Additive Functions on Convex Bodies

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Felix Klein's Erlangen Program 1872

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Groups acting on $\mathbb{R}^n$

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  where A is an $n \times n$ matrix of determinant 1
- General linear group $\mathrm{GL}(n)$: $x \mapsto Ax$
  where A is an $n \times n$ matrix of determinant $\neq 0$

Valuations

- $\mathcal{K}^n$ space of convex bodies (compact convex sets) in $\mathbb{R}^n$
- $\langle \mathbb{A}, + \rangle$ abelian semi-group
- A function $\mathrm{z} : \mathcal{K}^n \to \langle \mathbb{A}, + \rangle$ is a *valuation* (or an *additive function*) $\iff$

$$\mathrm{z}(K) + \mathrm{z}(L) = \mathrm{z}(K \cup L) + \mathrm{z}(K \cap L)$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$.

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- Hilbert's Third Problem:
  Dehn 1902, Sydler 1965, Jessen & Thorup 1978, . . .
- Theory of valuations:

Blaschke 1937, **Hadwiger** 1949, Sallee 1966, Schneider 1971, Groemer 1972, McMullen 1977, Goodey & Weil 1984, Betke & Kneser 1985, Klain 1995, Reitzner 1999, Alesker 1999, Fu 2006, Bernig 2006, Haberl 2006, Schuster 2006, . . .

Hadwiger's Classification Theorem 1952

Theorem

A functional $\mathrm{z} : \mathcal{K}^n \to \langle \mathbb{R}, + \rangle$ *is a continuous, rigid motion invariant valuation*

$$\Longleftrightarrow$$

$\exists \; c_0, c_1, \dots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{z}(K) = c_0 \, V_0(K) + \cdots + c_n \, V_n(K)$$

for every $K \in \mathcal{K}^n$.

$V_0(K), \dots, V_n(K)$	intrinsic volumes of K
V_n	n-dimensional volume
$2\, V_{n-1}(K) = S(K)$	surface area
$V_0(K) = 1$	Euler characteristic

Affine Surface Area of $K \in \mathcal{K}^n$

$$\Omega(K) = \int\limits_{\partial K} \kappa(K, x)^{\frac{1}{n+1}}\, dx$$

$\kappa(K, x)$ Gaussian curvature at $x \in \partial K$

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Applications in polytopal approximation

Fejes Tóth 1948, Schneider 1986, Gruber 1988, . . .

Theorem (L. 1999, L. & Reitzner 1999)

A functional $\mathrm{z} : \mathcal{K}^n \to \langle \mathbb{R}, + \rangle$ *is an upper semicontinuous,* $\mathrm{SL}(n)$ *and translation invariant valuation*

$$\iff$$

$\exists\ c_0, c_1 \in \mathbb{R}, c_2 \geq 0$ *such that*

$$\mathrm{z}(K) = c_0\, V_0(K) + c_1\, V_n(K) + c_2\, \Omega(K)$$

for every $K \in \mathcal{K}^n$.

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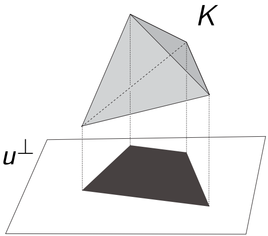
General affine surface areas

- Upper semicontinuous, SL(n) invariant valuations
- *A classification of* SL(n) *invariant valuations* (Jointly with M. Reitzner, Annals of Mathematics, in press)

Integral Affine Surface Area

- Cauchy's surface area formula

$$S(K) = \frac{1}{v_{n-1}} \int_{S^{n-1}} V_{n-1}(K|u^\perp)\, du$$



- $S(K)$ surface area of K
- $u^\perp$ hyperplane orthogonal to u
- $K|u^\perp$ projection of K to $u^\perp$
- $v_k = V_k(B^k)$; B^k k-dimensional unit ball

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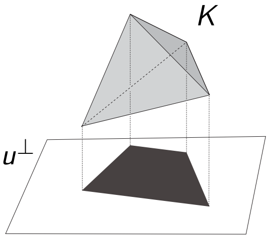

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- Integral affine surface area

$$\Phi(K) = \left(\frac{1}{n} \int_{S^{n-1}} V_{n-1}(K|u^\perp)^{-n}\, du\right)^{-1/n}$$

