Valuations in the Affine Geometry of Convex Bodies

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Abstract

A survey on $\mathrm{SL}(n)$ invariant and $\mathrm{SL}(n)$ covariant valuations is given.

1 Introduction

A function Φ defined on convex bodies in $\mathbb{R}^n$ and taking values in an Abelian semigroup is called a *valuation* if

$$\Phi(K) + \Phi(L) = \Phi(K \cup L) + \Phi(K \cap L) \quad \text{for } K, L, K \cup L \in \mathcal{K}^n,$$

where $\mathcal{K}^n$ is the set of convex bodies (convex, compact sets) in $\mathbb{R}^n$. Thus the notion of valuation is a generalization of the notion of measure. In the 1930s, Blaschke obtained the first classification of real valued valuations that are $\mathrm{SL}(n)$ invariant. This was greatly extended by Hadwiger in his famous classification of continuous and rigid motion invariant valuations.

Theorem 1 (Hadwiger [26]). *A functional $\Phi : \mathcal{K}^n \to \mathbb{R}$ is a continuous and rigid motion invariant valuation if and only if there are constants $c_0, c_1, \dots, c_n \in \mathbb{R}$ such that*

$$\Phi(K) = c_0\, V_0(K) + \ldots + c_n\, V_n(K)$$

for every $K \in \mathcal{K}^n$.

Here $V_0(K), \dots, V_n(K)$ are the intrinsic volumes of K; $V_n = V$ is the ordinary volume, V_{n-1} is proportional to the surface area and V_0 is the Euler Characteristic. The classical theory of valuations and their applications in integral geometry and geometric probability are described in the books and surveys [26, 35, 67, 70].

In recent years there have been many new developments in the theory of valuations (see [1, 7, 12, 13, 19, 32, 34, 68, 69, 82, 84]). Here we confine our attention to valuations in the affine geometry of convex bodies. Let $\mathrm{SL}(n)$ denote the special linear group, that is, the group of $n \times n$ matrices of determinant 1. We say that a functional Φ is $\mathrm{SL}(n)$ *invariant* if

$$\Phi(\alpha K) = \Phi(K) \qquad \forall K \in \mathcal{K}^n, \forall \alpha \in \mathrm{SL}(n).$$

We say that Φ is *equi-affine invariant* if it is $\mathrm{SL}(n)$ invariant and translation invariant. These notions are important for real valued valuations. We say that a functional Φ is $\mathrm{SL}(n)$ *covariant* if

$$\Phi(\alpha K) = \alpha\, \Phi(K) \qquad \forall K \in \mathcal{K}^n, \forall \alpha \in \mathrm{SL}(n).$$

This notion is important for vector and tensor valued valuations as well as for convex body and star body valued valuations. In the following, we describe classification theorems for $\mathrm{SL}(n)$ invariant and $\mathrm{SL}(n)$ covariant valuations and make some remarks on related results for rotation invariant and rotation covariant valuations.

2 Real valued valuations on polytopes

Clearly, the intrinsic volumes $V_1, \dots, V_{n-1}$ are not $\mathrm{SL}(n)$ invariant and Hadwiger's characterization theorem implies that every continuous, equi-affine invariant valuation on $\mathcal{K}^n$ is a linear combination of V_0 and V_n. It is very easy to see this directly. We sketch this well-known proof here. The result is far from best possible (there are characterizations of volume using only translation invariance, see [67] and [32]) but the arguments are used in one form or another in many of the proofs for characterization theorems.

