Automatic continuity and the Effros property

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Der Wissenschaftsfonds.

Preliminaries

- A function $\cdot : G \times X \longrightarrow X$, where G is a group and X is a set, is a *group action* if $e \cdot x = x$ for all $x \in X$ and $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$.
- A group action $\cdot$ is *transitive* if for all $(x, y) \in X \times X$ there exists $g \in G$ such that $g \cdot x = y$.
- The *graph* of a function $\varphi : G \longrightarrow H$ is

$$\mathrm{Gr}(\varphi) = \{(x, \varphi(x)) : x \in G\} \subseteq G \times H$$

  If φ is a group homomorphism, then $\mathrm{Gr}(\varphi)$ is a group (in fact, it is a subgroup of $G \times H$).

All spaces (including groups) are assumed to be separable and metrizable. All group actions are assumed to be continuous. (Separate continuity would be sufficient.) Since certain results involve determinacy, we will work in $\mathrm{ZF} + \mathrm{DC}$.

An established pattern in set theory

Many properties $\mathcal{P}$ behave as follows:

- Every Borel set of reals satisfies $\mathcal{P}$,
- Under AD, all sets of reals satisfy $\mathcal{P}$,
- Under AC, there exist counterexamples to $\mathcal{P}$,
- Under $V = L$, there exist definable (usually coanalytic) counterexamples to $\mathcal{P}$.

The classical regularity properties ($\mathcal{P} =$ "perfect set property", $\mathcal{P} =$ "Lebesgue measurable" and $\mathcal{P} =$ "Baire property") are the most famous instances of this pattern. More entertaining examples include $\mathcal{P} =$ "not a Hamel basis" and $\mathcal{P} =$ "not an ultrafilter." See my other talk for $\mathcal{P} =$ "σ-homogeneity." This talk is about

$$\mathcal{P} = \text{"Effros,"}$$

in the context of topological groups.

Micro-transitive group actions

Assume that $\cdot$ is an action of the group G on the space X.
Notice that, for every fixed $g \in G$, the function $g \cdot -$ is a homeomorphism of X. (Its inverse is $g^{-1} \cdot -$.)
But what about the function $- \cdot x$ for a fixed $x \in X$?
Let us denote this function by $\gamma_x : G \longrightarrow X$.

Definition (Ancel, 1987)

The action $\cdot$ is *micro-transitive* if $U \cdot x$ is a neighborhood of x for every $x \in X$ and every neighborhood U of the identity in G.

Proposition

Assume that the action $\cdot$ is transitive. Then the following conditions are equivalent:

- *G acts micro-transitively on X,*
- *γ_x is open for every $x \in X$,*
- *γ_x is open for some $x \in X$.*

The Effros property

Definition

We will say that a group G is *Effros* if every transitive action of G on a non-meager space is micro-transitive.

Theorem (Effros, 1965)

Every Polish group is Effros.

Theorem (van Mill, 2004)

Every analytic group is Effros.

Inspired by the "established pattern," we obtained the following:

Theorem

- *Under* AD, *every group is Effros,*
- *Under* AC, *there exists a non-Effros group,*
- *Under* $\mathrm{V} = \mathrm{L}$, *there exists a coanalytic non-Effros group.*

Automatic continuity

The idea of "automatic continuity" is that additional structure on a space makes it easier for functions on it to be continuous.
The following is the most important example in our context:

Theorem (Closed Graph Theorem)

Let G and H be Banach spaces, and let $\varphi : G \longrightarrow H$ be a linear function. If $\mathrm{Gr}(\varphi)$ is closed then φ is continuous.

Theorem (somewhat folklore)

Let G and H be groups, and let $\varphi : G \longrightarrow H$ be a homomorphism. If $\mathrm{Gr}(\varphi)$ is an Effros group and G is non-meager then φ is continuous.

In particular, (in the **separable** case!) the linear structure is irrelevant for the CGT. Also, the key property is not completeness but Effros. Furthermore, by van Mill's result, the **Analytic** Graph Theorem actually holds.

Proof of the somewhat folklore result

Assume that $\mathrm{Gr}(\varphi)$ is an Effros group and that G is non-meager. Consider the action $\cdot$ of $\mathrm{Gr}(\varphi)$ on G obtained by setting

$$(g, \varphi(g)) \cdot x = gx$$

for every $g, x \in G$. Obviously, the action $\cdot$ is transitive.

