Geometric Valuation Theory

Monika Ludwig

Technische Universität Wien

8ECM, Portorož, Slovenia

Introduction

- **Valuations on convex bodies**
  - **Classification theorems**
  - Algebraic integral geometry
    Alesker: JDG 2003, GAFA 2004; Fu: JDG 2006; Bernig & Fu: Ann. Math. 2011; Alesker & Bernig: AJM 2012, ...
- Valuations on star sets
  - Klain: AiM 1996, 1997; Villanueva: AiM 2016; Tradacete & Villanueva: AiM 2018, ...
- Valuations on Riemannian manifolds
  - Alesker: AiM 2006, GAFA 2007, 2010; Alesker & Fu: TAMS 2008; Alesker & Bernig: JDG 2017; Bernig, Fu & Solanes: GAFA 2014; Fu & Wannerer: GAFA 2019; Faifman & Wannerer: Selecta Math. 2021; Solanes & Wannerer: JDG 2021, ...
- **Valuations on functions spaces**

Valuations on Convex Bodies

- $\mathcal{K}^n$ space of convex bodies (non-empty, compact, convex sets) in $\mathbb{R}^n$

Valuations on Convex Bodies

- $\mathcal{K}^n$ space of convex bodies (non-empty, compact, convex sets) in $\mathbb{R}^n$
- $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ is a **valuation** $\iff$

$$\mathrm{Z}(K) + \mathrm{Z}(L) = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L)$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$.

Valuations on Convex Bodies

- $\mathcal{K}^n$ space of convex bodies (non-empty, compact, convex sets) in $\mathbb{R}^n$
- $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ is a **valuation** $\iff$

$$\mathrm{Z}(K) + \mathrm{Z}(L) = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L)$$

  for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$.

- Hilbert's Third Problem:
  Dehn 1902, Sydler 1965, Jessen & Thorup 1978, ...

Valuations on Convex Bodies

- $\mathcal{K}^n$ space of convex bodies (non-empty, compact, convex sets) in $\mathbb{R}^n$
- $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ is a **valuation** $\iff$

$$\mathrm{Z}(K) + \mathrm{Z}(L) = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L)$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$.

- Hilbert's Third Problem:
Dehn 1902, Sydler 1965, Jessen & Thorup 1978, ...
- Classification of valuations:

Blaschke 1937, **Hadwiger** 1949, Schneider 1971, Groemer 1972, McMullen 1977, Betke & Kneser 1985, Klain 1995, Ludwig 1999, Reitzner 1999, Alesker 1999, Bernig 2006, Fu 2006, Hug 2005, Haberl 2006, Schuster 2006, Tsang 2010, Wannerer 2010, Abardia 2011, Parapatits 2011, Faifman 2013, Solanes 2014, Böröczky 2015, Zeng 2018, ...

Felix Klein's Erlangen Program 1872

Geometry is the study of invariants of transformation groups.

Felix Klein's Erlangen Program 1872

Geometry is the study of invariants of transformation groups.

Groups acting on $\mathbb{R}^n$

Felix Klein's Erlangen Program 1872

Geometry is the study of invariants of transformation groups.

Groups acting on $\mathbb{R}^n$

- Group of translations: $x \mapsto x + y$
  where $y \in \mathbb{R}^n$

Felix Klein's Erlangen Program 1872

Geometry is the study of invariants of transformation groups.

Groups acting on $\mathbb{R}^n$

- Group of translations: $x \mapsto x + y$
  where $y \in \mathbb{R}^n$
- Group of rotations $\mathrm{SO}(n)$: $x \mapsto \vartheta x$
  where ϑ is an orthogonal $n \times n$ matrix of determinant 1

Felix Klein's Erlangen Program 1872

Geometry is the study of invariants of transformation groups.

Groups acting on $\mathbb{R}^n$

- Group of translations: $x \mapsto x + y$
  where $y \in \mathbb{R}^n$
- Group of rotations $SO(n)$: $x \mapsto \vartheta x$
  where ϑ is an orthogonal $n \times n$ matrix of determinant 1
- Special linear group $SL(n)$: $x \mapsto \vartheta x$
  where ϑ is an $n \times n$ matrix of determinant 1

Felix Klein's Erlangen Program 1872

Geometry is the study of invariants of transformation groups.

Groups acting on $\mathbb{R}^n$

- Group of translations: $x \mapsto x + y$
  where $y \in \mathbb{R}^n$
- Group of rotations $\mathrm{SO}(n)$: $x \mapsto \vartheta x$
  where ϑ is an orthogonal $n \times n$ matrix of determinant 1
- Special linear group $\mathrm{SL}(n)$: $x \mapsto \vartheta x$
  where ϑ is an $n \times n$ matrix of determinant 1
- General linear group $\mathrm{GL}(n)$: $x \mapsto \vartheta x$
  where ϑ is an $n \times n$ matrix of determinant $\neq 0$

Equi-Affine Classification Theorems

Theorem (Blaschke 1937)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, translation and* $\mathrm{SL}(n)$ *invariant valuation*

$$\Longleftrightarrow$$

$\exists\, c_0, c_n \in \mathbb{R}$*:*

$$\mathrm{Z}(K) = c_0 V_0(K) + c_n V_n(K)$$

for every $K \in \mathcal{K}^n$.

