Upper semicontinuous valuations on the space of convex discs

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Abstract

We show that every rigid motion invariant and upper semicontinuous valuation on the space of convex discs is a linear combination of the Euler characteristic, the length, the area, and a suitable curvature integral of the convex disc.

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1 Introduction and statement of results

Let $\mathcal{K}^2$ be the space of convex discs, i.e. of non-empty compact convex sets in the Euclidean plane $\mathbb{E}^2$. A functional $\mu : \mathcal{K}^2 \to \mathbb{R}$ is called additive or a *valuation*, if

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L)$$

whenever $K, L, K \cup L \in \mathcal{K}^2$. These valuations play an important role in convex geometry (see [16] and [15]) and have many applications in integral geometry (see [9] and [20]). One of the most important results in this field is Hadwiger's characterization theorem [7]. The planar case of this theorem states that every continuous and rigid motion invariant valuation $\mu : \mathcal{K}^2 \to \mathbb{R}$ can be written as a linear combination of the Euler characteristic χ, the length L, and the area A of the convex disc, i.e. there are constants $c_0, c_1, c_2 \in \mathbb{R}$ such that

$$\mu(K) = c_0\, \chi(K) + c_1\, L(K) + c_2\, A(K)$$

for every $K \in \mathcal{K}^2$. Here continuity is with respect to the usual topology on $\mathcal{K}^2$ induced by the Hausdorff metric.

Beside these continuous valuations, there are other valuations on $\mathcal{K}^2$ which are of geometrical interest. One example is the affine length λ of a convex disc, which is defined as

$$\lambda(K) = \int_{S^1} \rho(K,u)^{\frac{2}{3}}\, d\sigma(u),$$

where S^1 is the unit circle, σ is the one-dimensional Hausdorff measure and $\rho(K,u)$ is the curvature radius of the boundary of K at the point with normal $u \in S^1$. $\rho(K,u)$ exists for almost all $u \in S^1$ and is Lebesgue measurable. This affine length is well defined for every $K \in \mathcal{K}^2$, it is invariant with respect to area preserving affine transformations, and it is upper semicontinuous, i.e. for every sequence K_n of convex discs converging to K,

$$\lambda(K) \geq \limsup_{n\to\infty} \lambda(K_n)$$

(cf. [10]). In [11] it is shown that the affine length can be characterized by these properties. Namely, let $\mu : \mathcal{K}^2 \to \mathbb{R}$ be an upper semicontinuous valuation which is invariant with respect to area preserving affine transformations, then there are constants $c_0, c_1 \in \mathbb{R}$ and $c_2 \geq 0$ such that

$$\mu(K) = c_0\, \chi(K) + c_1\, A(K) + c_2\, \lambda(K)$$

for every $K \in \mathcal{K}^2$. The corresponding result in d-dimensional space was proved by [13], [17].

Further examples of valuations of geometric interest are the functionals

$$\int_{S^1} \rho(K,u)^p\, d\sigma(u) \tag{1}$$

with $0 < p < 1$. They are important in problems of asymptotic approximation by polygons (cf. [4], [14], [5], [6]). They are upper semicontinuous. This follows from the planar case of [12], which states the following. Let $\mathcal{D}$ be the set of functions $f : [0,\infty) \to [0,\infty)$ such that f is concave, $\lim_{t\to 0} f(t) = 0$, and $\lim_{t\to\infty} f(t)/t = 0$. Then, for $f \in \mathcal{D}$,

$$\int_{S^1} f(\rho(K,u))\, d\sigma(u) \tag{2}$$

depends upper semicontinuously on K. An equivalent way to represent the functionals defined in (2) is by

$$\int_{\operatorname{bd} K} g(\kappa(K,x))\, d\sigma(x) \tag{3}$$

where $g(t) = t\,f(1/t)$, $\operatorname{bd} K$ is the boundary of K, and $\kappa(K, x)$ is the curvature of $\operatorname{bd} K$ at x (see [12] and [8]). They are rigid motion invariant and because of (3) it is easy to see that they are valuations. We show that these functionals together with the Euler characteristic, length, and area are the only examples of rigid motion invariant and upper semicontinuous valuations.

Theorem. *Let $\mu : \mathcal{K}^2 \to \mathbb{R}$ be an upper semicontinuous and rigid motion invariant valuation. Then there are constants $c_0, c_1, c_2 \in \mathbb{R}$ and a function $f \in \mathcal{D}$ such that*

$$\mu(K) = c_0\, \chi(K) + c_1\, L(K) + c_2\, A(K) + \int_{S^1} f(\rho(K, u))\, d\sigma(u)$$

for every $K \in \mathcal{K}^2$.

A functional $\mu : \mathcal{K}^2 \to \mathbb{R}$ is called homogeneous of degree p, if

$$\mu(t\,K) = t^p\, \mu(K)$$

for every $t > 0$ and every $K \in \mathcal{K}^2$. It is easy to see that the functionals in (1) are homogeneous of degree p. The following simple consequence of our theorem holds.

