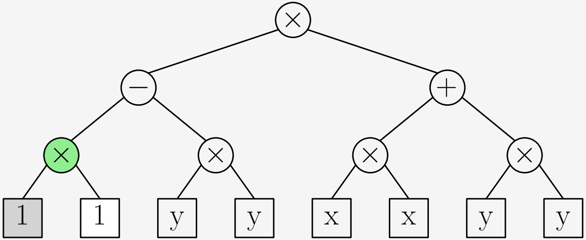

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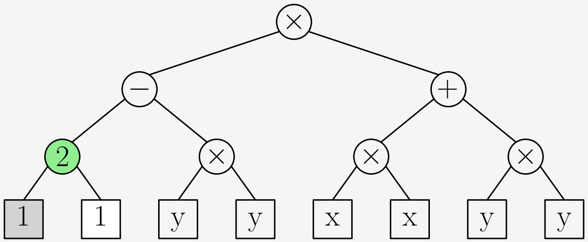

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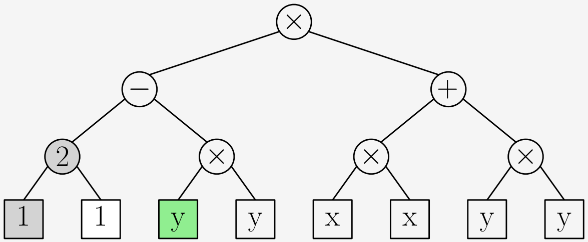

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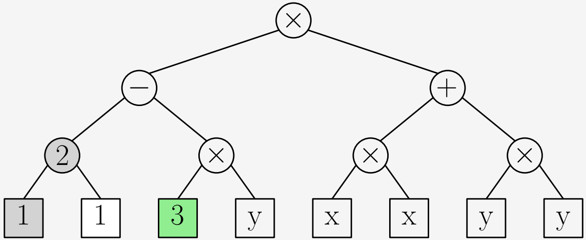

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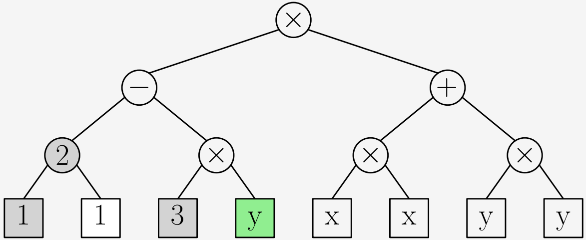

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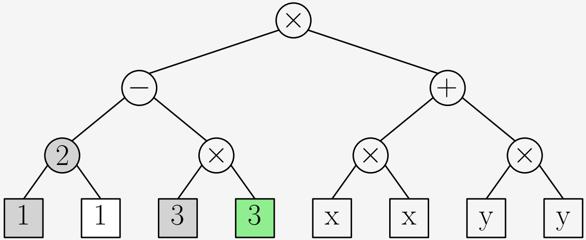

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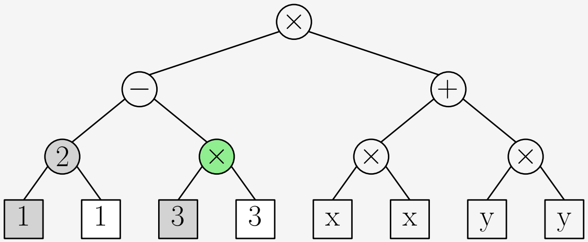

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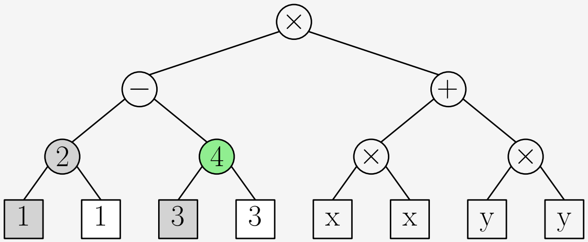

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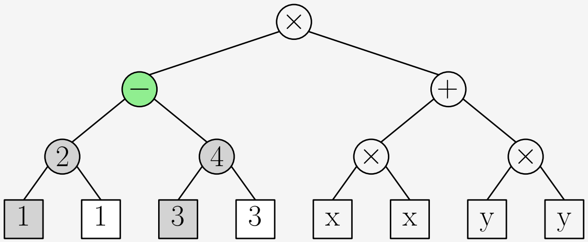

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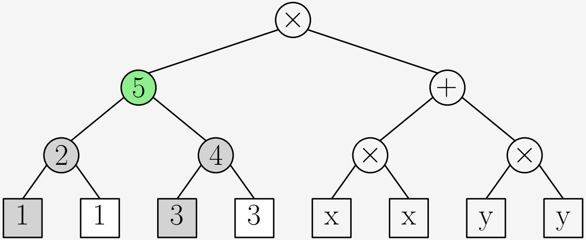

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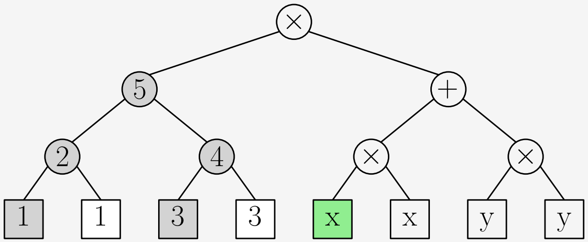

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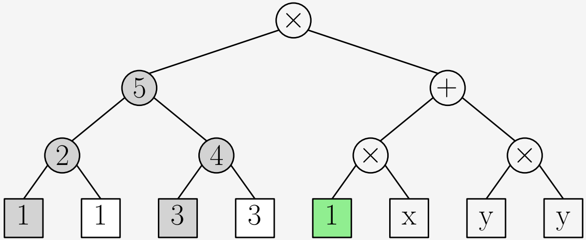

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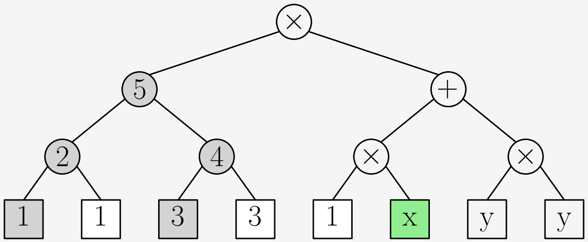

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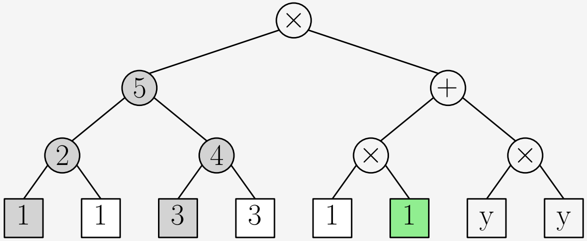

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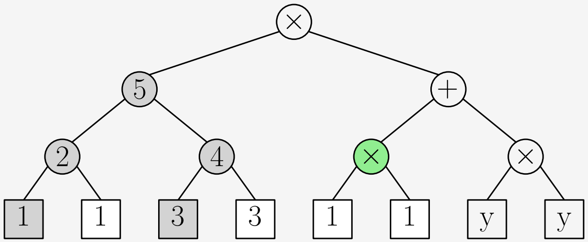

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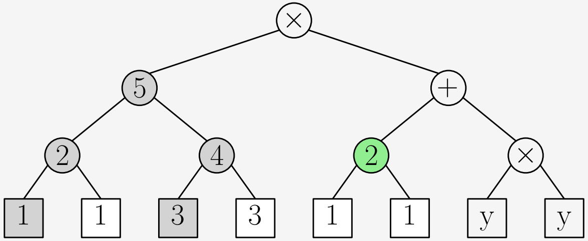

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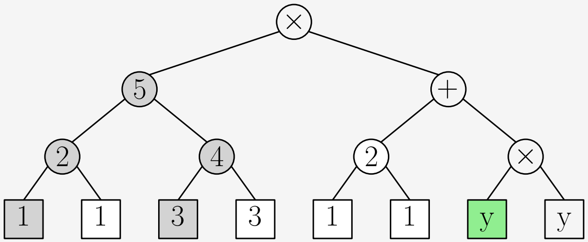

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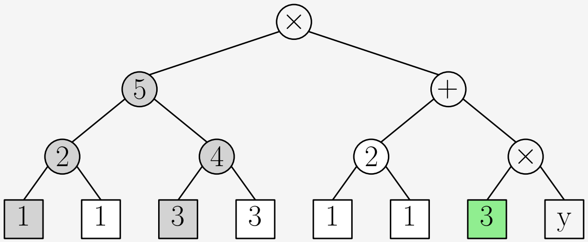

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Motivation: Efficiently store redundant information

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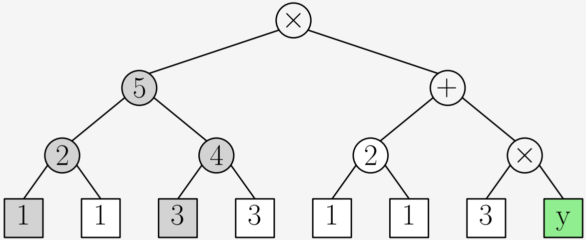

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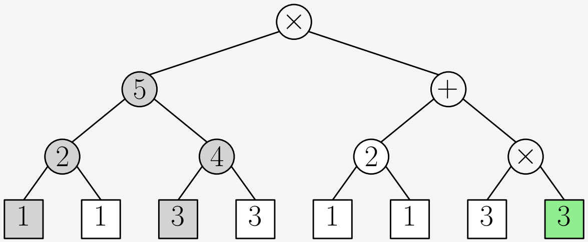

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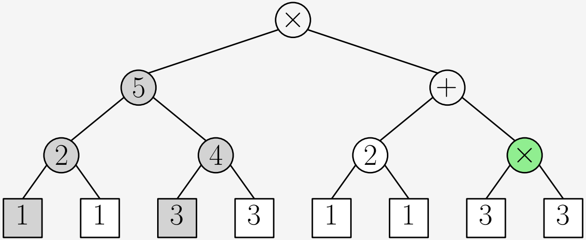

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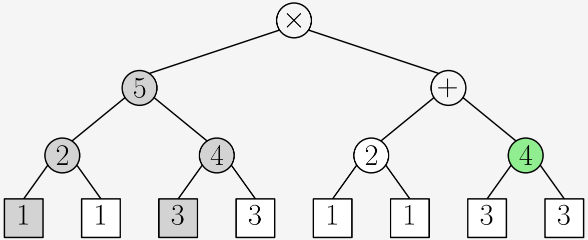