- $V_0(K)$ Euler characteristic of K (that is, $V_0(K) = 1$)
- $V_n(K)$ n-dimensional volume of K

Equi-Affine Classification Theorems

Theorem (Blaschke 1937)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, translation and* $\mathrm{SL}(n)$ *invariant valuation*

$$\Longleftrightarrow$$

$\exists\, c_0, c_n \in \mathbb{R}$*:*

$$\mathrm{Z}(K) = c_0 V_0(K) + c_n V_n(K)$$

for every $K \in \mathcal{K}^n$*.*

Theorem (L. & Reitzner 2017)

$\mathrm{Z} : \mathcal{P}^n \to \mathbb{R}$ *is a translation and* $\mathrm{SL}(n)$ *invariant valuation*

$$\Longleftrightarrow$$

$\exists\, c_0$ *and Cauchy function* $\zeta : [0, \infty) \to \mathbb{R}$*:*

$$\mathrm{Z}(P) = c_0 V_0(P) + \zeta\big(V_n(P)\big)$$

for every $P \in \mathcal{P}^n$*.*

- $\mathcal{P}^n$ convex polytopes in $\mathbb{R}^n$; $\zeta(x + y) = \zeta(x) + \zeta(y)$

Equi-Affine Classification Theorems

Theorem (Blaschke 1937)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, translation and* $\mathrm{SL}(n)$ *invariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_0, c_n \in \mathbb{R}$:

$$\mathrm{Z}(K) = c_0 V_0(K) + c_n V_n(K)$$

for every $K \in \mathcal{K}^n$.

Theorem (L. & Reitzner 2017)

$\mathrm{Z} : \mathcal{P}^n \to \mathbb{R}$ *is a Borel measurable, translation and* $\mathrm{SL}(n)$ *invariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_0, c_n \in \mathbb{R}$:

$$\mathrm{Z}(P) = c_0 V_0(P) + c_n V_n(P)$$

for every $P \in \mathcal{P}^n$.

- $\mathcal{P}^n$ convex polytopes in $\mathbb{R}^n$

Affine Surface Area of $K \in \mathcal{K}^n$

$$\Omega(K) = \int_{\partial K} \kappa(K,x)^{\frac{1}{n+1}} \, \mathrm{d}x$$

$\kappa(K,x)$ Gaussian curvature at $x \in \partial K$

Affine Surface Area of $K \in \mathcal{K}^n$

$$\Omega(K) = \int_{\partial K} \kappa(K,x)^{\frac{1}{n+1}} \,\mathrm{d}x$$

$\kappa(K,x)$ Gaussian curvature at $x \in \partial K$

Definition for smooth surfaces:
Georg Pick 1914, Wilhelm Blaschke 1923

Affine Surface Area of $K \in \mathcal{K}^n$

$$\Omega(K) = \int_{\partial K} \kappa(K,x)^{\frac{1}{n+1}}\,\mathrm{d}x$$

$\kappa(K,x)$ Gaussian curvature at $x \in \partial K$

Definition for smooth surfaces:
Georg Pick 1914, Wilhelm Blaschke 1923

Definition for general convex bodies:
Leichtweiß 1986, Schütt & Werner 1990, Lutwak 1991, Dolzmann & Hug 1995

Affine Surface Area of $K \in \mathcal{K}^n$

$$\Omega(K) = \int_{\partial K} \kappa(K,x)^{\frac{1}{n+1}} \, \mathrm{d}x$$

$\kappa(K,x)$ Gaussian curvature at $x \in \partial K$

Definition for smooth surfaces:
Georg Pick 1914, Wilhelm Blaschke 1923

Definition for general convex bodies:
Leichtweiß 1986, Schütt & Werner 1990, Lutwak 1991, Dolzmann & Hug 1995

Asymptotic best and random approximation by polytopes:
Fejes Tóth 1948, Rényi & Sulanke 1963, Schneider 1986, Gruber 1988, Bárány 1992, . . .

Properties of $\Omega : \mathcal{K}^n \to \mathbb{R}$

$$\Omega(K) = \int_{\partial K} \kappa(K,x)^{\frac{1}{n+1}} \, \mathrm{d}x$$

- Ω is translation and $\mathrm{SL}(n)$ invariant

$$\Omega(\vartheta K + y) \;=\; \Omega(K) \qquad \forall\, \vartheta \in \mathrm{SL}(n),\ y \in \mathbb{R}^n$$

- Ω is *upper semicontinuous* (Lutwak: AiM 1991)

$$\Omega(K) \geqslant \limsup_{j\to\infty} \Omega(K_j)$$

  if $K_j \to K$ as $j \to \infty$.