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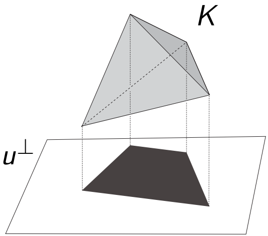

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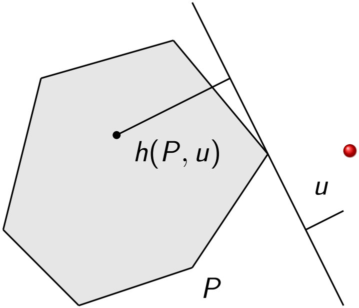
- Integral affine surface area

$$\Phi(K) = \left(\frac{1}{n} \int_{S^{n-1}} V_{n-1}(K|u^\perp)^{-n}\, du \right)^{-1/n}$$

  - $\Phi(\psi K) = |\det \psi|^{1-n}\, \Phi(K)$ for all $\psi \in \mathrm{GL}(n)$

Projection Body, ΠK, of K



- Support function:
  $h(P, u) = \max\{u \cdot x : x \in P\}$

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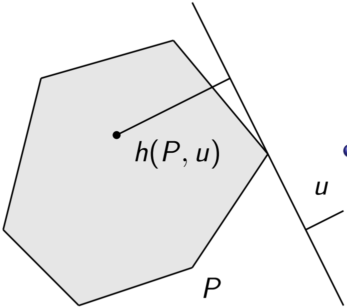

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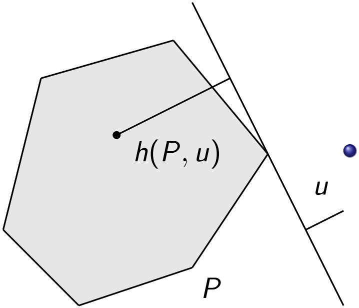
Definition (Minkowski 1901)

$$h(\Pi K, u) = V_{n-1}(K|u^{\perp}), \;\; u \in S^{n-1}$$

Projection Body, ΠK, of K



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Definition (Minkowski 1901)

$$h(\Pi K, u) = V_{n-1}(K|u^\perp), \;\; u \in S^{n-1}$$

- $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}$ polar body of K
- $\Pi^* K = (\Pi K)^*$ polar projection body of K
- $\Phi(K) = V(\Pi^* K)^{-1/n}$ integral affine surface area

Brunn Minkowski Theory

R. Schneider (*Convex Bodies: The Brunn Minkowski Theory*, 1993)
"Merging two elementary notions for point sets in Euclidean space: vector addition and volume"

- Minkowski sum (or vector sum) of $K, L \in \mathcal{K}^n$

$$\begin{aligned} K + L &= \{x + y : x \in K, y \in L\} \\ h(K + L, \cdot) &= h(K, \cdot) + h(L, \cdot) \end{aligned}$$

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- $\mathrm{Z} : \mathcal{K}^n \to \langle \mathcal{K}^n, + \rangle$ is a Minkowski valuation $\Leftrightarrow$

$$\mathrm{Z}\,K + \mathrm{Z}\,L = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L)$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$

Classification of Minkowski Valuations

Theorem (L. 2002)

$\mathrm{Z} : \mathcal{K}_0^n \to \langle \mathcal{K}^n, + \rangle$ *is a continuous,* $\mathrm{GL}(n)$ *contravariant Minkowski valuation*

$$\Longleftrightarrow$$

$\exists\, c \geq 0$:

$$\mathrm{Z}\, K = c\, \Pi\, K$$

for every $K \in \mathcal{K}_0^n$.