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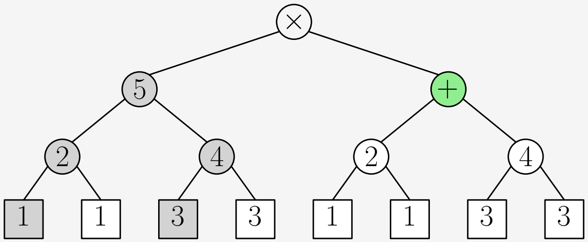

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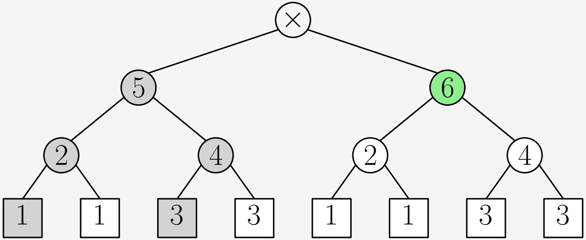

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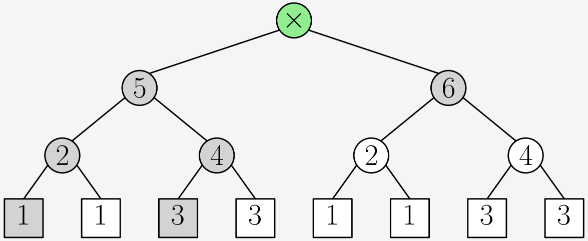

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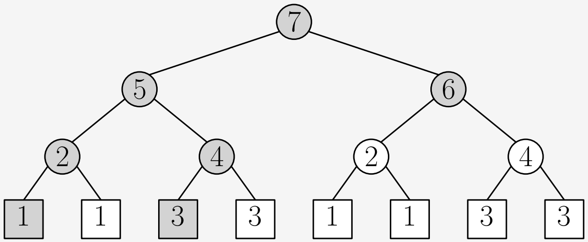

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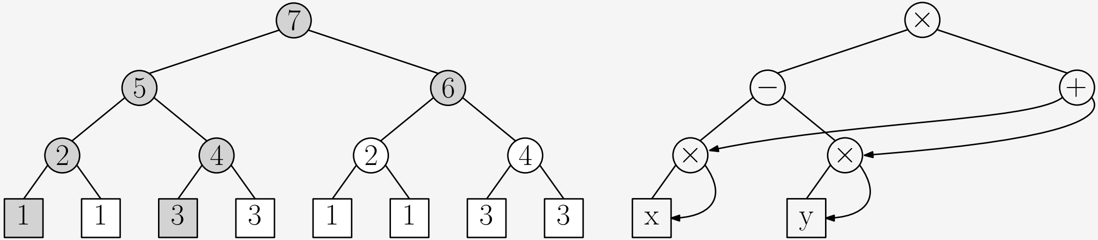

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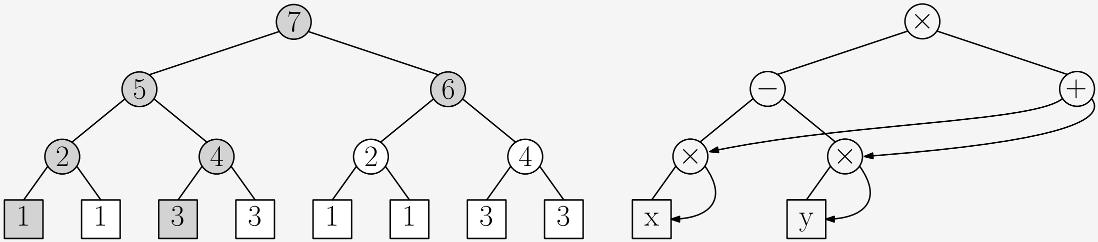

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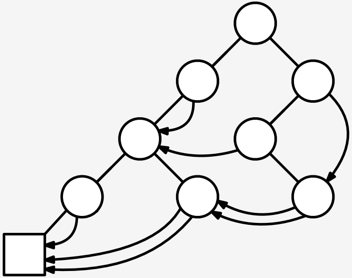
Definition

Compacted tree is the directed acyclic graph computed by this procedure.

Compacted binary trees are special DAGs



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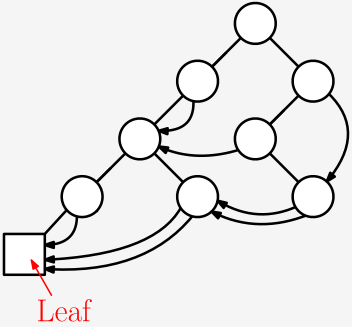

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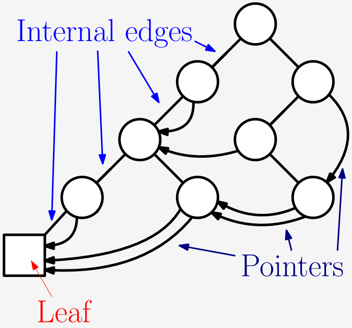

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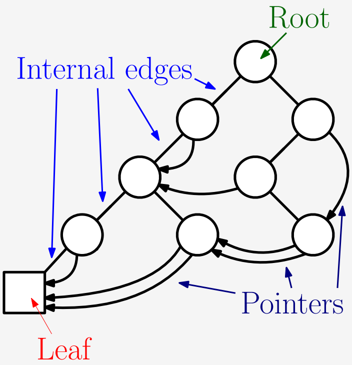

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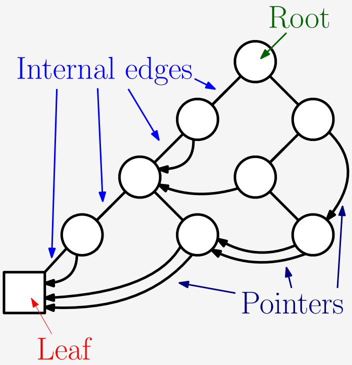

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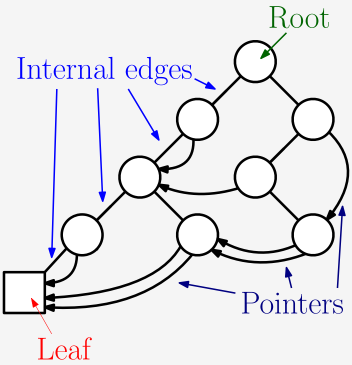

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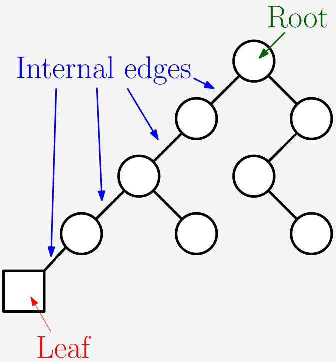

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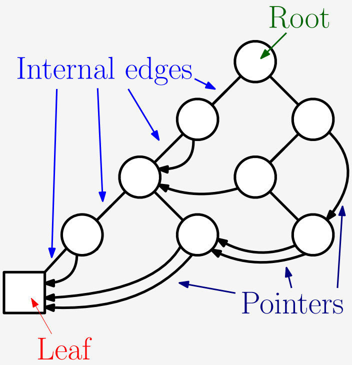

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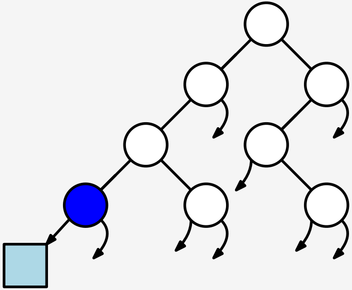

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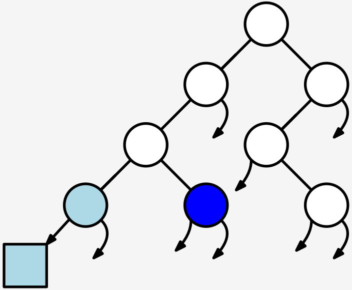

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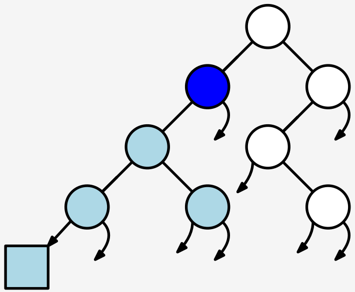

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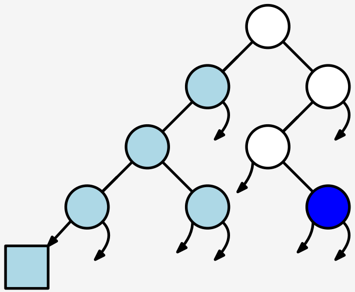

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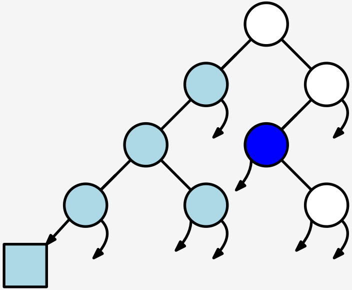

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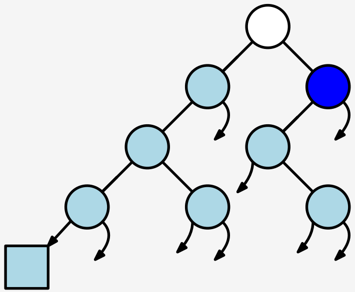

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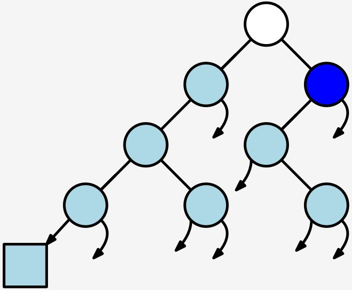

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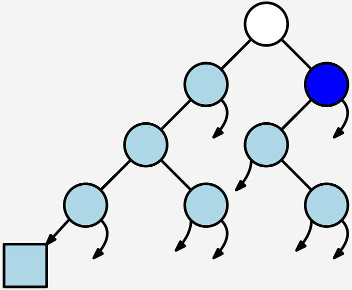
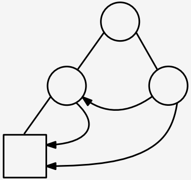
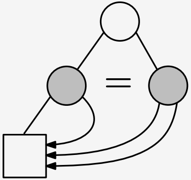




Valid compacted tree



Invalid compacted tree
A relaxed tree

Why are they interesting?