Corollary. *Let $\mu : \mathcal{K}^2 \to \mathbb{R}$ be an upper semicontinuous and rigid motion invariant valuation which is homogeneous of degree p. For $0 < p < 1$, there is a constant $c \geq 0$ such that*

$$\mu(K) = c \int_{S^1} \rho(K, u)^p\, d\sigma(u)$$

for every $K \in \mathcal{K}^2$. For $p = 0$, $\mu(K) = c\,\chi(K)$, for $p = 1$, $\mu(K) = c\,L(K)$, and for $p = 2$, $\mu(K) = c\,A(K)$ for every $K \in \mathcal{K}^2$ with a suitable constant $c \in \mathbb{R}$. In all other cases, $\mu(K) = 0$ for every $K \in \mathcal{K}^2$.

2 Proof of the Theorem

Since μ is translation invariant, we have for every $x \in \mathbb{E}^2$,

$$\mu(\{x\}) = c_0$$

with a suitable constant c_0. Define

$$\mu_0(K) = \mu(K) - c_0\, \chi(K).$$

Then μ_0 is an upper semicontinuous and rigid motion invariant valuation, which vanishes on singletons.

Let I be a one-dimensional convex disc, i.e. a line segment. Then $\mu_0(I)$ depends only on $L(I)$, the length of I, since μ_0 is rigid motion invariant. Hence there is a function $g : [0, \infty) \to \mathbb{R}$ such that

$$\mu_0(I) = g(L(I))$$

for every one-dimensional $I \in \mathcal{K}^2$. Since μ_0 vanishes on singletons, dividing I into two pieces I_1 and I_2 of length L_1 and L_2, respectively, shows that

$$g(L_1 + L_2) = g(L_1) + g(L_2)$$

holds for $L_1, L_2 \geq 0$. Thus g is a solution of Cauchy's functional equation and since μ_0 is upper semicontinuous, also g has this property. It is a well known property of solutions of Cauchy's functional equation (see, e.g., [1]) that this implies that there is a constant c_1 such that

$$g(L) = c_1\, L$$

for every $L \geq 0$. Define

$$\mu_1(K) = \mu_0(K) - c_1\, L(K).$$

Then μ_1 is an upper semicontinuous and rigid motion invariant valuation, which vanishes on every at most one-dimensional convex disc. Such a valuation is called *simple*. Set $\mu_1(\emptyset) = 0$. In the rest of the proof, we make use of the following property of simple valuations. Let $K \in \mathcal{K}^2$ and let $P_1, \dots, P_m$ be convex polygons with pairwise disjoint interiors and such that $K \subset P_1 \cup \dots \cup P_m$. Then

$$\mu_1(K) = \mu_1(K \cap P_1) + \dots + \mu_1(K \cap P_m).$$

This can be seen by suitably subdividing the polygons and using induction on the number of pieces like in the extension theorem [7], p. 81.

Let $T \in \mathcal{K}^2$ be a triangle. A well known theorem from elementary geometry (cf., e.g., [2]) states that in the plane every triangle is equi-dissectable to any other triangle with the same area, i.e. for triangles T and T' with $A(T) = A(T')$ there are triangles $T_1, \dots, T_m$ with pairwise disjoint interiors and triangles $T'_1, \dots, T'_m$ with pairwise disjoint interiors such that

$$T = \bigcup_{i=1}^{m} T_i \quad \text{and} \quad T' = \bigcup_{i=1}^{m} T'_i$$

and there are rigid motions $\varphi_1, \dots, \varphi_m$ such that

$$T'_i = \varphi(T_i)$$

holds for $i = 1, \dots, m$. Since μ_1 is a rigid motion invariant and simple valuation, this implies that $\mu_1(T) = \mu_1(T')$. Hence $\mu_1(T)$ depends only on $A(T)$ and consequently, there is a function $g : [0, \infty) \to \mathbb{R}$ such that

$$\mu_1(T) = g(A(T))$$

for every triangle T. Dissecting a triangle T into triangles T_1 and T_2 with area A_1 and A_2, respectively, now shows that

$$g(A_1 + A_2) = g(A_1) + g(A_2)$$

for $A_1, A_2 \geq 0$. Here we used the fact that μ_1 is a simple valuation. Therefore g is an upper semicontinuous solution of Cauchy's functional equation which implies that there is a constant c_2 such that

$$g(A) = c_2\, A$$

for every $A \geq 0$. Define

$$\mu_2(K) = \mu_1(K) - c_2\, A(K).$$

Then μ_2 is an upper semicontinuous and rigid motion invariant valuation, which vanishes on triangles and therefore, being simple, on polygons.

The above arguments show that proving the following statement implies our theorem.

Proposition 1. *Let $\mu : \mathcal{K}^2 \to \mathbb{R}$ be an upper semicontinuous and rigid motion invariant valuation with the property that $\mu(P) = 0$ for every polygon $P \in \mathcal{K}^2$. Then there is a function $f \in \mathcal{D}$ such that*

$$\mu(K) = \int_{S^1} f(\rho(K, u))\, d\sigma(u)$$

for every $K \in \mathcal{K}^2$.