Let $\mathcal{P}^n$ denote the set of convex polytopes in $\mathbb{R}^n$. Let $\Phi : \mathcal{P}^n \to \mathbb{R}$ be an equi-affine invariant valuation. Since Φ is translation invariant, we have $\Phi(\{x\}) = c_0$ for every singleton $\{x\}$, $x \in \mathbb{R}^n$. Therefore the functional $\Phi_0(P) = \Phi(P) - c_0$ is *simple*, that is, it vanishes on lower dimensional sets. Now let S be an n-dimensional simplex of volume s. Since Φ_0 is $\mathrm{SL}(n)$ invariant, $\Phi_0(S)$ depends only on s, that is, there is a function $f : [0, \infty] \to \mathbb{R}$ such that $\Phi_0(S) = f(s)$. We can subdivide S by cutting with a hyperplane containing an $(n-2)$-dimensional face of S into two simplices S_1, S_2 of volume s_1 and s_2, respectively. Since Φ_0 is a simple valuation, we have $\Phi_0(S) = \Phi_0(S_1) + \Phi_0(S_2)$ and therefore

$$f(s) = f(s_1 + s_2) = f(s_1) + f(s_2).$$

This holds for every $s_1, s_2 \ge 0$. Thus f is a solution of Cauchy's functional equation. If we assume that Φ is (Borel) measurable, we can conclude that $f(s) = c_1\, s$. Thus

$$\Phi(P) = c_0 + c_1\, V(P)$$

for every $P \in \mathcal{P}^n$.

Next, we consider the corresponding problem on $\mathcal{P}_0^n$, the set of convex polytopes that contain the origin in their interiors. Here the situation is not yet well understood. It is easy to see that on $\mathcal{P}_0^n$ there are additional examples of $\mathrm{SL}(n)$ invariant valuations. We describe the construction since it will also be used for tensor, convex body and star body valued valuations. For $\Phi : \mathcal{P}_0^n \to \mathbb{R}$ an $\mathrm{SL}(n)$ invariant valuation, set $\Psi(P) = \Phi(P^*)$. Here P^* is the polar body of $P \in \mathcal{P}_0^n$, that is,

$$P^* = \{y \in \mathbb{R}^n \,|\, x \cdot y \le 1 \ \text{ for all } \ x \in P\}$$

and $x \cdot y$ denotes the inner product x and y in $\mathbb{R}^n$. The functional $\Psi : \mathcal{P}_0^n \to \mathbb{R}$ has the following properties. For $P, Q, P \cup Q \in \mathcal{P}_0^n$, we have

$$(P \cup Q)^* = P^* \cap Q^* \ \text{ and } \ (P \cap Q)^* = P^* \cup Q^*.$$

Since Φ is a valuation,

$$\begin{aligned} \Psi(P) + \Psi(Q) = \quad & \Phi(P^*) + \Phi(Q^*) & = \\ & \Phi(P^* \cup Q^*) + \Phi(P^* \cap Q^*) & = \\ & \Phi((P \cap Q)^*) + \Phi((P \cup Q)^*) & = \Psi(P \cap Q) + \Psi(P \cup Q), \end{aligned}$$

that is, Ψ is also a valuation. For $\alpha \in \mathrm{SL}(n)$ and $P \in \mathcal{P}_0^n$, we have

$$(\alpha\, P)^* = \alpha^{-t}\, P^*,$$

where α^{-t} is the inverse of the transpose of α. Since Φ is $\mathrm{SL}(n)$ invariant,

$$\Psi(\alpha P) = \Phi((\alpha P)^*) = \Phi(\alpha^{-t} P^*) = \Phi(P^*) = \Psi(P),$$

that is, Ψ is also $\mathrm{SL}(n)$ invariant. We say that a functional Φ is *homogeneous of degree q* if

$$\Phi(t\, K) = t^q \Phi(K) \qquad \forall K \in \mathcal{K}^n, \forall t \ge 0.$$

If Φ is homogeneous of degree q, then

$$\Psi(t\,P) = \Phi((t\,P)^*) = \Phi(t^{-1}P^*) = t^{-q}\,\Psi(P),$$

that is, Ψ is homogeneous of degree $-q$. In particular, this shows that $K \mapsto V(K^*)$ is an $\mathrm{SL}(n)$ invariant and homogeneous valuation. The next result shows that there are no further examples.