- $\mathcal{K}_0^n$ convex bodies in $\mathbb{R}^n$ that contain the origin
- $\mathrm{Z} : \mathcal{K}_0^n \to \mathcal{K}^n$ is $\mathrm{GL}(n)$ contravariant $\Longleftrightarrow$

$$\mathrm{Z}(\phi K) = |\det \phi|^q \, \phi^{-t}\, \mathrm{Z}\, K$$

for all $\phi \in \mathrm{GL}(n), K \in \mathcal{K}_0^n$ for some $q \in \mathbb{R}$

Petty's Projection Inequality

$$V(K)^{n-1} V(\Pi^* K) \leq V(B^n)^{n-1} V(\Pi^* B^n)$$

- equality precisely for ellipsoids
- affine isoperimetric inequality
- Hölder's inequality $\Rightarrow$

$$\Big(\frac{1}{n v_n} \int_{S^{n-1}} V_{n-1}(K|u^\perp)^{-n}\, du\Big)^{-\frac{1}{n}} \leq \frac{1}{n v_n} \int_{S^{n-1}} V_{n-1}(K|u^\perp)\, du$$

$\Rightarrow$ isoperimetric inequality Clinton Petty 1972

Affine Sobolev Inequality (Gaoyong Zhang 1999)

$$\Big(\frac{1}{n\,v_n}\int_{S^{n-1}} \|D_u f\|_1^{-n}\,du\Big)^{-1/n} \geq 2v_{n-1}\,v_n^{1/n-1}\,\|f\|_{n/(n-1)}$$

- $f \in W^{1,1}(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n), \nabla f \in L^1(\mathbb{R}^n)\}$
- $D_u f$ directional derivative of f in direction u
  $\|f\|_p = (\int_{\mathbb{R}^n} |f(x)|\,dx)^{1/p}$

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$\Rightarrow$ sharp L^1 Sobolev inequality

$$\|\nabla f\|_1 \geq n\, v_n^{1/n} \|f\|_{n/(n-1)}$$

Federer & Fleming 1960, Maz′ya 1960

L^p Sobolev inequalities

$$\|\nabla f\|_p \geq \alpha_{n,p}\|f\|_q$$

- $f \in W^{1,p}(\mathbb{R}^n)$, $q = \frac{np}{n-p}$, $1 < p < n$
- $\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p}$
- equality for $f(x) = (a + b|x - x_0|^{p/(p-1)})^{1-n/p}$, $a, b > 0$, $x_0 \in \mathbb{R}^n$
- Aubin 1976, Talenti 1976

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Questions

Are there affine L^p Sobolev inequalities?
What are the corresponding geometric inequalities?

Affine L^p Sobolev Inequality

$$\Big(\int_{S^{n-1}} \|D_u f\|_p^{-n}\, du\Big)^{-1/n} \geq \gamma_{n,p}\, \|f\|_q$$

- $f \in W^{1,p}(\mathbb{R}^n)$, $1 < p < n$
- $D_u f$ directional derivative of f in direction u
- equality for $f(x) = (a + |\phi(x - x_0)|^{p/(p-1)})^{1-n/p}$, $a > 0$, $x_0 \in \mathbb{R}^n$, $\phi \in \mathrm{GL}(n)$
- affine isoperimetric inequality
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- $p \geq n$: *Affine Moser-Trudinger and Morrey-Sobolev inequalities* by Cianchi, Lutwak, Yang, Zhang 2009

L^p Brunn Minkowski Theory

"Merging two elementary notions for point sets in Euclidean space: L^p Minkowski addition and volume"

$K, L \in \mathcal{K}_0^n$ convex bodies in $\mathbb{R}^n$, $0 \in \operatorname{int} K, L$

- L^p Minkowski or vector sum (Firey 1962): $p > 1$, $\alpha, \beta \geq 0$

$$h^p(\alpha \cdot K +_p \beta \cdot L, u) = \alpha\, h^p(K, u) + \beta\, h^p(L, u)$$

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- L^p surface area measure, $S_p(K, \cdot)$, of K (Lutwak 1993)

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon} = \frac{p}{n} \int_{S^{n-1}} h^p(L, u)\, dS_p(K, u)$$

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- Bianchini, Campi, W. Chen, K.S. Chou, Colesanti, Haberl, C. Hu, Hug, Gardner, Giannopoulos, Grinberg, Gronchi, Guédon, Lutwak, S. Lv, X.N. Ma, Meyer, Oliker, Paouris, Reitzner, Ryabogin, Schneider, Schuster, Schütt, C. Shen, Stancu, Umanskiy, X.J. Wang, Werner, D. Yang, Xiao, D. Ye, G. Zhang, Zvavitch, ...