Since the polygons are dense in $\mathcal{K}^2$ and μ is upper semicontinuous, we have

$$\mu(K) \geq 0$$

for every $K \in \mathcal{K}^2$. Define the function $f : [0, \infty) \to [0, \infty)$ by

$$f(r) = \mu(B_r)/2\,\pi, \tag{4}$$

where B_r is the solid circle of radius r centered at the origin o. First, we prove the following result.

Lemma 1. $f \in \mathcal{D}$.

Proof. Since μ is upper semicontinuous and vanishes on singletons, we have for the origin o

$$0 = \mu(\{o\}) \geq \limsup_{r \to 0} \mu(B_r) = \limsup_{r \to 0} 2\,\pi\, f(r),$$

which implies that

$$\lim_{r \to 0} f(r) = 0.$$

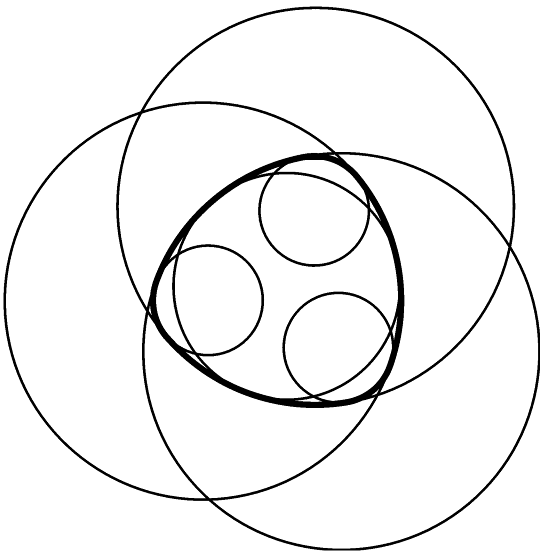

figure 1

Next, we show that f is concave. Let $0 \leq r < s < t$. We approximate the solid circle B_s of radius s by convex discs L_n constructed in the following way. We choose n translates $B_t^1, \ldots, B_t^n, B_t^{n+1} = B_t^1$ of the solid circle B_t of radius t such that $B_s \subset B_t^i$ for $i = 1, \ldots, n$, and such that the B_t^i's touch B_s from the exterior at consecutive points equally spaced on $\operatorname{bd} B_s$. Then we choose translates $B_r^0 = B_r^n, B_r^1, \ldots, B_r^n$ of the solid circle B_r of radius r such that B_r^i is contained in B_t^i and B_t^{i+1} and touches both of them from the interior. L_n is the convex disc whose boundary consists for $i = 1, \ldots, n$ of that part of $\operatorname{bd} B_t^i$ lying between the points where B_r^{i-1} and B_r^i touch B_t^i and that part of $\operatorname{bd} B_r^i$ lying between the points where B_t^i and B_t^{i+1} touch B_r^i (see figure 1).

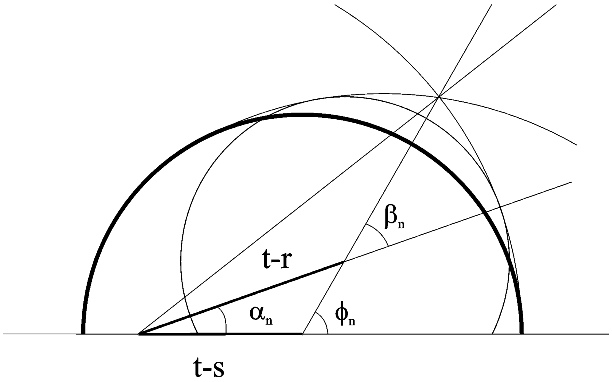

figure 2

For given n, we write $\phi_n = \frac{2\pi}{n}$, and we denote by $2\,\alpha_n$ the angle at the center of B_t^i between the lines to the points where B_r^{i-1} and B_r^i touch B_t^i, and by $2\,\beta_n$ the angle at the center of B_r^i between the lines to the points where B_t^i and B_t^{i+1} touch B_r^i (see figure 2). Then we have

$$\alpha_n + \beta_n = \phi_n$$

and by the sine theorem,

$$\frac{t-r}{\sin(\pi - \phi_n)} = \frac{t-s}{\sin(\beta_n)}.$$

Consequently,

$$\frac{\beta_n}{\phi_n} \to \frac{t-s}{t-r} \tag{5}$$

and

$$\frac{\alpha_n}{\phi_n} \to 1 - \frac{t-s}{t-r} \tag{6}$$

as $n \to \infty$.

Let $S_t(\alpha) \in \mathcal{K}^2$ be a *sector* with angle α, $0 \leq \alpha \leq \pi$, of the solid circle B_t, i.e. the intersection of B_t and two closed half-planes with the origin on their boundary which enclose an angle α. Since μ is rotation invariant, $\mu(S_t(\alpha))$ depends for t fixed only on α, i.e. there is a function $g : [0, \pi] \to [0, \infty)$ such that

$$\mu(S_t(\alpha)) = g(\alpha). \tag{7}$$

Choosing sectors $S_t(\alpha_1)$ and $S_t(\alpha_2)$ with disjoint interiors such that $S_t(\alpha_1) \cup S_t(\alpha_2) \in \mathcal{K}^2$ shows that