Since $\mathrm{Gr}(\varphi)$ is Effros and G is non-meager, it follows that $\cdot$ is micro-transitive. Therefore, the bijection $\gamma_e : \mathrm{Gr}(\varphi) \longrightarrow G$ associated to $\cdot$ is open, where e denotes the identity of G.

This means that γ_e^{-1} is continuous, hence so is $\varphi = \pi \circ \gamma_e^{-1}$, where $\pi : G \times H \longrightarrow H$ denotes the natural projection.

We remark that a similarly Effros-centric version of the Open Mapping Theorem also holds.

The result under AD

Theorem

Assume AD. *Then every group is Effros.*

The proof employs a sophisticated technique known as "stealing from Jan van Mill." (But the due credit was given!)

Also, this result does not need the full force of AD, but only the fact all subsets of a (Polish) space have the property of Baire.

The counterexample in ZFC

There are several examples of discontinuous group homomorphisms that we could have used in the following proof. (The one we chose yields a meager non-Effros group. For a Baire example, consider a non-principal ultrafilter on ω with its natural group structure.)

Theorem

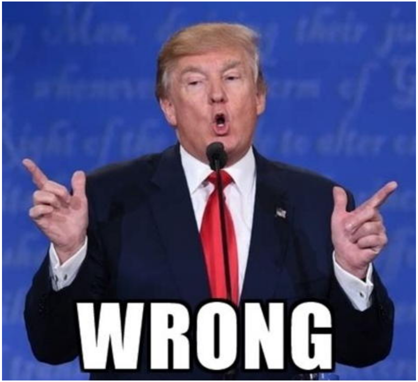
In ZFC, *there exists a non-Effros group.*

Proof. Using AC, we can fix a basis $\mathcal{H}$ for $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Since $\mathcal{H}$ is uncountable, we can pick $h_\infty \in \mathcal{H}$ and $h_n \in \mathcal{H} \setminus \{h_\infty\}$ for $n \in \omega$ such that $h_n \to h_\infty$.
Let $\varphi : \mathbb{R} \longrightarrow \mathbb{Q}$ be the unique linear functional such that

$$\varphi(h) = \begin{cases} 1 & \text{if } h = h_\infty, \\ 0 & \text{if } h \neq h_\infty \end{cases}$$

for every $h \in \mathcal{H}$. Since φ is discontinuous, $\mathrm{Gr}(\varphi)$ is non-Effros. ☕

Obviously, if you're a topologist, studying computability theory is a complete waste of time...



Definable counterexamples under V = L

In his 1989 paper, Miller sketched a method for constructing coanalytic versions of certain pathological sets of reals (in the spirit of Gödel's coanalytic set without the perfect set property).
In his Ph.D. thesis, Vidnyánszky gave a "black box" version of this.

Definition (Vidnyánszky, 2014)

Given $F \subseteq M^{\leq\omega} \times B \times M$, where M and B are sets of size ω_1, we will say that $X \subseteq M$ is *compatible with* F if there exist enumerations $B = \{p_\alpha : \alpha < \omega_1\}$, $X = \{x_\alpha : \alpha < \omega_1\}$ and, for every $\alpha < \omega_1$, a sequence $A_\alpha \in M^{\leq\omega}$ that is an enumeration of $\{x_\beta : \beta < \alpha\}$ in type $\leq \omega$ such that $x_\alpha \in F_{(A_\alpha, p_\alpha)}$ for every $\alpha < \omega_1$.

Here, given $(A, p) \in M^{\leq\omega} \times B$, we use the notation $F_{(A,p)} = \{x \in M : (A, p, x) \in F\}$. Intuitively, one should think of A_α as enumerating the portion of the desired set X constructed before stage α. The section $F_{(A_\alpha, p_\alpha)}$ consists of the admissible candidates to be added at stage α, where p_α encodes the current condition to be satisfied.

Suppose M is a space in which it makes sense to talk about computability. We will say that $S \subseteq M$ is *cofinal in the Turing degrees* if for every $a \in M$ there exists $x \in S$ such that $a \leq_{\mathrm{T}} x$ (that is, a can be computed using x as an *oracle*).
In the following result, for concreteness, assume that $M = B = 2^\omega$.

Theorem (Vidnyánszky, 2014)

Assume $\mathrm{V} = \mathrm{L}$. *Assume that* $F \subseteq M^{\leq\omega} \times B \times M$ *is coanalytic, and that for all* $(A, p) \in M^{\leq\omega} \times B$ *the section* $F_{(A,p)}$ *is cofinal in the Turing degrees. Then there exists a coanalytic* $X \subseteq M$ *that is compatible with* F.