Theorem 2 ([46])**.** *A functional $\Phi : \mathcal{P}_0^n \to \mathbb{R}$ is a measurable, $\mathrm{SL}(n)$ invariant valuation which is homogeneous of degree q if and only if there is a constant $c \in \mathbb{R}$ such that*

$$\Phi(P) = \begin{cases} c & \text{for } q = 0 \\ c\,V(P) & \text{for } q = n \\ c\,V(P^*) & \text{for } q = -n \\ 0 & \text{otherwise} \end{cases}$$

for every $P \in \mathcal{P}_0^n$.

It is not known if there are additional examples if Φ is not homogeneous. We conjecture that every $\mathrm{SL}(n)$ invariant and continuous valuation is a linear combination of a constant, the volume of the body and the volume of the polar body. Also the problem to classify rotation invariant valuations on $\mathcal{P}_0^n$ is open. Alesker [1] has obtained a classification of continuous, rotation invariant, polynomial valuations on $\mathcal{K}^n$.

3 Real valued valuations on convex bodies

There are $\mathrm{SL}(n)$ invariant valuations on convex bodies that vanish on polytopes. The *affine surface area* $\Omega : \mathcal{K}^n \to \mathbb{R}$, is such a functional. It is defined by

$$\Omega(K) = \int_{\partial K} \kappa(K, x)^{1/(n+1)}\,dx,$$

where $\kappa(K, x)$ is the generalized Gaussian curvature of K at x and ∂K is the boundary of K. Affine surface area was introduced by Pick and Blaschke at the beginning of the twentieth century in the context of Affine Differential Geometry. In the 1990s, it has been extended to a functional for general (not necessarily smooth) convex bodies by Leichtweiß, Lutwak, Schütt and Werner (see [39]). Affine surface area is an equi-affine invariant valuation. Lutwak [56] proved that Ω is upper semicontinuous.

Affine surface area has found a wide field of applications. In particular, affine surface area describes the quality of polytopal volume approximation (see [24, 41, 80, 86]). See also [89, 90, 91]. In the planar case, there is a nice geometric interpretation for affine surface area, see [10]. In general dimensions, floating bodies (see [39]) and related constructions (see [71, 92, 93]) are used to obtain geometric interpretations. There is the following characterization of Ω.

Theorem 3 ([42, 51])**.** *A functional $\Phi : \mathcal{K}^n \to \mathbb{R}$ is an upper semicontinuous and equi-affine invariant valuation if and only if there are constants c_0, c_1, and $c_2 \geq 0$ such that*

$$\Phi(K) = c_0 + c_1\,V(K) + c_2\,\Omega(K)$$

for every $K \in \mathcal{K}^n$.

This is an equi-affine analogue of Hadwiger's Characterization Theorem. The problem to determine all upper semicontinuous and rigid motion invariant valuations on $\mathcal{K}^n$ is open. Only in the planar case, there is a complete classification (see [43]).

Let $\mathcal{K}_0^n$ denote the space of convex bodies that contain the origin in their interiors. A classical notion is the *centro-affine surface area* $\Omega_c(K)$ that can be defined in the following way. If $K \in \mathcal{K}_0^n$ and the Gaussian curvature $\kappa(K,x)$ exists for $x \in \partial K$, set

$$\kappa_0(K,x) = \frac{\kappa(K,x)}{(x \cdot u(K,x))^{n+1}},$$

where $u(K,x)$ is the exterior normal unit vector to K at $x \in \partial K$. Note that $\kappa_0(K,x)^{-1/2}$ is (up to a constant) the volume of the centered ellipsoid that osculates K at x. Let

$$d\sigma_K(x) = (x \cdot u(K,x))\, dx$$

denote the cone measure of K. Then the centro-affine surface area is defined by

$$\Omega_c(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{1}{2}}\, d\sigma_K(x)$$

and it is $\mathrm{GL}(n)$ invariant. Here $\mathrm{GL}(n)$ denotes the general linear group, that is, the group of invertible $n \times n$ matrices. Similar to affine surface area, centro-affine surface area has applications in polytopal approximation. In particular, centro-affine surface area describes the quality of polytopal approximation with respect to the Banach-Mazur distance (see [23]). There is the following characterization of Ω_c.