$$g(\alpha_1 + \alpha_2) = g(\alpha_1) + g(\alpha_2) \tag{8}$$

for $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 \leq \pi$. Using (8), we can extend g to a function defined on $[0, \infty)$ that is a solution of Cauchy's functional equation. Since μ is upper semicontinuous, so is g. Thus there is a constant a such that $g(\alpha) = a\,\alpha$. By (7) and since μ is a simple valuation, $\mu(B_t) = 2\,\mu(S_t(\pi)) = a\,2\,\pi$, which shows that

$$\mu(S_t(\alpha)) = \frac{\alpha}{2\,\pi}\,\mu(B_t) = \alpha\,f(t). \tag{9}$$

L_n can be dissected into n rotated copies of a sector of B_r with angle β_n and n rotated copies of a sector of B_t with angle α_n. Since μ is a rotation invariant valuation and vanishes on polygons, this implies by (9) that

$$\mu(L_n) = n\,\frac{\beta_n}{2\,\pi}\,\mu(B_r) + n\,\frac{\alpha_n}{2\,\pi}\,\mu(B_t) = \frac{\beta_n}{\phi_n}\,\mu(B_r) + \frac{\alpha_n}{\phi_n}\,\mu(B_t). \tag{10}$$

Taking into account that μ is upper semicontinuous and that $L_n \to B_s$ as $n \to \infty$, we therefore obtain by (10), (9), (5), and (6)

$$\begin{aligned} \mu(B_s) = 2\,\pi\,f(s) \;&\geq\; \limsup_{n\to\infty} \mu(L_n) \\ &=\; \limsup_{n\to\infty} \left(\frac{\beta_n}{\phi_n}\,\mu(B_r) + \frac{\alpha_n}{\phi_n}\,\mu(B_t)\right) \\ &=\; 2\,\pi\,\left(\frac{t-s}{t-r}\,f(r) + (1 - \frac{t-s}{t-r})\,f(t)\right). \end{aligned}$$

Therefore, setting $\lambda = \frac{t-s}{t-r}$, we have $0 < \lambda < 1$ and

$$f(\lambda\,r + (1-\lambda)\,t) \geq \lambda\,f(r) + (1-\lambda)\,f(t),$$

which shows that f is concave.

Finally, we show that

$$\lim_{t\to\infty} \frac{f(t)}{t} = 0. \tag{11}$$

Let I be a line segment of length 1. We approximate I by segments C_t of solid circles B_t of radius t which go through the endpoints of I. Here a convex disc is called a *segment* of a circle B_t, if it is the intersection of B_t and a closed half-plane. A simple calculation using (9) shows that

$$\mu(C_t) = \mu(S_t(2\,\arcsin(\frac{1}{2\,t}))) = 2\,\arcsin(\frac{1}{2\,t})\,f(t).$$

Since $\mu(I) = 0$ and μ is upper semicontinuous, this implies that

$$\limsup_{t\to\infty} \arcsin(\frac{1}{2\,t}) f(t) = 0,$$

and therefore also (11). This completes the proof of Lemma 1. $\square$

Since for $f \in \mathcal{D}$ the functional

$$\mu_f(K) = \int_{S^1} f(\rho(K,u))\, d\sigma(u)$$

is an upper semicontinuous and rigid motion invariant valuation which vanishes on polygons and satisfies $\mu_f(B_r) = 2\pi f(r)$, it suffices to prove the following statement to show Proposition 1.

Proposition 2. *For a given $f \in \mathcal{D}$, there is a unique $\mu : \mathcal{K}^2 \to [0,\infty)$ with the following properties:*

(i) *μ is upper semicontinuous.*

(ii) *μ is rigid motion invariant.*

(iii) *μ is a valuation.*

(iv) *$\mu(P) = 0$ for every polygon $P \in \mathcal{K}^2$.*

(v) *$\mu(B_r) = 2\pi f(r)$.*

Let $\mu : \mathcal{K}^2 \to [0,\infty)$ have properties *(i)-(v)* and set $\mu(\emptyset) = 0$. Let $\mathcal{A} \subset \mathcal{K}^2$ be the set of convex discs which can be dissected into finitely many polygons and segments of solid circles. Since μ vanishes on polygons and is by (9) determined by f on sectors and segments of circles, $\mu(A)$ is determined by f for every $A \in \mathcal{A}$. Since the polygons belong to $\mathcal{A}$, $\mathcal{A}$ is dense in $\mathcal{K}^2$, and we can approximate every $K \in \mathcal{K}^2$ by elements of $\mathcal{A}$. The upper semicontinuity of μ implies that

$$\mu(K) \geq \limsup_{n\to\infty} \mu(A_n) \tag{12}$$

for every sequence A_n with $A_n \to K$. We will prove that for every $K \in \mathcal{K}^2$ there is a sequence $A_n \in \mathcal{A}$ such that we have equality in (12), i.e.

$$\mu(K) = \sup\{\limsup_{n\to\infty} \mu(A_n) : A_n \in \mathcal{A}, A_n \to K\}. \tag{13}$$

Showing this implies that μ is uniquely determined by f and therefore proves Proposition 2.