- Applications:
  - **XML-Compression** [Bousquet-Mélou, Lohrey, Maneth, Noeth 2015]
  - **Data storage** [Meinel, Theobald 1998], [Knuth 1968]
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Reverse question

How many compacted trees of (compacted) size n exist?

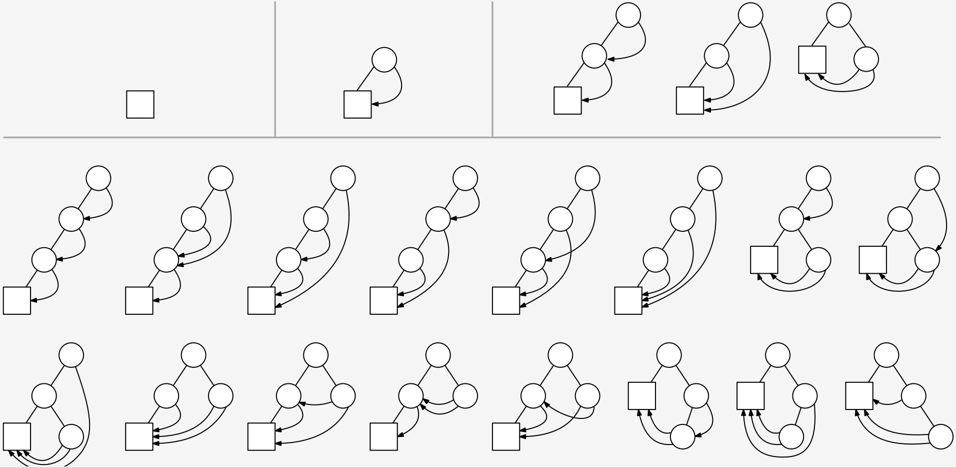
Compacted and relaxed binary trees

- **Size:** number of internal nodes
- $\boldsymbol{r_n}$: nr. of relaxed trees of size n
- $\boldsymbol{c_n}$: nr. of compacted trees of size n (unique subtrees)

$$(r_n)_{n\geq 0} = (1, 1, 3, 16, 127, 1363, 18628, \dots)$$
$$(c_n)_{n\geq 0} = (1, 1, 3, 15, 111, 1119, 14487, \dots)$$

Simple bounds

$$n! \leq c_n \leq r_n \leq \frac{1}{n+1}\binom{2n}{n} n!$$



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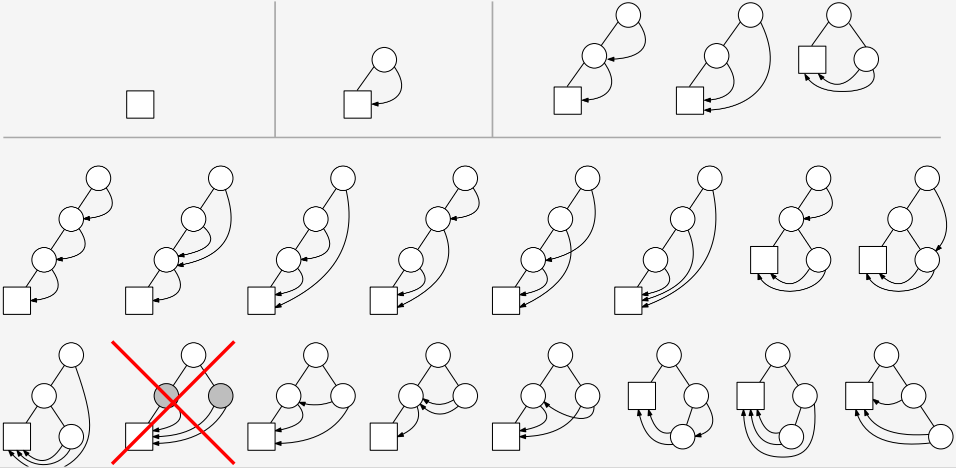
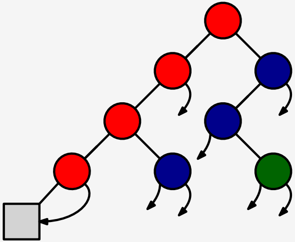


Bounded right height

The **right height** of a binary tree is the maximal number of right children on any path from the root to a leaf (not going through pointers).



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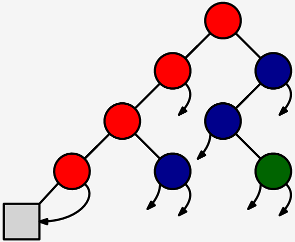

Theorem [Genitrini, Gittenberger, Kauers, W 2020]

The number $r_{k,n}$ ($c_{k,n}$) of relaxed (compacted) trees with right height at most k is for $n \to \infty$ asymptotically equivalent to

$$r_{k,n} \sim \gamma_k n! \left(4\cos\left(\frac{\pi}{k+3}\right)^2\right)^n n^{-\frac{k}{2}},$$

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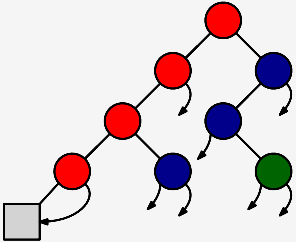

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Remarks:

- Uses exponential generating functions
- GFs are D-finite (order k)
- Methods from Analytic Combinatorics (Singularity analysis, etc.)
- Interesting combinatorics: E.g., $r_{1,n} = (2n-1)!!$

Asymptotics in the binary case

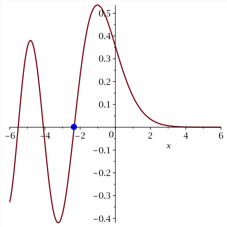
A *stretched exponential* μ^{n^σ} appears!

Theorem [Elvey Price, Fang, W 2021]

The number of relaxed and compacted **binary** trees satisfy for $n \to \infty$

$$r_n = \Theta\left(n!\, 4^n e^{3a_1 n^{1/3}} n\right) \qquad \text{and} \qquad c_n = \Theta\left(n!\, 4^n e^{3a_1 n^{1/3}} n^{3/4}\right),$$

where $a_1 \approx -2.338$ is the largest root of the Airy function $\text{Ai}(x)$.



$\text{Ai}''(x) = x\,\text{Ai}(x)$

Asymptotics in the binary case

A *stretched exponential* μ^{n^σ} appears!

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1. Bijective Comb.: Bijection to decorated Dyck paths
2. Enumerative Comb.: Two-parameter recurrence
3. Calculus + ODEs: Heuristic analysis of recurrence
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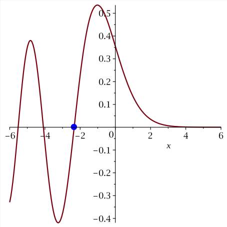

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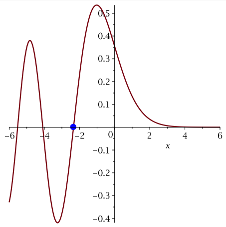

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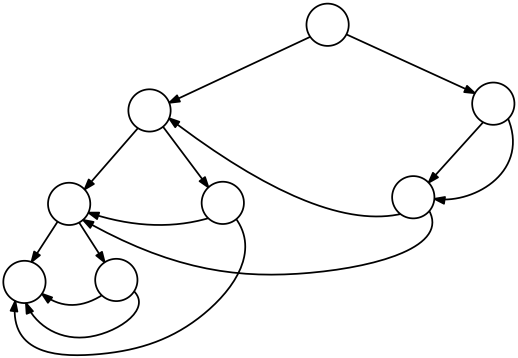

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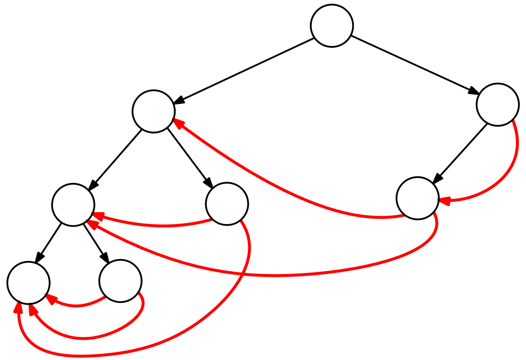

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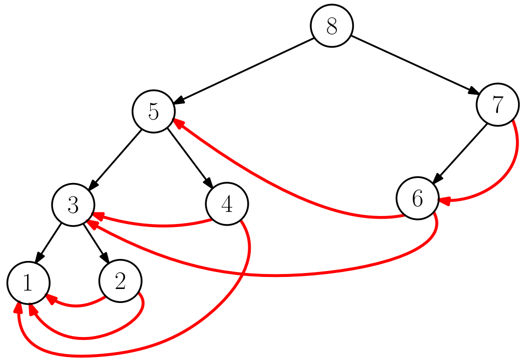

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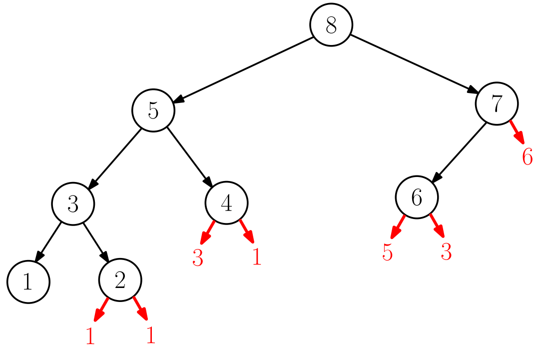

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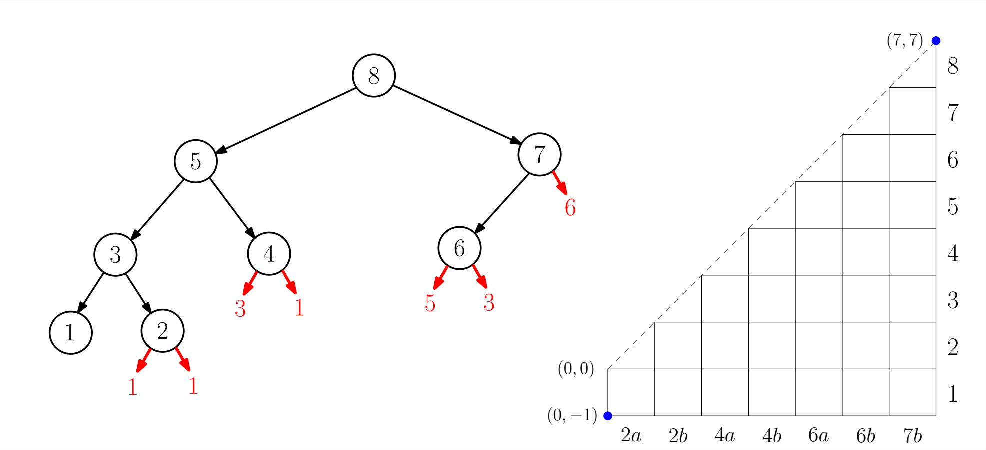