L^p Petty Projection Inequality

$$V(K)^{n-p} V(\Pi_p^* K)^p \le V(B^n)^{n-p} V(\Pi_p^* B^n)^p$$

- $\Pi_p K$ L^p projection body of K
- $h^p(\Pi_p K, u) = \int_{S^{n-1}} |u \cdot v|^p \, dS_p(K, v)$, $u \in S^{n-1}$, $p > 1$
- $S_p(K, \cdot) = h^{1-p}(K, \cdot) \, S(K, \cdot)$ L^p surface area measure of K
- equality precisely for ellipsoids centered at the origin
- affine isoperimetric inequality
- Erwin Lutwak, Deane Yang, Gaoyong Zhang 2000

Classification of L^p Minkowski Valuations

Theorem (L. 2005)

$\mathrm{Z} : \mathcal{K}_0^n \to \langle \mathcal{K}_0^n, +_p \rangle$ *is a continuous,* $\mathrm{GL}(n)$ *contravariant* L^p *Minkowski valuation*

$$\iff$$

$\exists\, c_1, c_2 \geq 0$*:*

$$\mathrm{Z}\, K = c_1 \cdot \Pi_p^+ K +_p c_2 \cdot \Pi_p^- K$$

for every $K \in \mathcal{K}_0^n$.

- $h^p(\Pi_p^+ K, u) = \displaystyle\int_{\{v \in S^{n-1} : u \cdot v \geq 0\}} |u \cdot v|^p \, dS_p(K, v)$
- $\Pi_p^- K = -\Pi_p^+ K$
- $\Pi_p^+ K, \Pi_p^- K$ asymmetric L^p projection bodies of K
- $\Pi_p K = \frac{1}{2} \cdot \Pi_p^+ K +_p \frac{1}{2} \cdot \Pi_p^- K$

Asymmetric L^p Projection Inequality

$$\mathrm{Z}_p K = c_1 \cdot \Pi_p^+ K +_p c_2 \cdot \Pi_p^- K$$

$$\Longrightarrow$$

$$V(K)^{n-p} V(\mathrm{Z}_p^* K)^p \le V(B^n)^{n-p} V(\mathrm{Z}_p^* B^n)^p$$

- equality precisely for ellipsoids centered at the origin
- $\Rightarrow$ L^p Petty Projection inequality
- Strongest inequalities for $\Pi_p^{\pm} K$
- Christoph Haberl & Franz Schuster (JDG, to appear)

Asymmetric L^p Affine Sobolev Inequality

$$\Big(\int_{S^{n-1}} \|D_u^+ f\|_p^{-n}\, du\Big)^{-1/n} \geq 2^{-1/p}\gamma_{n,p}\,\|f\|_q$$

- $D_u^+ f = \max\{D_u f, 0\}$
- equality for $f(x) = (a + |\phi(x - x_0)|^{p/(p-1)})^{1-n/p}$, $a > 0$, $x_0 \in \mathbb{R}^n$, $\phi \in \mathrm{GL}(n)$
- $\Rightarrow$ affine L^p Sobolev inequality
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- $p \geq n$: *Asymmetric Affine Pólya-Szegö Principles* by Haberl, Schuster, Xiao (preprint)

Valuations on Sobolev Spaces

Theorem (L. 2009)

$\mathrm{z}: W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}^n, + \rangle$ *is a continuous, affinely contravariant valuation*

$$\Longleftrightarrow$$

$\exists\, c \geq 0$:

$$\mathrm{z}(f) = c\, \Pi \langle f \rangle$$

for every $f \in W^{1,1}(\mathbb{R}^n)$.

- z is a valuation $\Longleftrightarrow$ $\mathrm{z}(f \vee g) + \mathrm{z}(f \wedge g) = \mathrm{z}(f) + \mathrm{z}(g)$ for all $f \in W^{1,1}(\mathbb{R}^n)$
- $f \vee g = \max\{f, g\}$ and $f \wedge g = \min\{f, g\}$

Valuations on Sobolev Spaces

Theorem (L. 2009)

$\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}^n, + \rangle$ *is a continuous, affinely contravariant valuation*

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- $V(\Pi^* \langle f \rangle) = \frac{2^n}{n} \int_{S^{n-1}} \| D_u f \|_1^{-n}\, du$
  left hand side of the affine Sobolev inequality

Thank you !!!