- Ω is a valuation

Equi-Affine Classification Theorem

Theorem (L. & Reitzner: AiM 1999)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is an upper semicontinuous, translation and* $\mathrm{SL}(n)$ *invariant valuation*

$$\Longleftrightarrow$$

$\exists\, c_0, c_n \in \mathbb{R}, a \geqslant 0$ *such that*

$$\mathrm{Z}(K) = c_0 V_0(K) + c_n V_n(K) + a\,\Omega(K)$$

for every $K \in \mathcal{K}^n$.

Equi-Affine Classification Theorem

Theorem (L. & Reitzner: AiM 1999)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is an upper semicontinuous, translation and* $\mathrm{SL}(n)$ *invariant valuation*

$$\Longleftrightarrow$$

$\exists\, c_0, c_n \in \mathbb{R}, a \geqslant 0$ *such that*

$$\mathrm{Z}(K) = c_0 V_0(K) + c_n V_n(K) + a\,\Omega(K)$$

for every $K \in \mathcal{K}^n$.

- Proof uses results from Geometric Measure Theory and approximation by piecewise ellipsoids

Centro-Affine Surface Area

$$\Omega_n(K) = \int_{\partial K} \kappa_0(K, x)^{\frac{1}{2}} \, \mathrm{d}V_K(x)$$

$\mathcal{K}^n_{(0)}$ convex bodies in $\mathbb{R}^n$ with origin in their interiors

Centro-Affine Surface Area

$$\Omega_n(K) = \int_{\partial K} \kappa_0(K, x)^{\frac{1}{2}} \, \mathrm{d}V_K(x)$$

$\mathcal{K}^n_{(0)}$ convex bodies in $\mathbb{R}^n$ with origin in their interiors

$u_K(x)$ outer unit normal vector

$\mathrm{d}V_K(x) = \langle x, u_K(x) \rangle \, \mathrm{d}x$ cone measure

$\kappa_0(K, x) = \dfrac{\kappa(K, x)}{\langle x, u_K(x) \rangle^{n+1}}$ centro-affine curvature

Centro-Affine Surface Area

$$\Omega_n(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{1}{2}} \,\mathrm{d}V_K(x)$$

$\mathcal{K}^n_{(0)}$ convex bodies in $\mathbb{R}^n$ with origin in their interiors

$u_K(x)$ outer unit normal vector

$\mathrm{d}V_K(x) = \langle x, u_K(x)\rangle \,\mathrm{d}x$ cone measure

$\kappa_0(K,x) = \dfrac{\kappa(K,x)}{\langle x, u_K(x)\rangle^{n+1}}$ centro-affine curvature

Properties of $\Omega_n : \mathcal{K}^n_{(0)} \to \mathbb{R}$

- Ω_n is $\mathrm{GL}(n)$ invariant: $\Omega_n(\vartheta K) = \Omega_n(K) \quad \forall\, \vartheta \in \mathrm{GL}(n)$
- Ω_n is *upper semicontinuous* (Lutwak: AiM 1996)
- Ω_n is a valuation

Centro-Affine Classification Theorem

Theorem (L. & Reitzner: Ann. Math. 2010)

$\mathrm{Z} : \mathcal{K}^n_{(0)} \to \mathbb{R}$ *is an upper semicontinuous and* $\mathrm{GL}(n)$ *invariant valuation*

$$\iff$$

$\exists\, c_0 \in \mathbb{R}, a \geqslant 0$ *such that*

$$\mathrm{Z}(K) = c_0 V_0(K) + a\,\Omega_n(K)$$

for every $K \in \mathcal{K}^n_{(0)}$.

Centro-Affine Classification Theorem

Theorem (L. & Reitzner: Ann. Math. 2010)

$\mathrm{Z} : \mathcal{K}^n_{(0)} \to \mathbb{R}$ *is an upper semicontinuous and* $\mathrm{GL}(n)$ *invariant valuation*

$$\Longleftrightarrow$$

$\exists\, c_0 \in \mathbb{R}, a \geqslant 0$ *such that*

$$\mathrm{Z}(K) = c_0\, V_0(K) + a\, \Omega_n(K)$$

for every $K \in \mathcal{K}^n_{(0)}$.