Theorem 4 ([52])**.** *A functional $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous,* $\mathrm{GL}(n)$ *invariant valuation if and only if there are constants $c_0 \in \mathbb{R}$ and $c_1 \geq 0$ such that*

$$\Phi(K) = c_0 + c_1\, \Omega_c(K)$$

for every $K \in \mathcal{K}_0^n$.

More generally, the following classification of $\mathrm{SL}(n)$ invariant and upper semicontinuous valuations holds.

Theorem 5 ([52])**.** *A functional $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous and* $\mathrm{SL}(n)$ *invariant valuation that vanishes on $\mathcal{P}_0^n$ if and only if there is a concave function $\phi : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0} \phi(t) = 0$ and $\lim_{t\to\infty} \phi(t)/t = 0$ such that*

$$\Phi(K) = \int_{\partial K} \phi(\kappa_0(K,x))\, d\sigma_K(x)$$

for every $K \in \mathcal{K}_0^n$.

Combined with Theorem 2 this gives a classification of upper semicontinuous, $\mathrm{SL}(n)$ invariant, homogeneous valuations on $\mathcal{K}_0^n$.

Theorem 6 ([52])**.** *A functional $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous,* $\mathrm{SL}(n)$ *invariant valuation that is homogeneous of degree q if and only if there are constants $c_0 \in \mathbb{R}$ and $c_1 \geq 0$ such that*

$$\Phi(K) = \begin{cases} c_0 + c_1\, \Omega_n(K) & \text{for } q = 0 \\ c_1\, \Omega_p(K) & \text{for } -n < q < n,\ q \neq 0, \\ c_0\, V(K) & \text{for } q = n \\ c_0\, V(K^*) & \text{for } q = -n \\ 0 & \text{otherwise} \end{cases}$$

for every $K \in \mathcal{K}_0^n$ where $p = n(n-q)/(n+q)$.

Here $\Omega_p(K)$ is the L_p*-affine surface area* of K. This notion was introduced by Lutwak [60] within the setting of the L_p-Brunn-Minkowski theory. For $K \in \mathcal{K}_0^n$ and $p \geq 1$, he defined

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}}\, du$$

where S^{n-1} is the unit sphere and $f_p(K,\cdot)$ is the L_p-curvature function of K. Lutwak [60] showed that Ω_p is $\mathrm{SL}(n)$ invariant and upper semicontinuous on $\mathcal{K}_0^n$. Hug [28] gave an equivalent definition of L_p-affine surface areas and extended Lutwak's definition from $p \geq 1$ to $p > 0$. Hug's definition can be written in the following way. For $p > 0$ and $K \in \mathcal{K}_0^n$,

$$\Omega_p(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{p}{n+p}}\, d\sigma_K(x).$$

For $p = 1$ we obtain the affine surface area $\Omega(K)$ and for $p = n$ the centro-affine surface area $\Omega_c(K)$. Geometric interpretations of L_p-affine surface areas can be found in [72, 87] and applications to partial differential equations in [61].

There is the following application of Theorem 6. Set $\Psi(K) = \Omega_p(K^*)$. Then Ψ is also an $\mathrm{SL}(n)$ invariant valuation and it is homogeneous of degree $-q = -n\,(n-p)/(n+p)$. Therefore by Theorem 6, there is a constant $c \geq 0$ such that $\Psi(K) = c\,\Omega_r(K)$ where $r = n^2/p$. Since for the ball B of radius 1 all L_p-affine surface areas coincide, we have $\Omega_p(B) = \Omega_p(B^*) = c\,\Omega_r(B)$ and $c = 1$. Thus

$$\Omega_p(K^*) = \Omega_{n^2/p}(K).$$

This result was obtained by Hug [29] using a different approach.