As a first step in the proof of (13), we show that it suffices to prove it for ε-smooth convex discs. Here we call a convex disc K *ε-smooth* if there is a convex disc K_0 such that

$$K = K_0 + \varepsilon B,$$

where B is the solid unit circle centered at the origin. Suppose that there is a $K \in \mathcal{K}^2$ such that

$$\mu(K) > \sup\{\limsup_{n\to\infty} \mu(A_n) : A_n \in \mathcal{A}, A_n \to K\}.$$

Then there is an $a > 0$ and a $\delta > 0$ such that

$$\mu(K) > \mu(A) + a\,\sigma(\mathrm{bd}\,K) \tag{14}$$

for every $A \in \mathcal{A}$ with $\delta(A, K) \leq \delta$, where $\delta(\cdot, \cdot)$ stands for the Hausdorff distance.

We need the following result. Let $L \in \mathcal{K}^2$ and let I be a line segment. Then

$$\mu(L + I) = \mu(L). \tag{15}$$

This can be seen in the following way. There are points in $\mathrm{bd}\,L$ with support lines parallel to I. Let H be a line connecting two such points in $\mathrm{bd}\,L$ and intersecting the interior of L, if this is non-empty. Denote by H^+, H^- the complementary closed half-planes bounded by H. Then $L + I$ can be dissected into translates of $L \cap H^+$, $L \cap H^-$ and a polygon. Since μ vanishes on polygons and is translation invariant, this implies that

$$\mu(L + I) = \mu(L \cap H^+) + \mu(L \cap H^-) = \mu(L),$$

which proves (15).

The solid unit circle B can be approximated by Minkowski sums S_n of finitely many line segments (cf. [19], Chapter 3.5). The upper semicontinuity of μ then implies that

$$\mu(K + \varepsilon\, B) \geq \limsup_{n\to\infty} \mu(K + \varepsilon\, S_n) \tag{16}$$

for every $\varepsilon > 0$. Since $\varepsilon\, S_n = I_1 + \ldots + I_m$ with suitable line segments I_k, we have by (15)

$$\mu(K + \varepsilon\, S_n) = \mu(K + I_1 + \ldots + I_m) = \mu(K + I_1 + \ldots + I_{m-1}) = \ldots = \mu(K)$$

for every n and $\varepsilon > 0$. Therefore it follows from (16) that for every $\varepsilon > 0$ we have

$$\mu(K + \varepsilon\, B) \geq \mu(K).$$

Thus for $\varepsilon \leq \frac{1}{2}\,\delta$, (14) implies that

$$\mu(K + \varepsilon\, B) \geq \mu(K) > \mu(A) + a\,\sigma(\mathrm{bd}\,K)$$

for every $A \in \mathcal{A}$ with $\delta(K + \varepsilon\, B, A) \leq \frac{1}{2}\,\delta$, since for such an $A \in \mathcal{A}$

$$\delta(K, A) \leq \delta(K, K + \varepsilon\, B) + \delta(K + \varepsilon\, B, A) \leq \delta.$$

Since σ depends continuously on K, it now follows that

$$\mu(K + \varepsilon\, B) > \mu(A) + \frac{a}{2}\,\sigma(\mathrm{bd}(K + \varepsilon\, B))$$

for every $A \in \mathcal{A}$ with $\delta(K + \varepsilon\, B, A) \leq \frac{1}{2}\,\delta$ and $\varepsilon \leq \frac{1}{2}\,\delta$ sufficiently small. If therefore (13) does not hold for a $K \in \mathcal{K}^2$, it also does not hold for an ε-smooth convex disc $K + \varepsilon\, B$ with a suitable $\varepsilon > 0$.

Thus it suffices to show the following proposition to prove (13) and thereby our theorem.

Proposition 3. *Let $K \in \mathcal{K}^2$ be ε-smooth with $\varepsilon > 0$. Then*

$$\mu(K) = \sup\{\limsup_{n\to\infty} \mu(A_n) : A_n \in \mathcal{A}, A_n \to K\}.$$

So let an ε-smooth $K \in \mathcal{K}^2$, $\delta > 0$ and $a > 0$ be given. Using suitable support triangles of K we construct an $A \in \mathcal{A}$ with $\delta(K, A) \le \delta$ such that

$$\mu(K) \le \mu(A) + a\,\sigma(\mathrm{bd}\,K) \tag{17}$$

holds. Here a triangle T is called a *support triangle* of K and $x, y \in \mathrm{bd}\,K$ are called its *endpoints*, if T is bounded by support lines to K at x and y and the chord connecting x and y. Further, we make use of the following simple version of Vitali's covering theorem (see, e.g., [3] or [18]). Let $N \subset \mathrm{bd}\,K$ and let $\mathcal{V}$ be a *Vitali class* for N of closed connected sets $V \subset \mathrm{bd}\,K$, i.e. for every $x \in N$ and $\tau > 0$ there exists a $V \in \mathcal{V}$ with $x \in V$ and $0 < \sigma(V) \le \tau$. Then Vitali's covering theorem states that for every $\eta > 0$ there are pairwise disjoint $V_1, \dots, V_m \in \mathcal{V}$ such that

$$\sigma(N) \le \sum_{i=1}^{m} \sigma(V_i) + \eta. \tag{18}$$

We will first show that for the set $N \subset \mathrm{bd}\,K$ of normal points, i.e. points where $\mathrm{bd}\,K$ is twice differentiable, there is a suitable Vitali class defined with the help of support triangles of K.