- Classification of $\mathrm{SL}(n)$ invariant and homogeneous valuations: Characterization of L_p affine surface areas (introduced by Lutwak 1996)
- Classification of $\mathrm{SL}(n)$ invariant valuations that vanish on polytopes

Classification of $\mathrm{SL}(n)$ Invariant Valuations

Theorem (L. & Reitzner; Haberl & Parapatits: JAMS 2014)

$\mathrm{Z} : \mathcal{K}^n_{(0)} \to \mathbb{R}$ *is an upper semicontinuous and* $\mathrm{SL}(n)$ *invariant valuation*

$$\Longleftrightarrow$$

$\exists\, c_0, c_n, c_{-n} \in \mathbb{R}$ *and* $\exists\, \zeta \in \mathrm{Conc}[0,\infty)$ *such that*

$$\mathrm{Z}(K) = c_0 V_0(K) + c_n V_n(K) + c_{-n} V_n^*(K) + \int_{\partial K} \zeta(\kappa_0(K,x))\, \mathrm{d}V_K(x)$$

for every $K \in \mathcal{K}^n_{(0)}$.

- $V_n^*(K) = V_n(K^*)$ polar volume
- $K^* = \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1\ \forall x \in K\}$ polar body
- $\mathrm{Conc}[0,\infty) = \{\zeta : [0,\infty) \to [0,\infty) : \zeta$ concave, $\lim_{t\to 0} \zeta(t) = \lim_{t\to\infty} \frac{\zeta(t)}{t} = 0\}$

Affine Classification Theorems

- Vector and tensor valuations:
  L.: DMJ 2003; Haberl & Parapatits: AJM 2016, AiM 2017;
  L. & Silverstein: AiM 2017; Zeng & Ma: TAMS 2018; Abardia, Böröczky, Domokos, Kertész: JFA 2019; Ma & Wang: CJM 2021
- Convex-body-valued valuations:
  L.: AiM 2002, TAMS 2005, AJM 2006, JDG 2010;
  Haberl & L.: IMRN 2006; Haberl: AJM 2011, JEMS 2012;
  Wannerer: IUMJ 2011; Wannerer & Schuster: TAMS 2012;
  Abardia & Bernig: AiM 2011; Abardia: JFA 2012, IMRN 2015;
  Parapatits: TAMS 2014, JLMS 2014; Li, Yuan & Leng: TAMS 2015;
  Li & Leng: AiM 2016, IUMJ 2017
- Function-valued valuations:
  Li & Ma: JFA 2017; Li: IMRN 2020, AiM 2021

Rigid Motion Invariant Valuations

Theorem (Hadwiger 1952)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, translation and rotation invariant valuation*

$$\iff$$

$\exists\, c_0, \dots, c_n \in \mathbb{R}$ *:*

$$\mathrm{Z}(K) = c_0 V_0(K) + \cdots + c_n V_n(K)$$

for every $K \in \mathcal{K}^n$*.*

- $V_0(K), \dots, V_n(K)$ intrinsic volumes of K
- $2\, V_{n-1}(K)$ surface area of K
- $\frac{2}{n} \frac{\kappa_n}{\kappa_{n-1}}\, V_1(K)$ mean width of K
- Klain 1995; Klain & Rota 1997: Introduction to Geometric Probability

Intrinsic Volumes

- Steiner formula

$$V_n(K + \lambda\, B^n) = \sum_{j=0}^{n} \lambda^{n-j} \kappa_{n-j} V_j(K)$$

- Crofton and Cauchy-Kubota Formulas

$$V_j(K) = \int_{Graff(n,j)} V_0(K \cap E)\,\mathrm{d}\mu_j(E) = \int_{Gr(n,j)} V_j(K|E)\,\mathrm{d}\nu_j(E)$$

- K convex body with smooth boundary

$$V_j(K) = \frac{\binom{n}{j}}{n\kappa_{n-j}} \int_{\mathbb{S}^{n-1}} s_j(K,u)\,\mathrm{d}u = \frac{\binom{n}{j}}{n\kappa_{n-j}} \int_{\partial K} H_{n-j-1}(K,x)\,\mathrm{d}x$$

Application: Principal Kinematic Formula

For $K, L \in \mathcal{K}^n$,

$$\int_{\phi \in \mathrm{SO}(n) \ltimes \mathbb{R}^n} V_0(K \cap \phi L)\, d\phi = \sum_{i=0}^{n} \frac{\kappa_i\, \kappa_{n-i}}{\binom{n}{i}\, \kappa_n}\, V_i(K)\, V_{n-i}(L)$$

- $d\phi$ normalized Haar measure on $\mathrm{SO}(n) \ltimes \mathbb{R}^n$
- Blaschke, Chern, Hadwiger, Santaló, ...