4 Tensor valued valuations

For vector valued valuations, Schneider [78] proved the following analogue of Hadwiger's characterization theorem: Every continuous, rotation covariant, vector valued valuation z on $\mathcal{K}^n$ with the property that $z(K+x) - z(K)$ is parallel to x for every $x \in \mathbb{R}^n$ is a linear combination of quermassvectors (see also [27] and [81], Chapter 5.4). Here we are interested in $\mathrm{SL}(n)$ covariant vector valued valuations on $\mathcal{P}_0^n$. For this question, the fundamental notion is the *moment vector*

$$m(P) = \int_P x\, dx$$

of $P \in \mathcal{P}_0^n$, that is, $m(P)$ is the centroid of P multiplied by the volume of P. The moment vector is an $\mathrm{SL}(n)$ covariant valuation on $\mathcal{P}_0^n$. The problem to classify $\mathrm{SL}(n)$ covariant valuations from $\mathcal{P}_0^n$ to $\mathbb{R}^n$ is not completely solved but there is a classification of $\mathrm{GL}(n)$ covariant valuations. Here a function $z : \mathcal{P}_0^n \to \mathbb{R}^n$ is called $\mathrm{GL}(n)$ *covariant*, if there is a real number q such that

$$z(\alpha P) = |\det \alpha|^q\, \alpha z(P) \qquad \forall P \in \mathcal{P}_0^n, \forall \alpha \in \mathrm{GL}(n).$$

A function is (Borel) measurable if the pre-image of every open set is a Borel set.

Theorem 7 ([44]). *A function $z : \mathcal{P}_0^n \to \mathbb{R}^n$, $n \geq 3$, is a $\mathrm{GL}(n)$ covariant, measurable valuation if and only if there is a constant $c \in \mathbb{R}$ such that*

$$z(P) = c\, m(P)$$

for every $P \in \mathcal{P}_0^n$.

More general tensor valued valuations on $\mathcal{K}^n$ were studied and classified in [2, 68, 69, 83, 84]. Here we consider only symmetric tensors of rank 2, that is, functions $Z : \mathcal{P}_0^n \to \mathcal{M}^n$, where $\mathcal{M}^n$ is the set of real symmetric $n \times n$ matrices. Note that to every positive definite matrix $A \in \mathcal{M}^n$ corresponds an ellipsoid E_A defined by

$$E_A = \{x \in \mathbb{R}^n : x \cdot Ax \leq 1\}. \tag{1}$$

A classical concept from mechanics is the *Legendre ellipsoid* or *ellipsoid of inertia* $\Gamma_2 K$ associated with a convex body $K \subset \mathbb{R}^n$ (see [39, 40, 73]). It can be defined as the unique ellipsoid centered at the center of mass of K such that the ellipsoid's moment of inertia about any axis passing through the center of mass is the same as that of K. The Legendre ellipsoid can also be defined by the *moment matrix* $M_2(K)$ of K. This is the $n \times n$ matrix with coefficients

$$\int_K x_i\, x_j\, dx,$$

where we use coordinates $x = (x_1, \ldots, x_n)$ for $x \in \mathbb{R}^n$. For a convex body K with non-empty interior, $M_2(K)$ is a positive definite symmetric $n \times n$ matrix and using (1) we have

$$\Gamma_2 K = \sqrt{\frac{n+2}{V(K)}}\, E_{M_2(K)^{-1}}.$$

Note that $M_2 : \mathcal{K}^n \to \mathcal{M}^n$ is $\mathrm{GL}(n)$ covariant of weight $q = 1$, where a function $Z : \mathcal{P}_0^n \to \mathcal{M}^n$ is $\mathrm{GL}(n)$ covariant if there is a real number q such that

$$Z(\alpha P) = |\det \alpha|^q\, \alpha Z(P) \alpha^t \qquad \forall K \in \mathcal{K}^n, \forall \alpha \in \mathrm{GL}(n).$$

Here α^t denotes the transpose of α. There is the following classification of $\mathrm{GL}(n)$ covariant matrix valued valuations.