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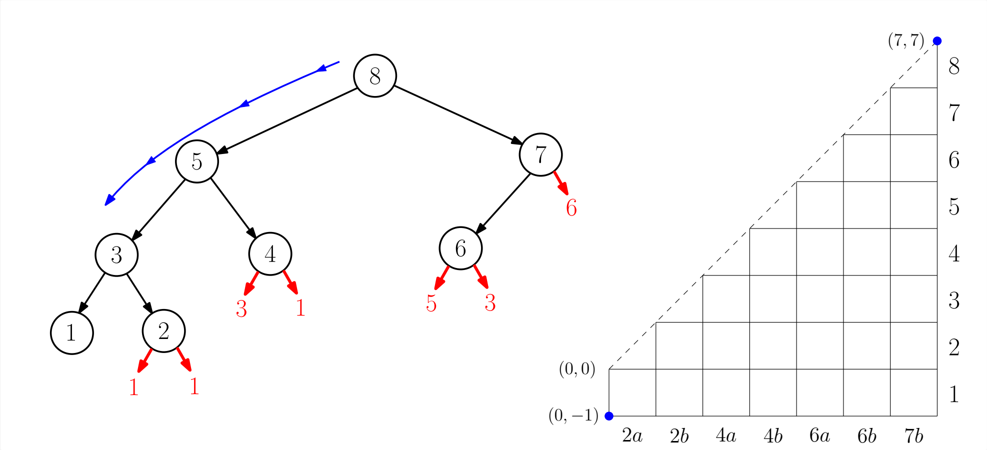

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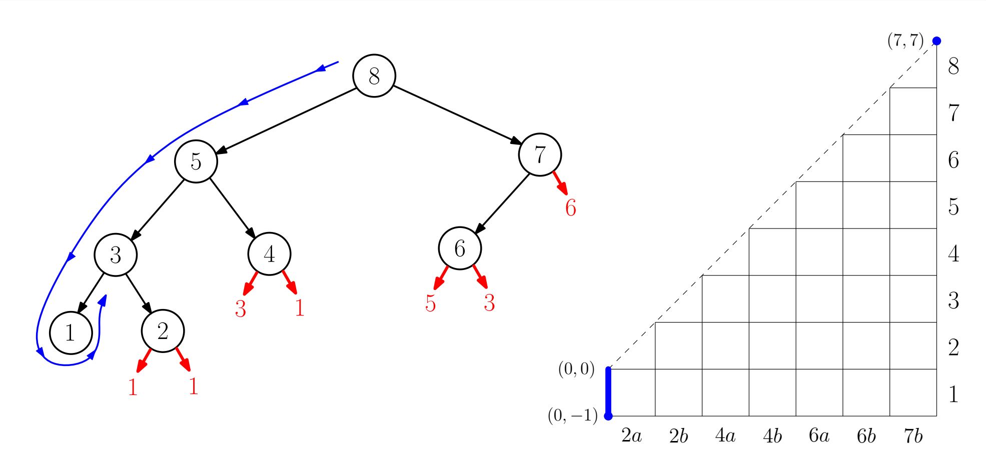

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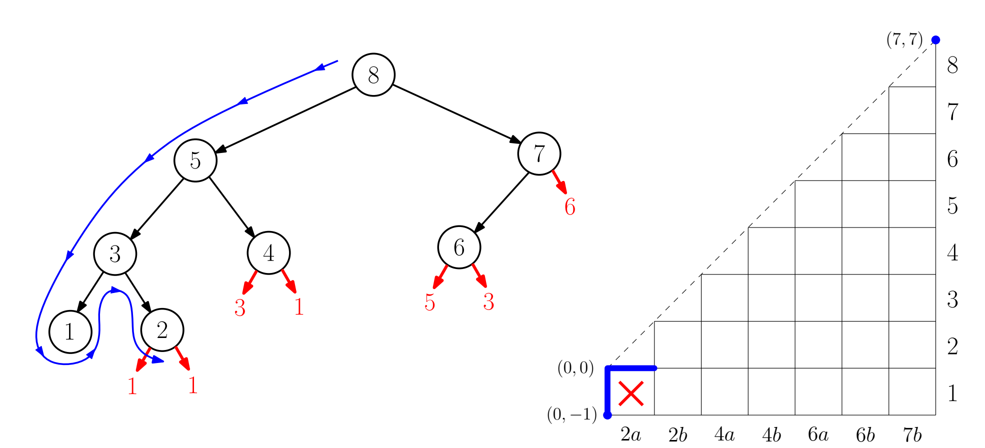

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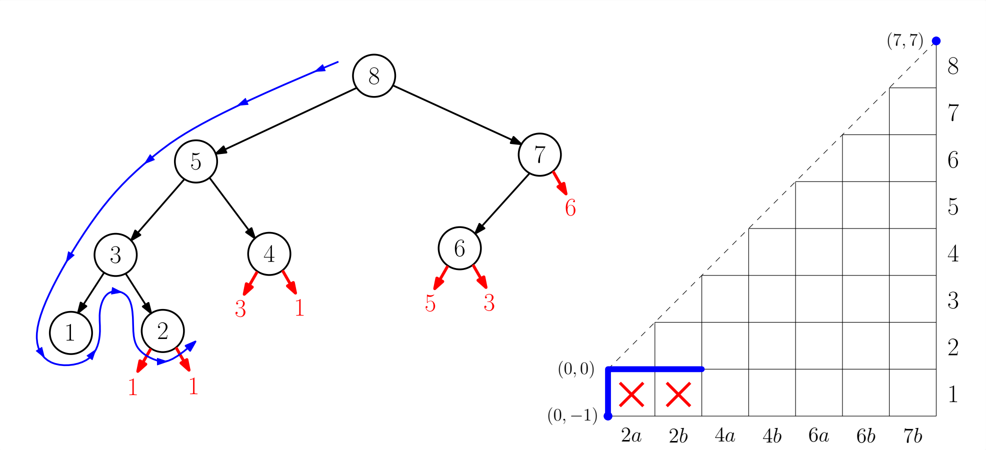

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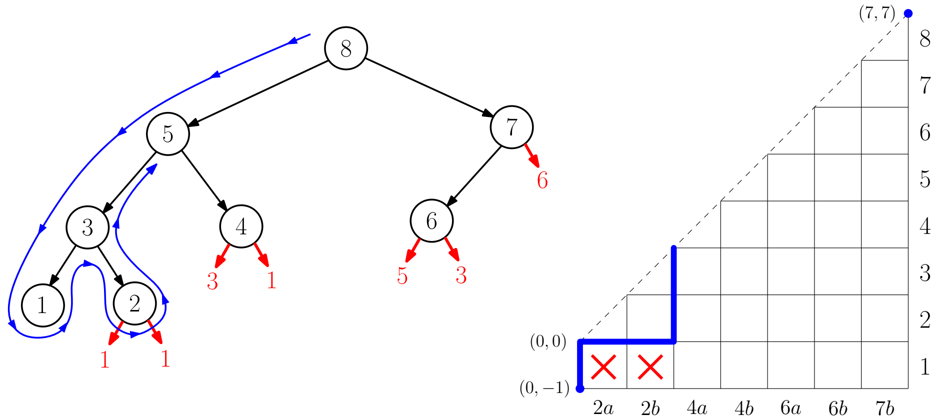

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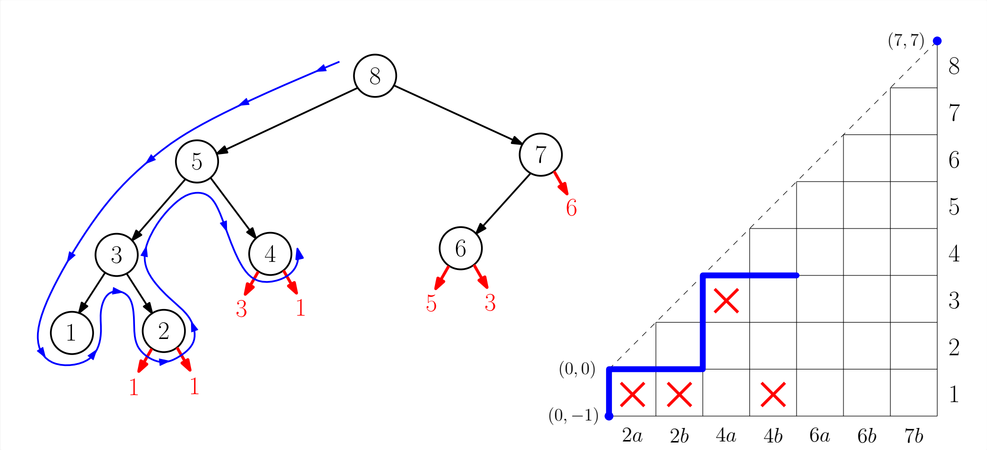

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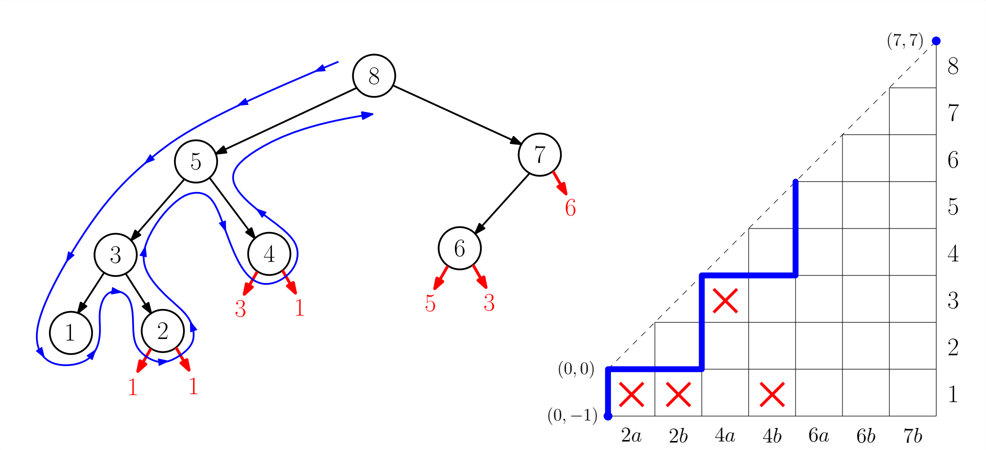