Application: Principal Kinematic Formula

For $K, L \in \mathcal{K}^n$,

$$\int_{\phi \in \mathrm{SO}(n) \ltimes \mathbb{R}^n} V_0(K \cap \phi L)\, d\phi = \sum_{i=0}^n \frac{\kappa_i\, \kappa_{n-i}}{\binom{n}{i}\, \kappa_n}\, V_i(K)\, V_{n-i}(L)$$

- $d\phi$ normalized Haar measure on $\mathrm{SO}(n) \ltimes \mathbb{R}^n$
- Blaschke, Chern, Hadwiger, Santaló, ...
- **Proof.** $\mathrm{Z}(K, L) = \int_{\mathrm{SO}(n) \ltimes \mathbb{R}^n} V_0(K \cap \phi L)\, d\phi$
  - $\mathrm{Z}(K, \cdot), \mathrm{Z}(\cdot, L)$ continuous valuations on $\mathcal{K}^n$
  - $\mathrm{Z}(K, \cdot), \mathrm{Z}(\cdot, L)$ rigid motion invariant

Application: Principal Kinematic Formula

For $K, L \in \mathcal{K}^n$,

$$\int_{\phi \in \mathrm{SO}(n) \ltimes \mathbb{R}^n} V_0(K \cap \phi L)\, d\phi = \sum_{i=0}^n \frac{\kappa_i\, \kappa_{n-i}}{\binom{n}{i}\, \kappa_n} V_i(K)\, V_{n-i}(L)$$

- $d\phi$ normalized Haar measure on $\mathrm{SO}(n) \ltimes \mathbb{R}^n$
- Blaschke, Chern, Hadwiger, Santaló, ...
- **Proof.** $\mathrm{Z}(K, L) = \int_{\mathrm{SO}(n) \ltimes \mathbb{R}^n} V_0(K \cap \phi L)\, d\phi$
  - $\mathrm{Z}(K, \cdot), \mathrm{Z}(\cdot, L)$ continuous valuations on $\mathcal{K}^n$
  - $\mathrm{Z}(K, \cdot), \mathrm{Z}(\cdot, L)$ rigid motion invariant

$$\Rightarrow \quad \mathrm{Z}(K, L) = \sum_{i,j=0}^n c_{ij} V_i(K)\, V_j(L)$$

Determine c_{ij} by choosing suitable bodies! □

Rigid Motion Invariant Valuations

Theorem (Hadwiger 1952)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, translation and rotation invariant valuation*

$$\Longleftrightarrow$$

$\exists\, c_0, \dots, c_n \in \mathbb{R} : \quad \mathrm{Z} = c_0 V_0 + \cdots + c_n V_n$

Open problems

- Classification of continuous, rotation invariant valuations on convex bodies in $\mathbb{S}^n$ and $\mathbb{H}^n$ for $n \geqslant 3$
- Classification of continuous, translation and rotation invariant valuations on $\mathcal{P}^n$ for $n \geqslant 3$
- Classification of upper semicontinuous, translation and rotation invariant valuations on $\mathcal{K}^n$ for $n \geqslant 3$

Abstract Hadwiger Theorem

Theorem (Alesker: AiM 2000, GAFA 2007)

For a compact subgroup G of $\mathrm{O}(n)$, the space of continuous, translation and G invariant valuations on $\mathcal{K}^n$ is finite dimensional

$$\iff$$

G acts transitively on $\mathbb{S}^{n-1}$.

Abstract Hadwiger Theorem

Theorem (Alesker: AiM 2000, GAFA 2007)

For a compact subgroup G of $\mathrm{O}(n)$, the space of continuous, translation and G invariant valuations on $\mathcal{K}^n$ is finite dimensional

$$\iff$$

G acts transitively on $\mathbb{S}^{n-1}$.

- $\mathrm{U}(n)$ invariance (Alesker: GAFA 2001, Fu: JDG 2006, Bernig & Fu: Ann. Math. 2001, Wannerer: JDG 2014, AiM 2014)
- $\mathrm{SU}(n)$ invariance (Bernig: GAFA 2009)
- G_2, $\mathrm{Spin}(7)$, $\mathrm{Spin}(9)$ invariance (Bernig: Israel J. 2011, Bernig & Voide: Israel J. 2016)
- $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\cdot\mathrm{U}(1)$, $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ invariance (Bernig: JIMJ 2012, Bernig & Solanes: JFA 2014, PLMS 2017)

Valuations on Function Spaces

- $\mathcal{F}(X) = \{f : X \to \mathbb{R}\}$ space of real valued functions on X
- $f \vee g = \max\{f, g\}$, $f \wedge g = \min\{f, g\}$

Valuations on Function Spaces

- $\mathcal{F}(X) = \{f : X \to \mathbb{R}\}$ space of real valued functions on X
- $f \vee g = \max\{f, g\}$, $f \wedge g = \min\{f, g\}$
- $\mathrm{Z} : \mathcal{F}(X) \to \mathbb{R}$ is a **valuation** $\iff$

$$\mathrm{Z}(f) + \mathrm{Z}(g) = \mathrm{Z}(f \vee g) + \mathrm{Z}(f \wedge g)$$

for all $f, g \in \mathcal{F}(X)$ such that $f \vee g, f \wedge g \in \mathcal{F}(X)$.