Theorem 8 ([47]). *A function $Z : \mathcal{P}_0^n \to \mathcal{M}^n$, $n \geq 3$, is a measurable,* $\mathrm{GL}(n)$ *covariant valuation if and only if there is a constant $c \in \mathbb{R}$ such that*

$$Z(P) = c\, M_2(P) \quad \text{or} \quad Z(P) = c\, M_{-2}(P^*)$$

for every $P \in \mathcal{P}_0^n$.

Here $M_{-2}(P^*)$ is the matrix with coefficients

$$\sum_u \frac{a(P^*, u)}{h(P^*, u)}\, u_i\, u_j$$

where the sum is taken over all unit normals u of facets of P^* and where $a(P^*, u)$ is the $(n-1)$-dimensional volume of the facet with normal u and $h(P^*, u)$ is the distance from the origin of the hyperplane containing this facet. This matrix corresponds to the ellipsoid $\Gamma_{-2} P^*$ recently introduced Lutwak, Yang, and Zhang [63]. Using (1), this *LYZ ellipsoid* is given by

$$\Gamma_{-2} P^* = \sqrt{V(K)}\, E_{M_{-2}(P^*)}.$$

More information on this ellipsoid, its applications, and its connection to the Fisher information from information theory can be found in [25, 63, 65].

5 Convex body valued valuations

The basic notion of addition for convex bodies is Minkowski addition. For $K_1, K_2 \in \mathcal{K}^n$, the Minkowski sum is

$$K_1 + K_2 = \{x_1 + x_2 : x_1 \in K_1, x_2 \in K_2\}$$

and $K_1 + K_2 \in \mathcal{K}^n$. Minkowski addition can also be described by using the *support function* $h(K, \cdot)$, which is defined for $u \in S^{n-1}$ by

$$h(K, u) = \max\{x \cdot u : x \in K\}.$$

Note that $h(K, \cdot)$ on S^{n-1} determines K and that the support function of the Minkowski sum is given by

$$h(K_1 + K_2, \cdot) = h(K_1, \cdot) + h(K_2, \cdot).$$

Minkowski addition and volume are the fundamental notions in the Brunn-Minkowski theory (see [81]). We remark that there are important extensions of the concepts of the Brunn-Minkowski theory in the L_p-Brunn-Minkowski theory (see [57, 60]).

Here we consider convex body valued functions on $\mathcal{K}^n$ and $\mathcal{K}^n_0$ that are valuations with respect to Minkowski addition. Since we are interested in the affine geometry of convex bodies, we confine our attention to operators $\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ that are $\mathrm{SL}(n)$ covariant or $\mathrm{SL}(n)$ contravariant. Here an operator is called $\mathrm{SL}(n)$ *contravariant* if

$$\mathrm{Z}(\alpha K) = \alpha^{-t}\, \mathrm{Z}\, K \qquad \forall K \in \mathcal{K}^n_0, \forall \alpha \in \mathrm{SL}(n),$$

where α^{-t} is the transpose of the inverse of α.

The classical example of an $\mathrm{SL}(n)$ contravariant operator is the *projection operator* $\Pi : \mathcal{K}^n \to \mathcal{K}^n$. It is defined in the following way. The *projection body*, ΠK, of K is the convex body whose support function is given by

$$h(\Pi K, u) = \mathrm{vol}(K|u^\perp) \quad \text{for } u \in S^{n-1},$$

where vol denotes $(n-1)$-dimensional volume and $K|u^\perp$ denotes the image of the orthogonal projection of K onto the subspace orthogonal to u. Projection bodies were introduced by Minkowski at the turn of the last century. They are an important tool for studying projections. Petty [75] showed that

$$\Pi(\alpha K) = |\det \alpha|\, \alpha^{-t} \Pi K \quad \text{and} \quad \Pi(K + x) = \Pi K \tag{2}$$

for every $K \in \mathcal{K}^n$, $\alpha \in \mathrm{GL}(n)$, and $x \in \mathbb{R}^n$. It follows from (2) that the volume of ΠK and of the polar of ΠK are affine invariants, and there are important affine isoperimetric inequalities for these quantities (see [76, 94, 58, 20, 64, 96]). There is the following characterization of Π.