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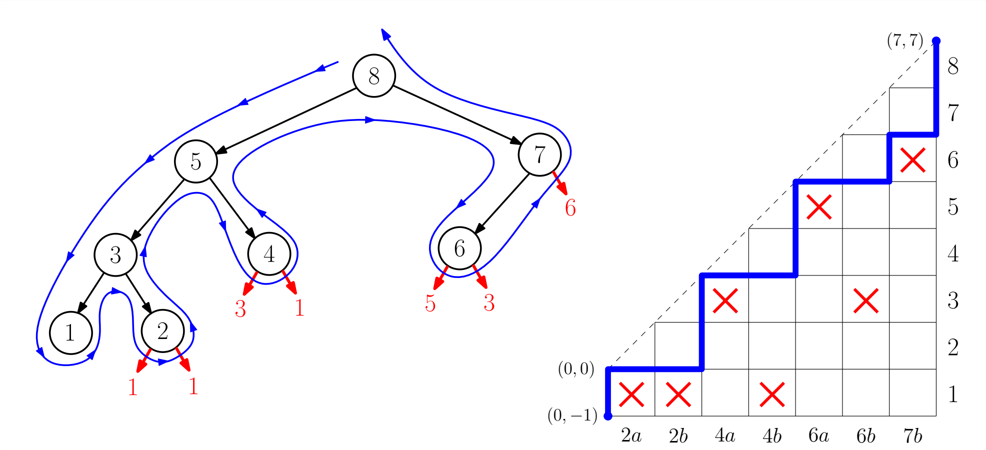

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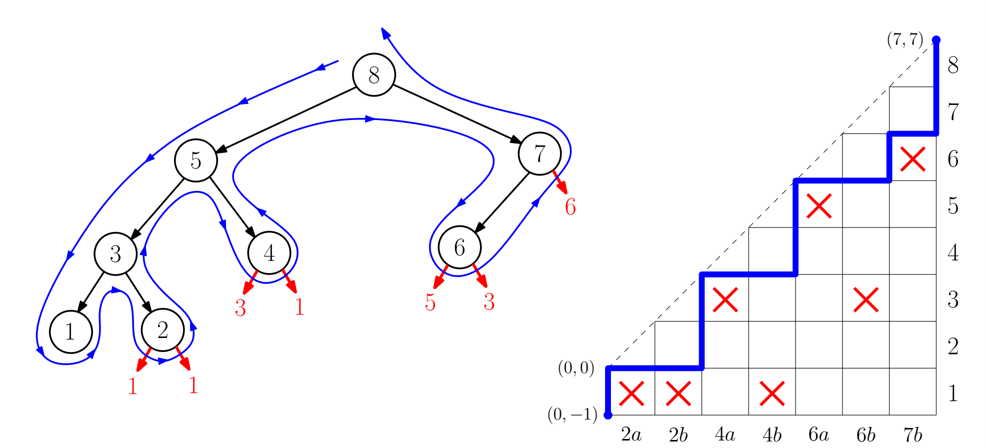

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$$\Rightarrow \quad a_{m,n} = (n+1)a_{m-1,n} + a_{m,n-1} \qquad \text{for } m \geq n \geq 0$$

Most general result: k-ary trees

Theorem [Ghosh Dastidar, W 2024]

The number r_n of relaxed ***k*-ary trees** with n internal nodes satisfies

$$r_n = \Theta\left((n!)^{k-1}\, \gamma(k)^n\, e^{3a_1 \beta(k) n^{1/3}}\, n^{\alpha(k)}\right),$$

with $a_1 \approx -2.338$ is the largest root of the Airy function $\mathrm{Ai}(x)$ and

$$\gamma(k) = \frac{k^k}{(k-1)^{k-1}}, \qquad \beta(k) = \left(\frac{k(k-1)}{2}\right)^{1/3}, \qquad \alpha(k) = \frac{7k-8}{6}.$$

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Conjecture

Experimentally, we find in the binary case ($k = 2$) that

$$r_n \sim \gamma_r n! 4^n e^{3a_1 n^{1/3}} n \qquad \text{and} \qquad c_n \sim \gamma_c n! 4^n e^{3a_1 n^{1/3}} n^{3/4},$$

where

$$\gamma_r \approx 166.95208957 \qquad \text{and} \qquad \gamma_c \approx 173.12670485.$$

Minimal Deterministic Finite Automata

Deterministic finite automata (DFA)

DFA on alphabet $\{a, b\}$

Graph with

- two outgoing edges from each node (state), labelled a and b
- An initial state q_0
- A set F of *final states* (coloured green).

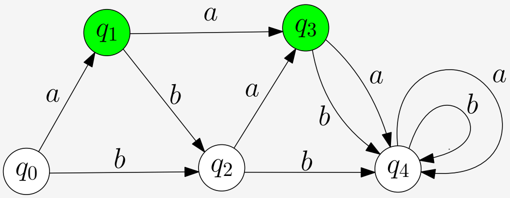

Figure: DFA

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DFA on alphabet $\{a, b\}$

Graph with

- two outgoing edges from each node (state), labelled *a* and *b*
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Properties

- **Language:** the set of accepted words
- **Minimal:** no DFA with fewer states accepts the same language
- **Acyclic:** no cycles (except loops at unique sink)

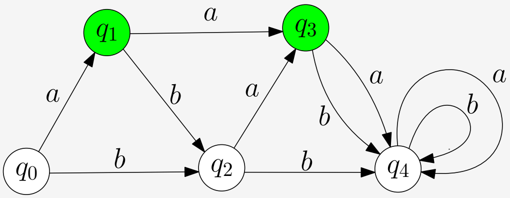
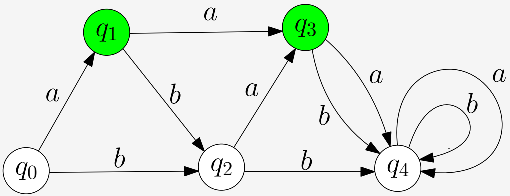


Figure: DFA, which is the minimal DFA recognizing the language $\{a, aa, ba, aba\}$.

Counting minimal acyclic DFAs

- Enumeration studied by Domaratzki, Kisman, Shallit, and Liskovets 2002–2006
- **Open problem:** Asymptotics
- Best bounds were out by an exponential factor



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Main result minimal DFAs

A *stretched exponential* μ^{n^σ} appears again!

Theorem [Elvey Price, Fang, W 2020]

The number m_n of minimal DFAs with $n+1$ states recognizing a finite binary language satisfies for $n \to \infty$

$$m_n = \Theta\left(n!\, 8^n e^{3a_1 n^{1/3}} n^{7/8}\right),$$

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Conjecture

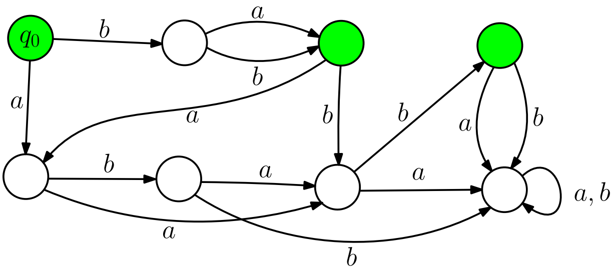
Experimentally we find

$$m_n \sim \gamma n! 8^n e^{3a_1 n^{1/3}} n^{7/8},$$

where

$$\gamma \approx 76.438160702.$$

Bijection to decorated paths



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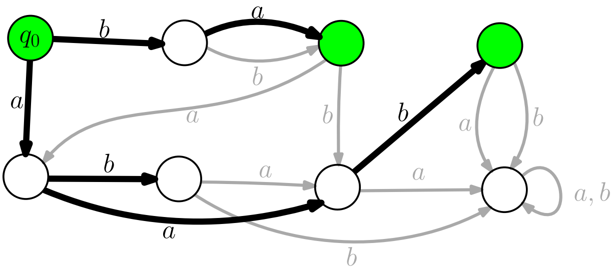

- Highlight spanning tree given by depth first search (ignoring the sink)
- I.e., black path to each vertex is first in lexicographic order

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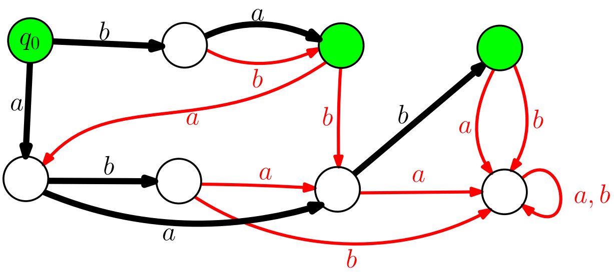

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- Colour other edges red

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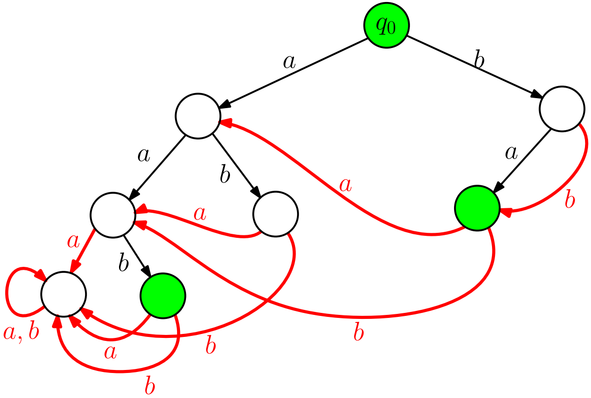
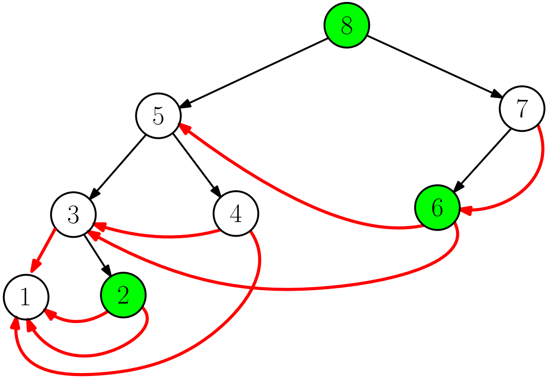


- Highlight spanning tree given by depth first search (ignoring the sink)
- I.e., black path to each vertex is first in lexicographic order
- Colour other edges red
- Draw as a binary tree with *a* edges pointing left and *b* edges pointing right

Bijection to decorated paths



- Label nodes in post-order. By construction red edges point from a larger number to a smaller number