Lemma 2. *For every $\tau > 0$ and every normal point $x_0 \in \mathrm{bd}\,K$, there is a support triangle T of K and an $A_T \in \mathcal{A}$ such that*

(i) $x_0 \in \mathrm{bd}\,K \cap T$ *and* $0 < \sigma(\mathrm{bd}\,K \cap T) < \tau$

(ii) $A_T \subset T$ *and T is a support triangle of* A_T

(iii) $\mu(K \cap T) \le \mu(A_T) + \frac{a}{2}\,\sigma(\mathrm{bd}\,K \cap T)$.

Proof. By choosing a suitable coordinate system we can represent $\mathrm{bd}\,K$ locally around x_0 by a convex function $g(s)$ such that $x_0 = (0, g(0))$ and such that as $s \to 0$

$$g(s) = \frac{1}{2}\,\kappa(K, x_0)\,s^2 + o(s^2), \tag{19}$$

where $\kappa(K, x_0)$ is the curvature of $\mathrm{bd}\,K$ at x_0.

We first consider the case $\kappa(K, x_0) > 0$. Let $x = x(s)$ be the point with coordinates $(-s, g(-s))$, let $y = y(s)$ be the point $(s, g(s))$, and let $T = T(s)$ be the support triangle with endpoints $x(s)$ and $y(s)$. Then *(i)* holds for $s > 0$

sufficiently small. Let $H(x)$ and $H(y)$ be support lines at x and y, respectively, and let $w = w(s)$ be the point where $H(x)$ and $H(y)$ intersect. Without loss of generality, we may assume that

$$|x - w| \geq |y - w|.$$

Define $y' = y'(s)$ as the point on $H(y)$ such that

$$|x - w| = |y' - w|$$

and $y \in [w, y']$, where $[w, y']$ is the closed line segment with endpoints w and y'. The triangle $T' = T'(s)$ with vertices x, w, and y' is isosceles. Hence there is a solid circle $B(z, r)$ with center $z = z(s)$ and radius $r = r(s)$ such that $H(x)$ is tangent to $B(z, r)$ at x and $H(y)$ is tangent to $B(z, r)$ at y' (see figure 3). A simple calculation using (19) shows that as $s \to 0$

$$B(z, r) \to B(z_0, r_0), \tag{20}$$

where $r_0 = 1/\kappa(K, x_0)$ is the radius of the circle of curvature to $\operatorname{bd} K$ at x_0 and z_0 is its center, and that

$$\lim_{s \to 0} \frac{|x(s) - w(s)|}{|y(s) - w(s)|} = 1. \tag{21}$$

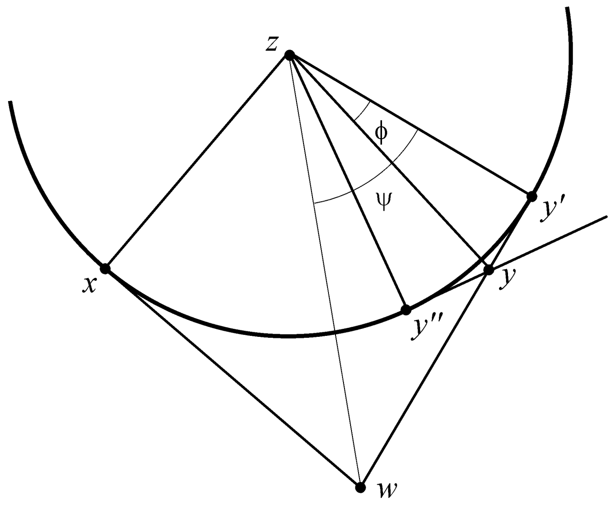

figure 3

The point y does not lie in the interior of $B(z, r)$ and $[y, y']$ is tangent to $B(z, r)$. Let $y'' = y''(s)$ be the second point on $\operatorname{bd} B(z, r)$ such that $[y, y'']$ is tangent to $B(z, r)$, and let $T'' = T''(s)$ be the triangle with vertices x, w and y''. We define $A_T = A_T(s)$ as

$$A_T = (B(z, r) \cap T'') \cup \operatorname{conv}\{x, y'', y\},$$

where conv denotes convex hull. Then $A_T \in \mathcal{A}$ and *(ii)* holds. That also *(iii)* holds, can be seen in the following way.

Let $\psi = \psi(s)$ be the angle between $[z,w]$ and $[z,y']$, and $\phi = \phi(s)$ the angle between $[z,y]$ and $[z,y']$. Then using (9) we have

$$\mu(A_T) = \frac{2\,(\psi-\phi)}{2\,\pi}\,\mu(B(z,r)) = \frac{2\,\psi}{2\,\pi}\left(\mu(B(z,r)) - \frac{\phi}{\psi}\,\mu(B(z,r))\right).$$

By (21) it follows that

$$\begin{aligned}\lim_{s\to 0}\frac{\phi}{\psi} = \lim_{s\to 0}\frac{\tan\phi}{\tan\psi} = \lim_{s\to 0}\frac{|y'-y|}{|y'-w|} &= \lim_{s\to 0}\frac{|y'-w|-|y-w|}{|y'-w|}\\ &= 1-\lim_{s\to 0}\frac{|y-w|}{|x-w|} = 0.\end{aligned}$$

Therefore, for every $\eta > 0$,

$$\frac{2\,\psi}{2\,\pi}\,(\mu(B(z,r)) - \eta) \le \mu(A_T) \tag{22}$$

holds for $s > 0$ sufficiently small.