Essentially, the above says that if "the construction process is coanalytic" and the set of possible candidates is always rich enough, then the desired set can be made coanalytic.

Unfortunately, there are situations in which more than one element must be added at every stage (even more unfortunately, this is the case when constructing a group).

Let $Z = 2^\omega$, and observe that Z^ξ can be identified with Z whenever $2 \leq \xi \leq \omega$ for the purposes of computability. When the space M is in the form Z^ξ, we will say that $S \subseteq M$ is *equicofinal in the Turing degrees* if for every $a \in Z$ there exists $x \in S$ such that the following conditions are satisfied:

- $a \leq_T x(n)$ for every $n \in \xi$,
- $x(m) \equiv_T x(n)$ for every $m, n \in \xi$.

Theorem

Assume $V = L$. *Let* $M = Z^\xi$, *where* $2 \leq \xi \leq \omega$. *Assume that* $F \subseteq M^{\leq\omega} \times B \times M$ *is coanalytic, and that for all* $(A, p) \in M^{\leq\omega} \times B$ *the section* $F_{(A,p)}$ *is equicofinal in the Turing degrees. Then there exists* $X \subseteq M$ *such that the following conditions are satisfied:*

- X *is compatible with* F,
- $\{x(n) : x \in X \text{ and } n \in \xi\}$ *is coanalytic.*

A definable non-Effros group under $\mathsf{V} = \mathsf{L}$

Theorem

Assume $\mathsf{V} = \mathsf{L}$. *Then there exists a discontinuous group homomorphism* $\varphi : 2^\omega \longrightarrow 2^\omega$ *with coanalytic graph.*

Corollary

Assume $\mathsf{V} = \mathsf{L}$. *Then there exists a coanalytic non-Effros group.*

Proof of the theorem. The strategy is to accomplish the goal at step 0, then survive until the end (that is, keep the construction within the constraints of Vidnyánszky's method.)

Set $Z = 2^\omega \times 2^\omega$ and $M = Z^\omega$ (at each stage, we will add an element of Z and all of its sums with previous elements).

We will begin by constructing a countable dense subgroup G_0 of Z that is the graph of a homomorphism φ_0 between (countable) subgroups of 2^ω.

First define $\mathbf{e}_n \in 2^\omega$ for $n \in \omega$ by setting

$$\mathbf{e}_n(m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Let E denote the subgroup of 2^ω generated by $\{\mathbf{e}_n : n \in \omega\}$.
Fix $\pi : \omega \longrightarrow \omega$ such that $\pi^{-1}(n)$ is infinite for every $n \in \omega$.
Let $\varphi_0 : E \longrightarrow 2^\omega$ be the unique homomorphism such that $\varphi_0(\mathbf{e}_n) = \mathbf{e}_{\pi(n)}$ for each n. Then $G_0 = \mathrm{Gr}(\varphi_0)$ will be as desired.

Our plan is to construct a set $X \subseteq M$ such that

$$G = G_0 \cup \{x(n) : x \in X \text{ and } n \in \omega\}$$

is the graph of a homomorphism $\varphi : 2^\omega \longrightarrow 2^\omega$. Since $G_0 \subseteq G$ and G_0 is dense in Z, it is clear that φ will be discontinuous.

The set of "conditions" to be satisfied will be $B = 2^\omega$.
More precisely, we will make sure that each $p \in B$ will be added to the domain of our homomorphism at some stage.

Declare $(A, p, x) \in F$, if one of the following conditions holds, where we denote by $\{(z_n, w_n) : n \in \omega\}$ all the pairs that are either in G_0 or are enumerated by A:

- p is not new (that is, $p = z_n$ for some $n \in \omega$). Then there must be $(z, w) \in 2^\omega \times 2^\omega$ such that z is new (that is, $z \neq z_n$ for every $n \in \omega$) and x enumerates $\{(z_n + z, w_n + w) : n \in \omega\}$.
- p is new. As above, but we must have $z = p$.

It is straightforward to check that F is coanalytic (in fact, Borel).

Also notice that we are completely free in our choice of w. This makes it possible to code enough information into it to satisfy the equicofinality condition.

To conclude the proof, apply the "multivariable' version of Vidnyánszky's theorem.

Thank you for listening!