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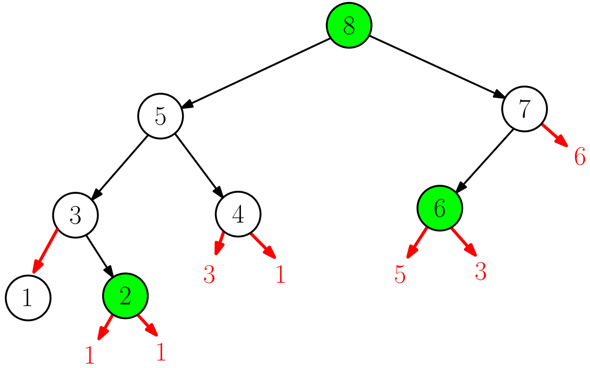

- Label nodes in post-order. By construction red edges point from a larger number to a smaller number
- $\rightarrow$ Label pointers

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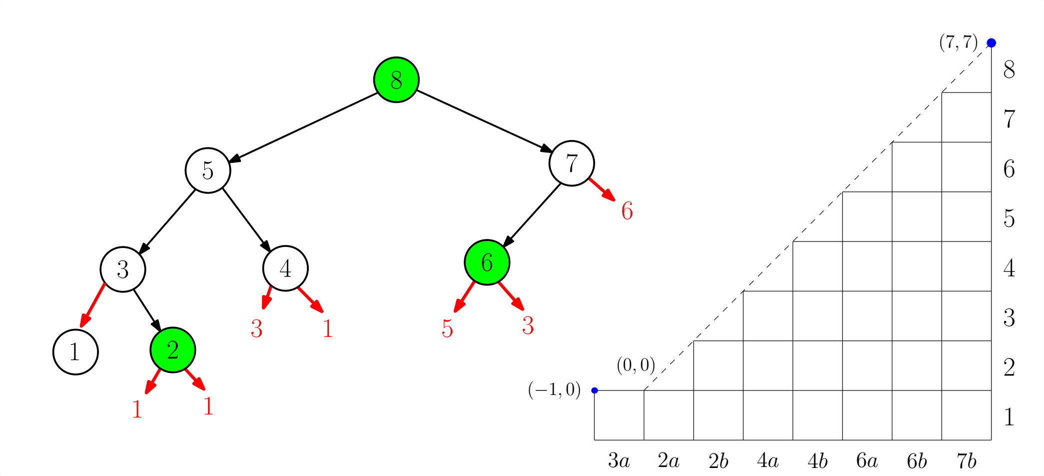

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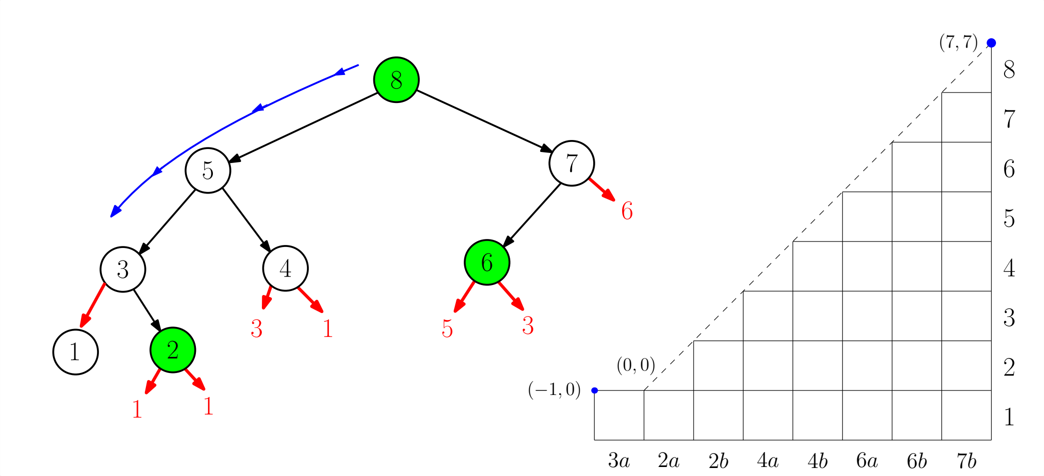

When the tree traversal...

- goes up: add up step with color matching the corresponding node.
- passes a pointer:
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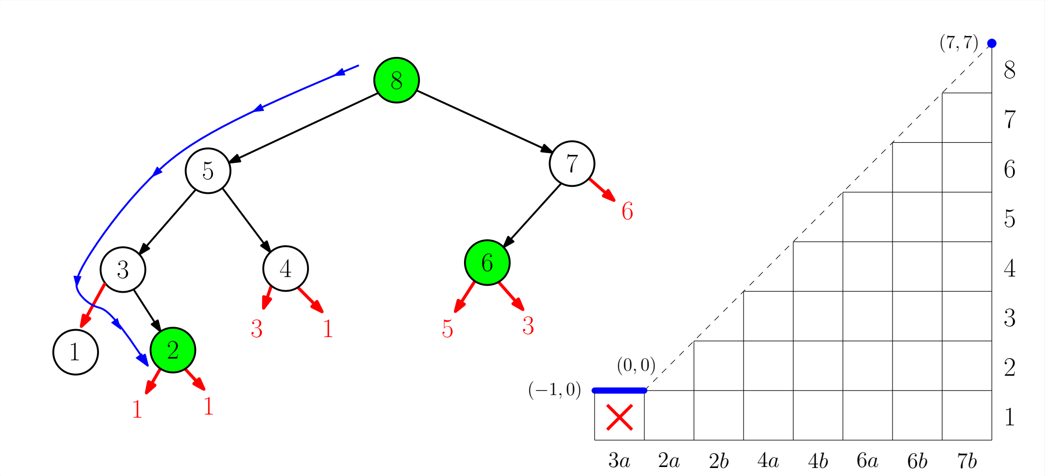
Bijection to decorated paths



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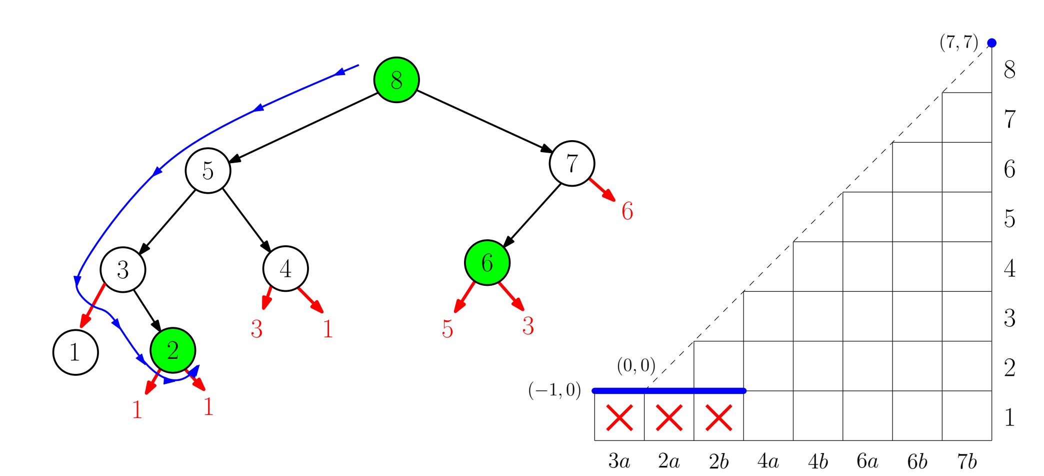

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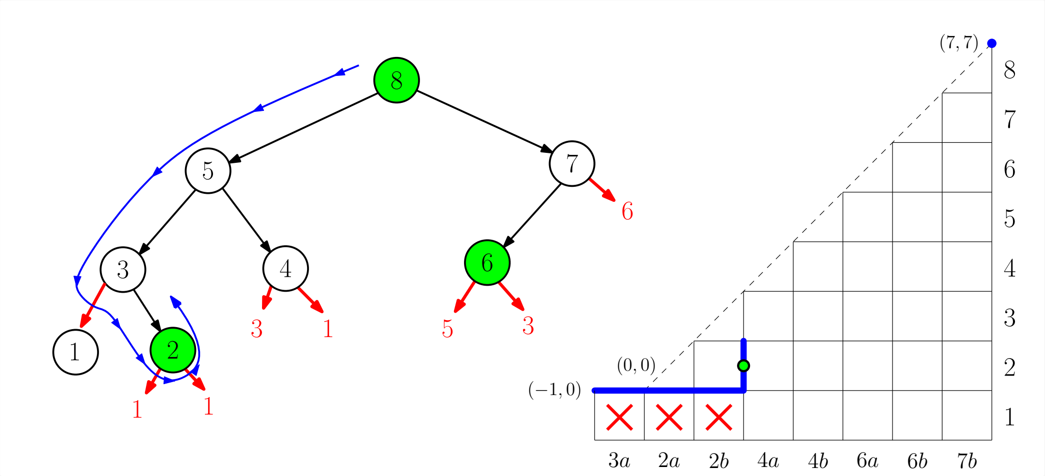

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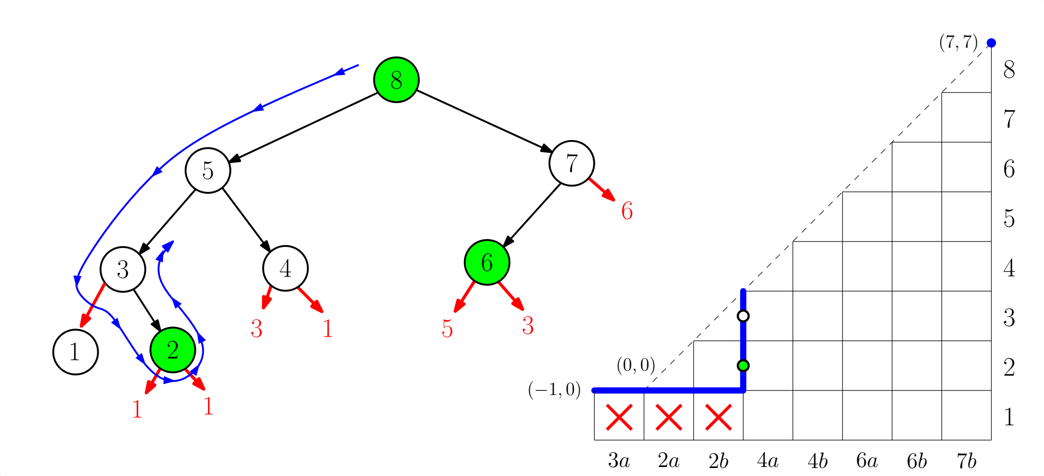