For $\mu(K\cap T)$ we have the following. T' is a support triangle of

$$(K\cap T)\cup \mathrm{conv}\{x,y,y'\},$$

which is convex. Since T' is also a support triangle of $B(z,r)$, there are rotations $\varphi_1,\dots,\varphi_n$ with $n \le 2\,\pi/(2\,\psi) < n+1$ such that the $\varphi_i(T')$'s have pairwise disjoint interiors and are support triangles of $B(z,r)$. Define

$$L_s = \bigcup_{i=1}^{n}\varphi_i\Big((K\cap T)\cup \mathrm{conv}\{x,y,y'\}\Big)\cup\Big(B(z,r)\backslash\bigcup_{i=1}^{n}\varphi_i(T')\Big). \tag{23}$$

Then our construction implies that $L_s \in \mathcal{K}^2$, that $\mu(L_s) \ge n\,\mu(K\cap T)$, and by (20), that

$$L_s \to B(z_0,r_0)$$

as $s\to 0$. Since μ is upper semicontinuous, this implies that

$$\mu(B(z_0,r_0)) \ge \limsup_{s\to 0}\mu(L_s) \ge \limsup_{s\to 0}\frac{2\,\pi}{2\psi}\,\mu(K\cap T).$$

Hence for every $\eta > 0$

$$\mu(K\cap T) \le \frac{2\psi}{2\,\pi}\,(\mu(B(z,r)) + \eta) \tag{24}$$

for $s > 0$ sufficiently small, where we used that $\mu(B(z,r)) = 2\,\pi\,f(r)$ is continuous. This, (22) and (20) now imply that

$$\begin{aligned}\mu(K\cap T) &\le \mu(A_T) + \frac{2\,\psi}{2\,\pi}\,2\,\eta\\ &\le \mu(A_T) + \frac{4\,\eta}{2\,\pi\,r}\,\sigma(\mathrm{bd}\,K\cap T)\\ &\le \mu(A_T) + \frac{8\,\eta}{2\,\pi\,r_0}\,\sigma(\mathrm{bd}\,K\cap T)\end{aligned}$$

for $s > 0$ sufficiently small. Here we used the simple estimate that $\sigma(\operatorname{bd} K \cap T) \geq r\,\psi$ for $s > 0$ sufficiently small. Setting $\eta = a\,\pi\,r_0/8$ now shows that *(iii)* holds for $s > 0$ sufficiently small.

Now, let $\kappa(K, x_0) = 0$. Let $T = T(s)$ be the support triangle of K with endpoints $x = x_0$ and $y = y(s) = (s, g(s))$ and let $A_T = T$. Then *(i)* and *(ii)* hold. For every $r > 0$, there is a solid circle $B(z, r)$ with $x_0 \in \operatorname{bd} B(z, r)$ which is locally contained in K. We choose r so large that

$$16\,\frac{f(r)}{r} \leq \frac{a}{2}, \tag{25}$$

which is possible, since $\lim_{r\to\infty} f(r)/r = 0$. Let $w = w(s)$ be the point on the support line to K at x such that $y \in [z, w]$ and let $y' = y'(s)$ be the point on $\operatorname{bd} B(z, r)$ such that $[y', w]$ is tangent to $B(z, r)$ (see figure 4). Let $\psi = \psi(s)$ be the angle between $[z, x]$ and $[z, w]$.

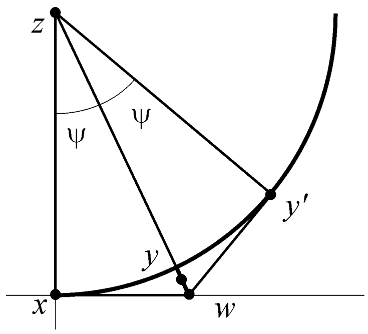

figure 4

Then the triangle $T' = T'(s)$ with the vertices x, w, and y' is a support triangle of $B(z, r)$. Since $B(z, r)$ is locally contained in K, the support lines of K at y do not intersect $B(z, r)$ for $s > 0$ sufficiently small. Therefore

$$(K \cap T) \cup \operatorname{conv}\{x, y, y'\}$$

is convex for $s > 0$ sufficiently small and T' is also a support triangle of this set. Define L_s as in (23). Then $L_s \to B(z, r)$ as $s \to 0$ and $\mu(L_s) \geq n\,\mu(K \cap T)$. Since μ is upper semicontinuous and $n \leq 2\,\pi/(2\,\psi) < n + 1$, we have for every $\eta > 0$,

$$\frac{2\,\pi}{2\,\psi}\,\mu(K \cap T) \leq \mu(B(z, r)) + 2\,\pi\,\eta = 2\,\pi\,(f(r) + \eta)$$

for $s > 0$ sufficiently small. Using the simple estimate that

$$\psi \leq \frac{4}{r}\,\sigma(\operatorname{bd} K \cap T)$$

for $s > 0$ sufficiently small, we have

$$\mu(K \cap T) \leq \frac{8}{r}\,(f(r) + \eta)\,\sigma(\operatorname{bd} K \cap T). \tag{26}$$

Setting $\eta = a\,r/32$, we obtain by (25) that *(iii)* holds. □

Further, we need the following result.