Valuations on Function Spaces

- $\mathcal{F}(X) = \{f : X \to \mathbb{R}\}$ space of real valued functions on X
- $f \vee g = \max\{f, g\}$, $f \wedge g = \min\{f, g\}$
- $\mathrm{Z} : \mathcal{F}(X) \to \mathbb{R}$ is a **valuation** $\iff$

$$\mathrm{Z}(f) + \mathrm{Z}(g) = \mathrm{Z}(f \vee g) + \mathrm{Z}(f \wedge g)$$

for all $f, g \in \mathcal{F}(X)$ such that $f \vee g, f \wedge g \in \mathcal{F}(X)$.

Examples

- Valuations on convex bodies (via indicator or support functions)
- Valuations on star sets (via indicator or radial functions)

Valuations on Function Spaces

- $\mathcal{F}(X) = \{f : X \to \mathbb{R}\}$ space of real valued functions on X
- $f \vee g = \max\{f, g\}$, $f \wedge g = \min\{f, g\}$
- $\mathrm{Z} : \mathcal{F}(X) \to \mathbb{R}$ is a **valuation** $\iff$

$$\mathrm{Z}(f) + \mathrm{Z}(g) = \mathrm{Z}(f \vee g) + \mathrm{Z}(f \wedge g)$$

for all $f, g \in \mathcal{F}(X)$ such that $f \vee g, f \wedge g \in \mathcal{F}(X)$.

Examples

- Valuations on convex bodies (via indicator or support functions)
- Valuations on star sets (via indicator or radial functions)

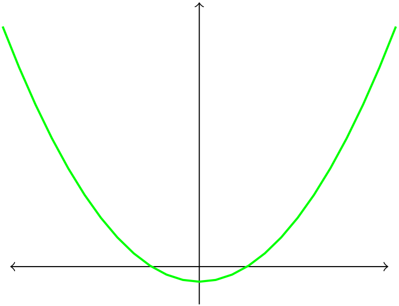
Question (L. 2010):

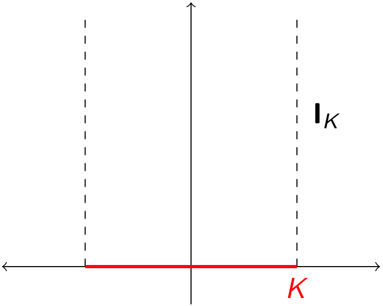
- Classification of interesting valuations on classical function spaces

Valuations on Classical Function Spaces

- Valuations on L_p and Orlicz functions:
  Tsang: IMRN 2010, TAMS 2012; L.: AAM 2013;
  Ober: JMAA 2014; Kone: AAM 2014; Li & Ma: JFA 2017
- Valuations on Sobolev and BV functions:
  L.: AIM 2011, AJM 2012; Wang: IUMJ 2014; Ma: SCM 2016
- **Valuations on convex functions:**
  Cavallina & Colesanti: AGMS 2015; Colesanti, L. & Mussnig:
  IMRN 2017, CVPDE 2017, IUMJ 2020, JFA 2020; Alesker: AG 2019;
  Knörr 2019; Mussnig: AiM 2019, CJM 2019, JGA 2020
- Valuations on quasi-concave functions:
  Colesanti & Lombardi: 2017; Colesanti, Lombardi & Parapatits: 2017
- Valuations on continuous and Lipschitz functions:
  Villanueva: AiM 2016; Tradacete & Villanueva: JMAA 2017,
  AiM 2018, IMRN 2020; Colesanti, Pagnini, Tradacete & Villanueva:
  AiM 2020, JFA 2021

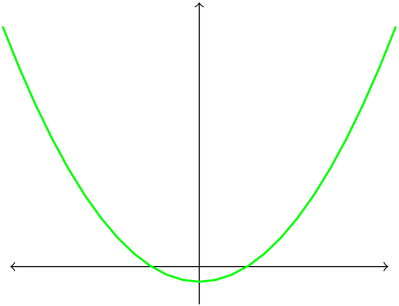
Valuations on Convex Functions

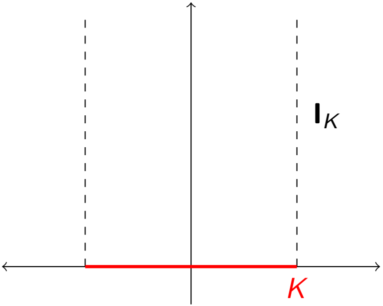




- Convex functions
  $\mathrm{Conv}(\mathbb{R}^n) = \{u : \mathbb{R}^n \to (-\infty, \infty] : u \text{ convex, l.s.c., proper}\}$

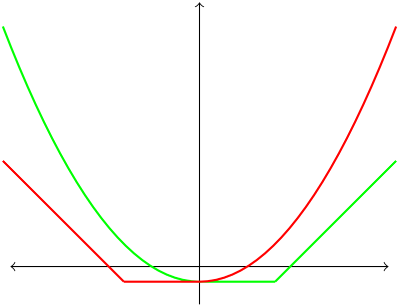
Valuations on Convex Functions





- Convex functions
  $\mathrm{Conv}(\mathbb{R}^n) = \{u : \mathbb{R}^n \to (-\infty, \infty] : u \text{ convex, l.s.c., proper}\}$
- u_k is epi-convergent to u in $\mathrm{Conv}(\mathbb{R}^n) \Leftrightarrow$
  - $u(x) \leqslant \liminf_{k\to\infty} u_k(x_k)$ for every (x_k) with $x_k \to x$
  - $\forall x$, $\exists (x_k)$ with $x_k \to x$ such that $u(x) = \lim_{k\to\infty} u_k(x_k)$