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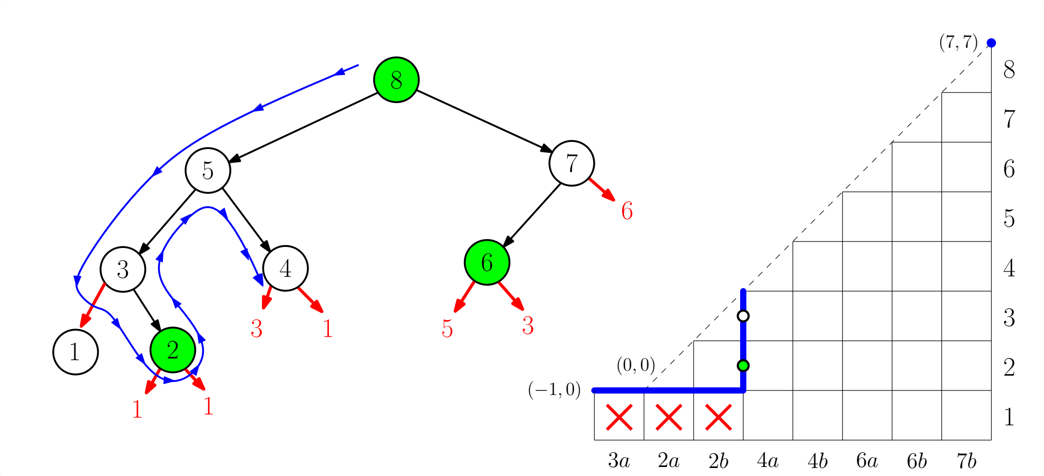

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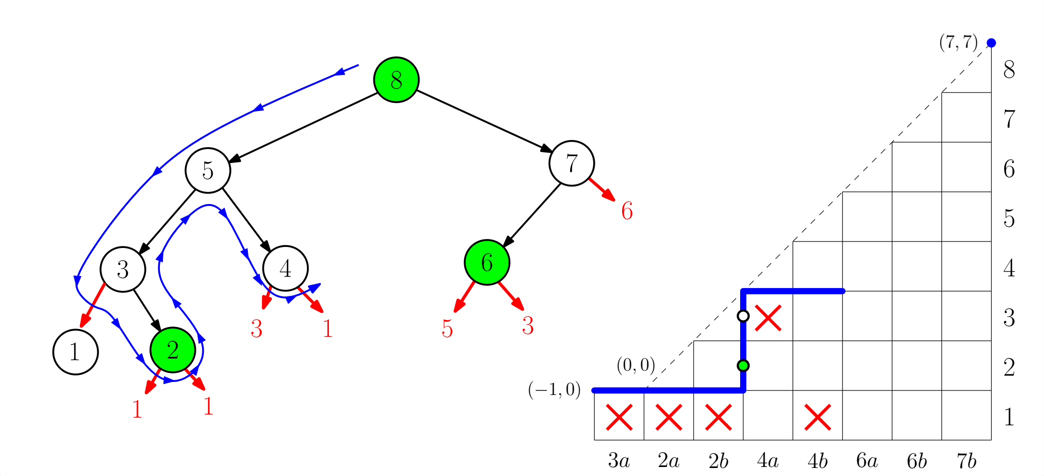

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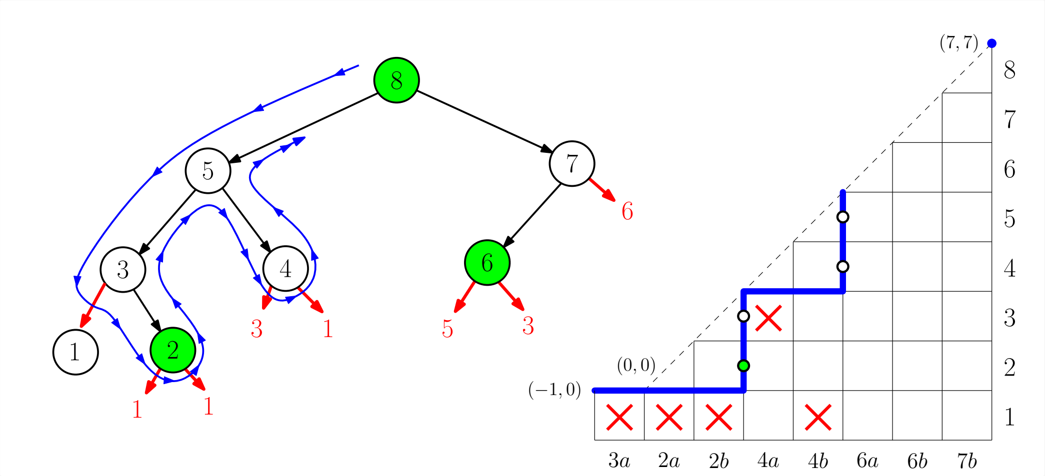

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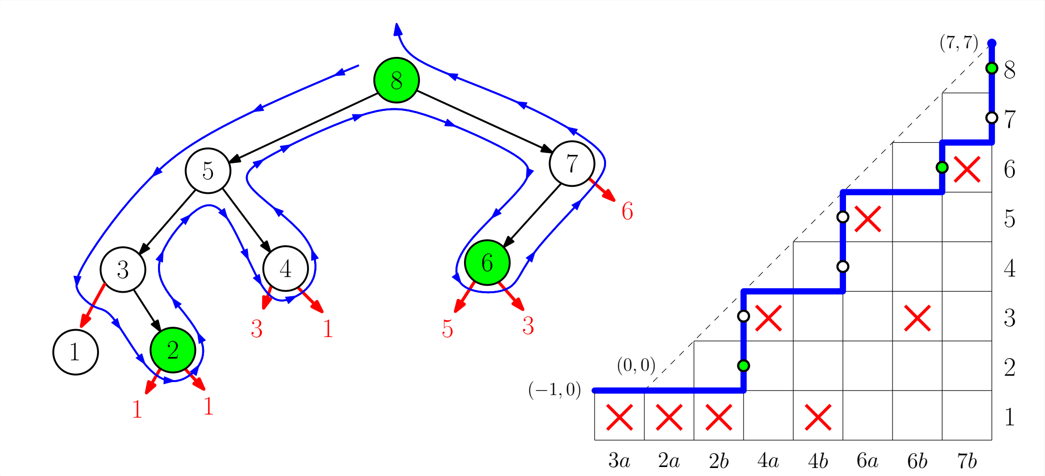

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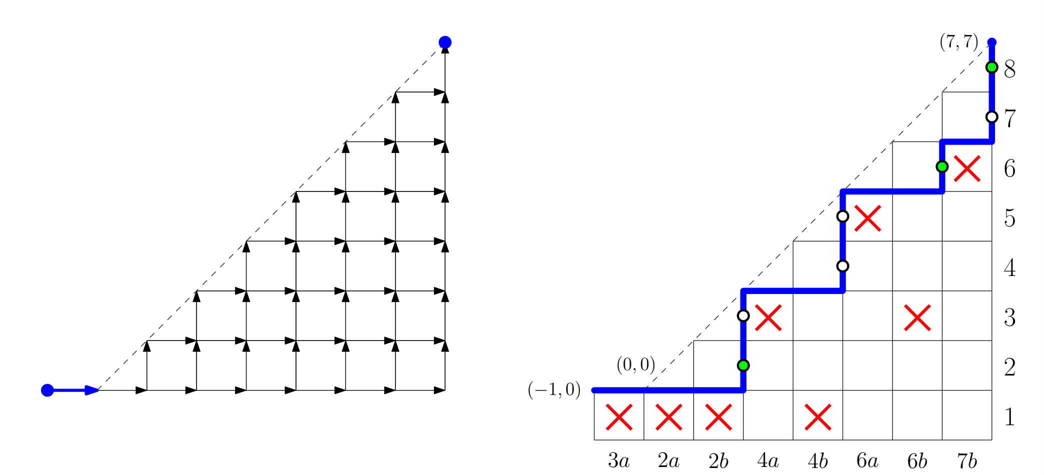
Decorated paths



- Path starts at $(-1, 0)$ and ends at (n, n)
- Path stays below diagonal (after first step)

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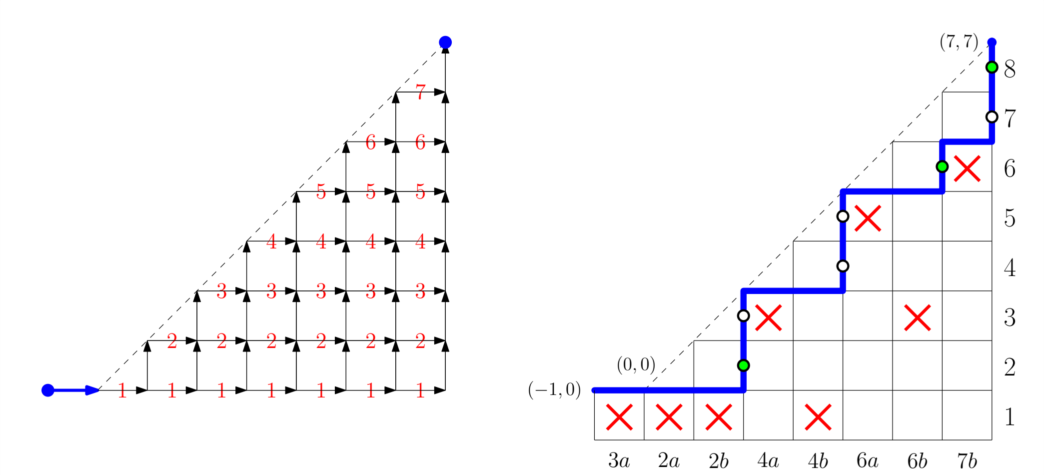