Theorem 9 ([45, 48]). *An operator $Z : \mathcal{P}^n \to \mathcal{K}^n$ is an $\mathrm{SL}(n)$ contravariant and translation invariant valuation if and only if there is a constant $c \geq 0$ such that*

$$ZP = c\, \Pi P$$

for every $P \in \mathcal{P}^n$.

A simple consequence of this characterization is that every continuous, $\mathrm{SL}(n)$ contravariant, translation invariant valuation on $\mathcal{K}^n$ is a multiple of the projection operator.

The corresponding result for $\mathrm{SL}(n)$ covariant operators is the following.

Theorem 10 ([48])**.** *An operator $Z : \mathcal{P}^n \to \mathcal{K}^n$ is an* $\mathrm{SL}(n)$ *covariant and translation invariant valuation if and only if there is a constant $c \geq 0$ such that*

$$ZP = c\,\mathrm{D}\,P$$

for every $P \in \mathcal{P}^n$.

Here $\mathrm{D}\,P = P + (-P)$ is the *difference body* of P, which is an important concept in the affine geometry of convex bodies. The fundamental affine isoperimetric inequality for difference bodies is the Rogers-Shephard inequality [77].

It is an open problem to establish a classification of rigid motion covariant convex body valued valuations. But there are some important results. An operator $\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ is *Minkowski additive* if $\mathrm{Z}(K_1 + K_2) = \mathrm{Z}\,K_1 + \mathrm{Z}\,K_2$ for $K_1, K_2 \in \mathcal{K}^n$. Note that every Minkowski additive operator is a valuation with respect to Minkowski addition but not vice versa. Continuous Minkowski additive operators that commute with rigid motions are called endomorphisms. Schneider [79] (see also [81]) showed that there is a great variety of these operators. He obtained a complete classification of endomorphisms in $\mathcal{K}^2$ and characterizations of special endomorphisms in $\mathcal{K}^n$. These results were further extended by Kiderlen [31]. Also operators that map Blaschke sums of convex bodies to Minkowski sums are examples of valuations with respect to Minkowski addition. For these operators, classification results were obtained by Schuster [85].

Next, we consider operators on $\mathrm{Z} : \mathcal{K}_0^n \to \mathcal{K}^n$. Such an operator is called $\mathrm{GL}(n)$ *covariant*, if there is a real number q such that

$$\mathrm{Z}(\alpha K) = |\det \alpha|^q\, \alpha\, \mathrm{Z}\,K \qquad \forall K \in \mathcal{K}_0^n, \forall \alpha \in \mathrm{GL}(n).$$

It is called $\mathrm{GL}(n)$ *contravariant*, if there is a real number q such that

$$\mathrm{Z}(\alpha K) = |\det \alpha|^q\, \alpha^{-t}\, \mathrm{Z}\,K \qquad \forall K \in \mathcal{K}_0^n, \forall \alpha \in \mathrm{GL}(n).$$

Note that the projection operator is $\mathrm{GL}(n)$ contravariant of weight $q = 1$ and that the operator $K \mapsto \Pi K^*$ is $\mathrm{GL}(n)$ covariant of weight $q = -1$. Further examples of $\mathrm{GL}(n)$ covariant operators are the *trivial* operators $K \mapsto c_0\,K + c_1(-K)$, $c_0, c_1 \geq 0$.

Theorem 11 ([48, 50])**.** *An operator $\mathrm{Z} : \mathcal{P}_0^n \to \mathcal{K}^n$ is a non-trivial* $\mathrm{GL}(n)$ *covariant valuation if and only if there are constants $c_0 \geq 0$ and $c_1 \in \mathbb{R}$ such that*

$$\mathrm{Z}\,P = c_0\,\mathrm{M}\,P + c_1\,m(P) \quad \textit{or} \quad \mathrm{Z}\,P = c_0\,\Pi\,P^*$$

for every $P \in \mathcal{P}_0^n$.