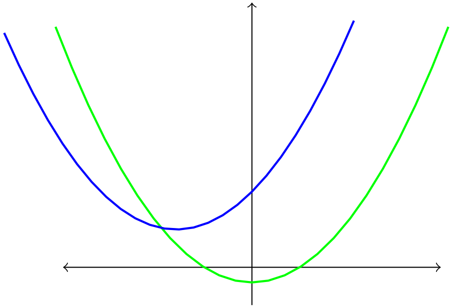
Valuations on Convex Functions



- $\mathcal{C}^{n+1}$ closed, convex sets in $\mathbb{R}^{n+1}$
- $\operatorname{epi} u := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \colon u(x) \leqslant t\}$ epi-graph of $u \in \mathrm{Conv}(\mathbb{R}^n)$
- $\mathcal{C}^{n+1}_{epi} = \{C \subset \mathbb{R}^{n+1} : C$ epi-graph of $u \in \mathrm{Conv}(\mathbb{R}^n)\} \subset \mathcal{C}^{n+1}$
- $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ is a valuation
  $\Leftrightarrow \bar{\mathrm{Z}} : \mathcal{C}^{n+1}_{epi} \to \mathbb{R}$, defined by $\bar{\mathrm{Z}}(\operatorname{epi}(u)) = \mathrm{Z}(u)$, is a valuation

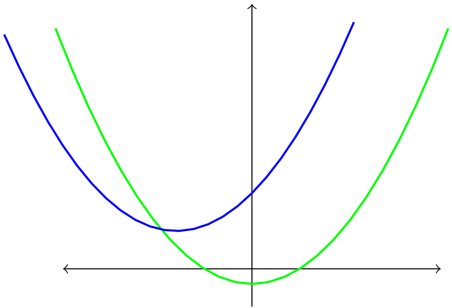
Epi-Translation and Rotation Invariant Valuations

- $\mathcal{F}(\mathbb{R}^n) \subset \mathrm{Conv}(\mathbb{R}^n)$
- $\mathrm{Z} : \mathcal{F}(\mathbb{R}^n) \to \mathbb{R}$ is epi-translation invariant
  $\Leftrightarrow \mathrm{Z}(u \circ \tau^{-1} + c) = \mathrm{Z}(u)$ for all translations $\tau : \mathbb{R}^n \to \mathbb{R}^n$ and $c \in \mathbb{R}$



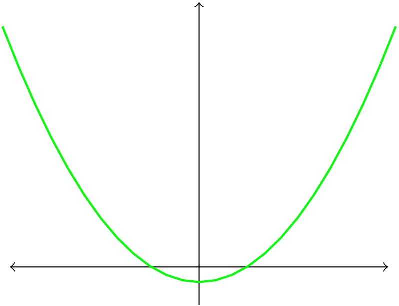
Epi-Translation and Rotation Invariant Valuations

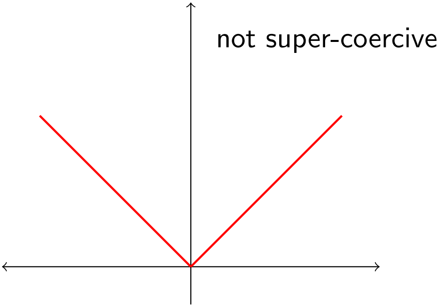
- $\mathcal{F}(\mathbb{R}^n) \subset \mathrm{Conv}(\mathbb{R}^n)$
- $\mathrm{Z} : \mathcal{F}(\mathbb{R}^n) \to \mathbb{R}$ is epi-translation invariant
  $\Leftrightarrow \mathrm{Z}(u \circ \tau^{-1} + c) = \mathrm{Z}(u)$ for all translations $\tau : \mathbb{R}^n \to \mathbb{R}^n$ and $c \in \mathbb{R}$



- $\mathrm{Z} : \mathcal{F}(\mathbb{R}^n) \to \mathbb{R}$ is rotation invariant
  $\Leftrightarrow \mathrm{Z}(u \circ \vartheta^{-1}) = \mathrm{Z}(u)$ for all $\vartheta \in \mathrm{SO}(n)$

Valuations on Super-Coercive Convex Functions





- $u \in \mathrm{Conv}(\mathbb{R}^n)$ super-coercive
  $\Leftrightarrow \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty$
- $\mathrm{Conv_{sc}}(\mathbb{R}^n) = \big\{u \in \mathrm{Conv}(\mathbb{R}^n) : u \text{ super-coercive}\big\}$

Functional Intrinsic Volumes

Theorem (Colesanti, L. & Mussnig 2020+)

For $j \in \{0, \ldots, n\}$ and $\zeta \in D_j^n$, there exists a unique, continuous, epi-translation and rotation invariant valuation $\mathrm{V}_{j,\zeta} \colon \mathrm{Conv_{sc}}(\mathbb{R}^n) \to \mathbb{R}$ such that

$$\mathrm{V}_{j,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-j}\, \mathrm{d}x$$

for every $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$.