Lemma 3. *There is a $c(\varepsilon)$ such that*

$$\mu(K \cap P) \leq c(\varepsilon)\, \sigma(\operatorname{bd} K \cap P)$$

for every polygon $P \in \mathcal{K}^2$ and every ε-smooth $K \in \mathcal{K}^2$ with $\varepsilon > 0$.

Proof. Since K is ε-smooth, for every $x_0 \in \operatorname{bd} K$ there is a $B(z, \varepsilon) \subset K$ such that $x_0 \in \operatorname{bd} B(z, \varepsilon)$. Let T be a support triangle of K with endpoints $x = x_0$ and y. Then we can construct a support triangle T' of $B(z, \varepsilon)$ with vertices x, w, y' as in the second part of the proof of Lemma 2 (see figure 4). As in (26) we therefore have with $\eta = 1$

$$\mu(K \cap T) \leq \frac{8}{\varepsilon}\,(f(\varepsilon) + 1)\, \sigma(\operatorname{bd} K \cap T) \tag{27}$$

for every T sufficiently small, and since in the proof of (26) only the circle $B(z, r)$ and the angle between $[z, x]$ and $[z, y]$ are used, this holds uniformly for every $x_0 \in \operatorname{bd} K$. We can therefore dissect P into finitely many polygons which are either support triangles for which (27) holds or lie entirely in K or outside of K. Since μ vanishes on polygons, (27) therefore proves the lemma. □

Since a convex function is almost everywhere twice differentiable (see, e.g., [19]), the set N of points, where $\operatorname{bd} K$ is twice differentiable, has measure

$$\sigma(N) = \sigma(\operatorname{bd} K).$$

By Lemma 2 the sets $\operatorname{bd} K \cap T$ defined in Lemma 2 are a Vitali class for N and this remains true if we only take triangles T with $\sigma(\operatorname{bd} T) \leq \delta$. Let

$$0 < \eta \leq \frac{a}{2\, c(\varepsilon)}\, \sigma(\operatorname{bd} K) \tag{28}$$

and $\eta \leq \delta$. Then we can choose by Vitali's theorem (18) support triangles $T_1, \ldots, T_m$ such that

$$\sigma(\operatorname{bd} K) = \sigma(N) \leq \sum_{i=1}^{m} \sigma(\operatorname{bd} K \cap T_i) + \eta \tag{29}$$

and such that the sets $\operatorname{bd} K \cap T_i$ are pairwise disjoint. Let $A_{T_1}, \ldots, A_{T_m}$ be the elements of $\mathcal{A}$ corresponding to $T_1, \ldots, T_m$ as defined in Lemma 2 and define

$$A = \operatorname{conv}\{A_{T_1} \cup \ldots \cup A_{T_m}\}.$$

Then our construction using support triangles implies that

$$\mu(A \cap T_i) = \mu(A_{T_i}) \tag{30}$$

for $i = 1, \ldots, m$, and that $\delta(K, A) \leq \delta$ holds. Let x be an interior point of K and let P_i be the convex hull of x and T_i. We choose polygons $Q_1, \ldots, Q_n$ such that $P_1, \ldots, P_m, Q_1, \ldots, Q_n$ have pairwise disjoint interiors and such that A and K are contained in

$$P_1 \cup \ldots \cup P_m \cup Q_1 \cup \ldots \cup Q_n.$$

Since μ vanishes on polygons and by (30), we have

$$\mu(A) = \sum_{i=1}^{m} \mu(A \cap P_i) + \sum_{j=1}^{n} \mu(A \cap Q_j) = \sum_{i=1}^{m} \mu(A_{T_i}).$$

For K we have by Lemma 2 and Lemma 3 and since μ vanishes on polygons,

$$\begin{aligned} \mu(K) &= \sum_{i=1}^{m} \mu(K \cap P_i) + \sum_{j=1}^{n} \mu(K \cap Q_j) \\ &\leq \sum_{i=1}^{m} \Big(\mu(A_{T_i}) + \frac{a}{2}\,\sigma(\mathrm{bd}\,K \cap T_i)\Big) + c(\varepsilon) \sum_{j=1}^{n} \sigma(\mathrm{bd}\,K \cap Q_j) \\ &\leq \mu(A) + \frac{a}{2}\,\sigma(\mathrm{bd}\,K) + c(\varepsilon)\,\eta, \end{aligned}$$

where we used (29). Consequently, by (28)

$$\mu(K) \leq \mu(A) + a\,\sigma(\mathrm{bd}\,K).$$

This shows that (17) holds and therefore concludes the proof of our theorem.

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