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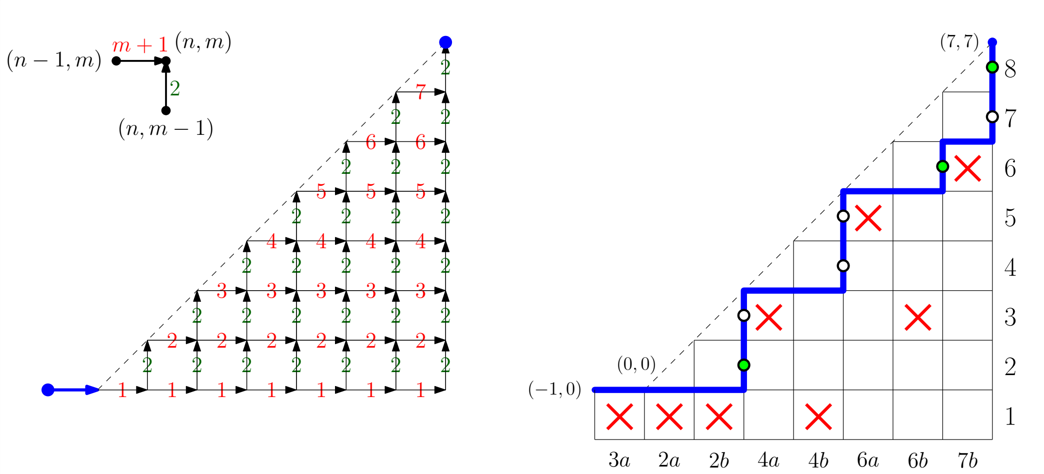

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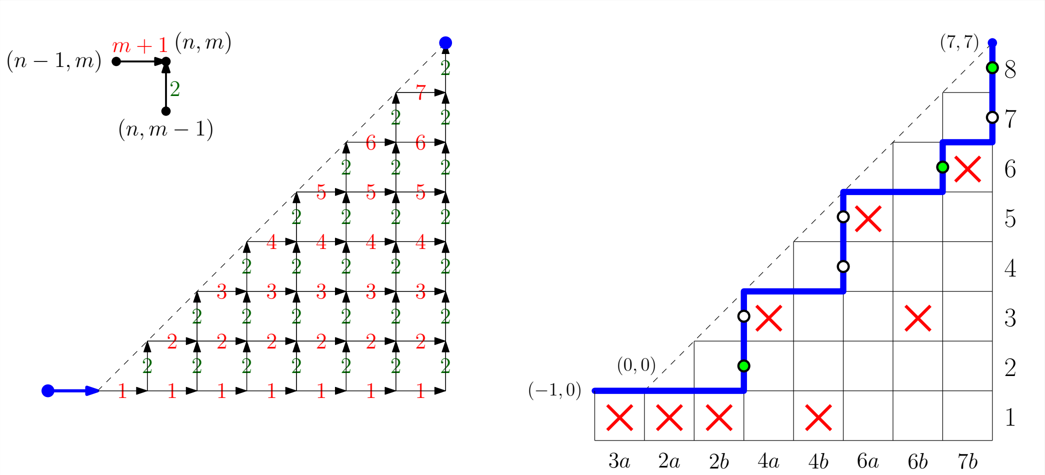
- Each vertical step is colored white or green

By the bijection: The number of these paths is the number d_n of acyclic DFAs with $n+1$ nodes.

Decorated paths



Recurrence: Denote by $a_{n,m}$ the number of paths ending at (n, m).

$$a_{n,m} = 2a_{n,m-1} + (m+1)a_{n-1,m}, \qquad \text{for } n \geq m$$

$$a_{-1,0} = 1.$$

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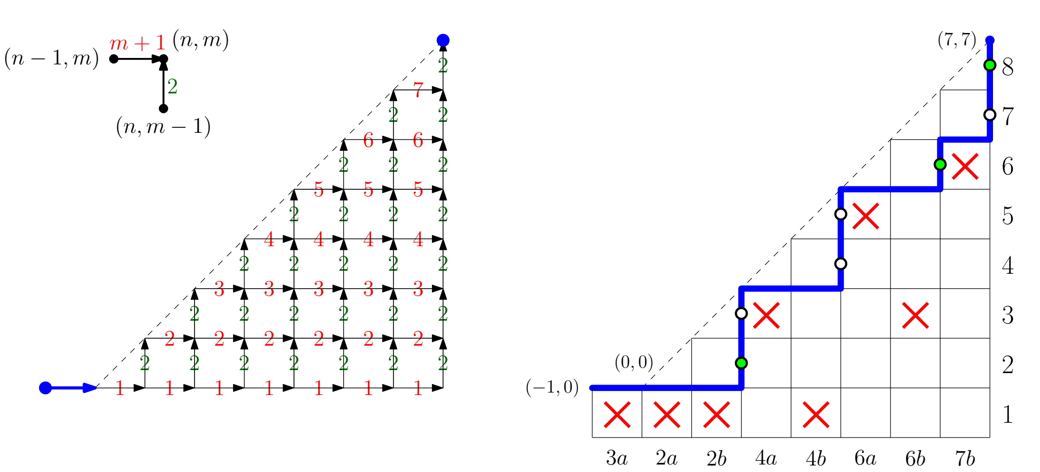

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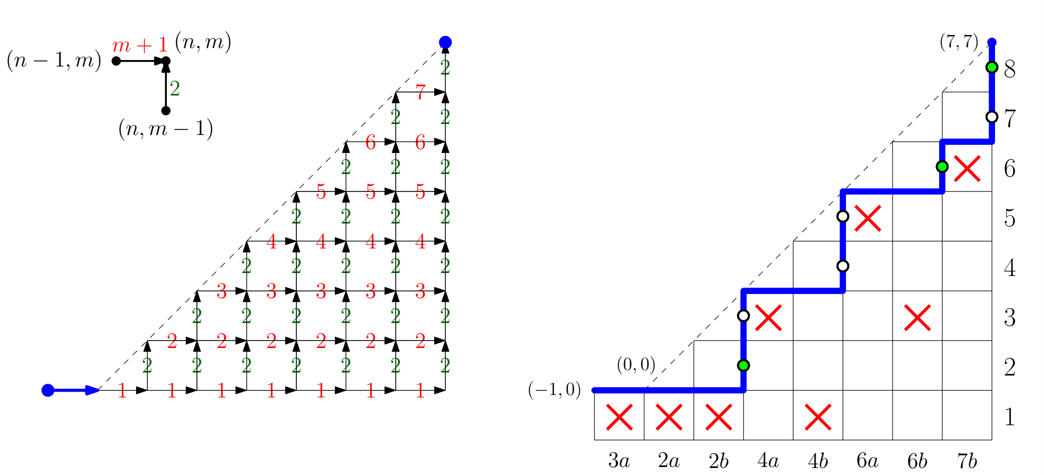
By the bijection: $d_n = a_{n,n}$ is the number of acyclic DFAs with $n + 1$ nodes. What about minimality?

Recurrence for minimal DFAs



Recurrence: Denote by $b_{n,m}$ the number of paths ending at (n, m).

$$b_{n,m} = 2b_{n,m-1} + (m+1)b_{n-1,m} - mb_{n-2,m-1}, \qquad \text{for } n \geq m,$$
$$b_{-1,0} = 1.$$

Now: $m_n = b_{n,n}$ is the number of minimal acyclic DFAs with $n + 1$ nodes.

Phylogenetic tree-child networks

Biology: *d*-combining tree-child networks

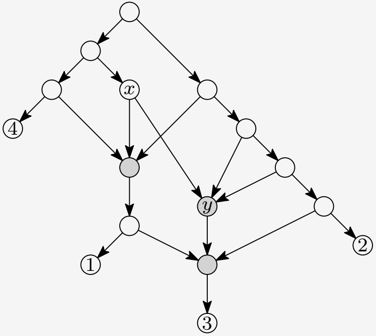
Definition

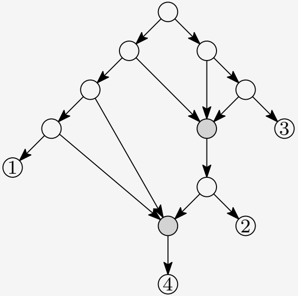
A *d-ary rooted phylogenetic network* is a DAG with nodes of the type:

- *unique root*: indegree 0, outdegree 2
- *leaf*: indegree 1, outdegree 0
- *tree node*: indegree 1, outdegree 2
- *reticulation node*: indegree d, outdegree 1

Furthermore, the n leaves are labeled bijectively by $\{1, \dots, n\}$.

Tree-child: every non-leaf node has at least one child that is not a reticulation.





Asymptotics of d-combining tree-child networks

A *stretched exponential* μ^{n^σ} appears!

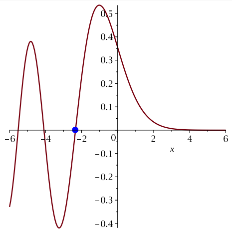
Theorem [Chang, Fuchs, Liu, W, Yu 2023]

The number $\mathrm{TC}_n^{(d)}$ of d-combining tree-child networks with n leaves satisfies

$$\mathrm{TC}_n^{(d)} = \Theta\left((n!)^d \, \gamma(d)^n \, e^{3a_1 \beta(d) n^{1/3}} \, n^{\alpha(d)}\right) \qquad \text{for } n \to \infty,$$

with $a_1 \approx -2.338$: largest root of the Airy function $\mathrm{Ai}(x)$ and

$$\alpha(d) = -\frac{d(3d-1)}{2(d+1)}, \qquad \beta(d) = \left(\frac{d-1}{d+1}\right)^{2/3}, \qquad \gamma(d) = 4\frac{(d+1)^{d-1}}{(d-1)!}.$$



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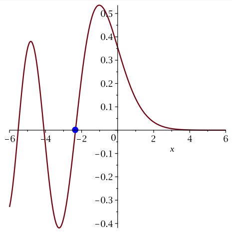
Proof strategy

1. Bijective Comb.: Bijection to Young tableaux with walls
2. Enumerative Comb.: Two-parameter recurrence
3. Calculus + ODEs: Heuristic analysis of recurrence
4. Computer algebra: Inductive proof of asymptotically tight bounds



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$$e_{n,m} = \mu_{n,m}\, e_{n-1,m+1} + \nu_{n,m}\, e_{n-1,m-1}$$

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$$\mu_{n,m} = 1 + \frac{2(d-1)}{(d+1)n + (d-1)m - 2(d+1)} \qquad \text{and} \qquad \nu_{n,m} = \prod_{i=2}^{d} \left(1 - \frac{2(m+i)}{(d+1)(n+m)}\right).$$

We are interested in $e_{2n,0}$, as $\mathrm{TC}_n^{(d)} = \Theta\left((n!)^d \left(\frac{\gamma(d)}{4}\right)^n n^{1-d}\, e_{2n,0}\right)$.

Many new natural appearances of stretched exponentials

Theorem

The number c_n of compressed binary trees,

satisfy for $n \to \infty$

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Characterized by Dyck-like recurrences with rational weight functions:

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- Multiplicative constant? Does it exist?
- Limit shapes: expected height, typical shape, etc.
- Further applications in computer science, biology, physics, etc.

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