Here $\mathrm{M}\,P$ is the *moment body* of $P \in \mathcal{P}_0^n$, that is, the convex body whose support function is given by

$$h(\mathrm{M}\,K, u) = \int_K |u \cdot x|\,dx \quad \text{for } u \in S^{n-1}.$$

If the n-dimensional volume $V(K)$ of K is positive, then the *centroid body* $\Gamma\,K$ of K is defined by

$$\Gamma\,K = \frac{1}{V(K)}\,\mathrm{M}\,K.$$

Centroid bodies are a classical notion from geometry (see [16, 39, 81]). If K is centrally symmetric, then $\Gamma\,K$ is the body whose boundary consists of the locus of the centroids of the halves of K formed when K is cut by hyperplanes through the origin. The fundamental affine isoperimetric inequality for centroid bodies is the Busemann-Petty centroid inequality [74]. Recent results on centroid bodies can be found in [11, 17, 22, 53, 55, 62, 66, 73].

6 Star body valued valuations

The basic notion of addition for star bodies is radial addition. Here a set $L \subset \mathbb{R}^n$ is a star body, if it is sharshaped with respect to the origin and has a continuous *radial function* $\rho(L, \cdot)$, which is defined for $u \in S^{n-1}$ by

$$\rho(L, u) = \max\{t \geq 0 : t\,u \in L\}.$$

Note that $\rho(L, \cdot)$ on S^{n-1} determines L. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$. Then the radial sum $L_1 \tilde{+} L_2$ of $L_1, L_2 \in \mathcal{S}^n$ is given by

$$\rho(L_1 \tilde{+} L_2, \cdot) = \rho(L_1, \cdot) + \rho(L_2, \cdot)$$

and $L_1 \tilde{+} L_2 \in \mathcal{S}^n$. Radial addition and volume are the fundamental notions in the dual Brunn-Minkowski theory (see [16]). Here we consider star body valued functional on $\mathcal{K}_0^n$ that are valuations with respect to radial addition. Note that the trivial operators, $K \mapsto c_0\, K \tilde{+} c_1(-K)$, $c_0, c_1 \geq 0$, are $\mathrm{GL}(n)$ covariant and valuations with respect to radial addition.

Theorem 12 ([49])**.** *An operator* $\mathrm{Z} : \mathcal{P}_0^n \to \mathcal{S}^n$ *is a non-trivial* $\mathrm{GL}(n)$ *covariant valuation if and only if there is a constant* $c \geq 0$ *such that*

$$\mathrm{Z}\,P = c\,\mathrm{I}\,P^*$$

for every $P \in \mathcal{P}_0^n$*.*

Here $\mathrm{I}\,P^*$ is the *intersection body* of $P^* \in \mathcal{P}_0^n$, that is, the star body whose radial function is given by

$$\rho(\mathrm{I}\,P^*, u) = \mathrm{vol}(P^* \cap u^\perp) \quad \text{for} \quad u \in S^{n-1},$$

where $P^* \cap u^\perp$ denotes the intersection of P^* with the subspace orthogonal to u. Intersections bodies first appear in Busemann's [8] theory of area in Finsler spaces and they were first explicitly defined and named by Lutwak [54]. Intersection bodies turned out to be critical for the solution of the *Busemann-Petty problem*: If the central hyperplane sections of an origin-symmetric convex body in $\mathbb{R}^n$ are always smaller in volume than those of another such body, is its volume also smaller? Lutwak [54] showed that the answer to the Busemann-Petty problem is affirmative if the body with the smaller sections is an intersection body of a star body. This led to the final solution that the answer is affirmative if $n \leq 4$ and negative otherwise (see [14, 15, 18, 36, 37, 95, 97]). Further applications of intersection bodies can be found in [9, 21, 22, 30, 38, 73].

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