- For $j \in \{0, \ldots, n-1\}$,
$$D_j^n := \Big\{\zeta \in C_b((0,\infty)) \colon \lim_{s \to 0^+} s^{n-j}\zeta(s) = 0, \\ \lim_{s \to 0^+} \int_s^\infty t^{n-j-1}\zeta(t)\, \mathrm{d}t \text{ finite}\Big\}$$
- $D_n^n := \Big\{\zeta \in C_b((0,\infty)) \colon \lim_{s \to 0^+} \zeta(s) \text{ finite}\Big\}$

Functional Intrinsic Volumes

Theorem (Colesanti, L. & Mussnig 2020+)

For $j \in \{0, \dots, n\}$ and $\zeta \in D_j^n$, there exists a unique, continuous, epi-translation and rotation invariant valuation $\mathrm{V}_{j,\zeta} \colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ such that

$$\mathrm{V}_{j,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-j} \,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$.

- $[\mathrm{D}^2 u(x)]_k$ kth elementary symmetric function of the eigenvalues of the Hessian matrix $\mathrm{D}^2 u(x)$, kth trace of $\mathrm{D}^2 u(x)$

Functional Intrinsic Volumes

Theorem (Colesanti, L. & Mussnig 2020+)

For $j \in \{0, \dots, n\}$ and $\zeta \in D_j^n$, there exists a unique, continuous, epi-translation and rotation invariant valuation $\mathrm{V}_{j,\zeta} : \mathrm{Conv_{sc}}(\mathbb{R}^n) \to \mathbb{R}$ such that

$$\mathrm{V}_{j,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-j} \, \mathrm{d}x$$

for every $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$.

- $[\mathrm{D}^2 u(x)]_k$ kth elementary symmetric function of the eigenvalues of the Hessian matrix $\mathrm{D}^2 u(x)$, kth trace of $\mathrm{D}^2 u(x)$
- Hessian measures: Trudinger & Wang (Ann. Math. 1999), Colesanti & Hug (TAMS 2000)
- Hessian valuations: Colesanti, L. & Mussnig (IUMJ 2020)
- Singular Hessian valuations, Moreau-Yosida approximation

The Hadwiger Theorem for Convex Functions

Theorem (Colesanti, L. & Mussnig 2020+)

$\mathrm{Z} : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, epi-translation and rotation invariant valuation*

$$\iff$$

$\exists\ \zeta_0 \in D_0^n, \ldots, \zeta_n \in D_n^n$:

$$\mathrm{Z}(u) = \mathrm{V}_{0,\zeta_0}(u) + \cdots + \mathrm{V}_{n,\zeta_n}(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

- $\mathrm{V}_{j,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-j}\,\mathrm{d}x$

  for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$

The Hadwiger Theorem for Convex Functions

Theorem (Colesanti, L. & Mussnig 2020+)

$\mathrm{Z} : \mathrm{Conv_{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, epi-translation and rotation invariant valuation*

$$\Longleftrightarrow$$

$\exists\ \zeta_0 \in D_0^n, \dots, \zeta_n \in D_n^n$:

$$\mathrm{Z}(u) = \mathrm{V}_{0,\zeta_0}(u) + \cdots + \mathrm{V}_{n,\zeta_n}(u)$$

for every $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$.

- Orthogonal cylinder functions, Hadwiger's canonical simplex decomposition, induction on the dimension
- Reduction to the case of epi-additive functionals and epi-symmetrization

The Hadwiger Theorem for Convex Functions

Theorem (Colesanti, L. & Mussnig 2020+)

$\mathrm{Z} : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, epi-translation and rotation invariant valuation*

$$\Longleftrightarrow$$

$\exists\ \zeta_0 \in D_0^n, \dots, \zeta_n \in D_n^n$:

$$\mathrm{Z}(u) = \mathrm{V}_{0,\zeta_0}(u) + \cdots + \mathrm{V}_{n,\zeta_n}(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Theorem (Hadwiger 1952)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, translation and rotation invariant valuation*

$$\Longleftrightarrow$$

$\exists\ c_0, \dots, c_n \in \mathbb{R}$:

$$\mathrm{Z}(K) = c_0 V_0(K) + \cdots + c_n V_n(K)$$

for every $K \in \mathcal{K}^n$.

Thank you!