Approximation of convex bodies and a momentum lemma for power diagrams

Károly Böröczky, Jr. and Monika Ludwig

The research was partially supported by Österreichisches Ost-West Programm. The first named author was also supported by OTKA.

Abstract

The volume of the symmetric difference of a smooth convex body in $\mathbb{E}^3$ and its bestapproximating polytope with n vertices is asymptotically a constant multiple of $\frac{1}{n}$. We determine this constant and the similarly defined constant for approximation with a given number of facets by solving two isoperimetric problems for planar tilings.

1991 AMS subject classification: 52C20, 52A27

Keywords: asymptotic approximation, power diagram, Laguerre tiling, momentum lemma

1 Introduction and statement of results

Let C be a convex body in Euclidean d-space $\mathbb{E}^d$, i.e., a compact convex set with non-empty interior, and denote by $\mathcal{P}_n^i$ and $\mathcal{P}_{(n)}^c$ the set of polytopes with at most n vertices inscribed to C and the set of polytopes with at most n facets circumscribed to C, respectively. Denote by $\delta(.,.)$ the symmetric difference metric. Beginning with the work of L. Fejes Tóth [2], there are many investigations (cf. the survey [5]) on the asymptotic behavior as $n \to \infty$ of the distance of C to its *best approximating* polytopes with at most n vertices or facets, i.e., of

$$\delta(C, \mathcal{P}_n^i) = \inf\{\delta(C, P) : P \in \mathcal{P}_n^i\}$$

and
$$\delta(C,\mathcal{P}^c_{(n)}) = \inf\{\delta(C,P) : P \in \mathcal{P}^c_{(n)}\}.$$
For $C \subset \mathbb{E}^3$ with boundary of differentiability class $\mathcal{C}^2$ and positive Gaussian curvature κ_C, L. Fejes Tóth [2], p. 152, indicated that
$$\delta(C,\mathcal{P}^i_n) \sim \frac{1}{4\sqrt{3}} \left(\int_{\mathrm{bd}\, C} \kappa_C(x)^{\frac{1}{4}} d\sigma(x) \right)^2 \frac{1}{n} \tag{1}$$
and
$$\delta(C,\mathcal{P}^c_{(n)}) \sim \frac{5}{36\sqrt{3}} \left(\int_{\mathrm{bd}\, C} \kappa_C(x)^{\frac{1}{4}} d\sigma(x) \right)^2 \frac{1}{n} \tag{2}$$
as $n \to \infty$, where σ is the surface area measure in $\mathbb{E}^d$. These formulae were proved by P.M. Gruber in [3] and [4], for the planar case see [8] and for $d > 3$ [6].

We are interested in the analogues of (1) and (2) for the problem of approximation by *general* polytopes, i.e., polytopes that are not necessarily inscribed or circumscribed to C. Let $\mathcal{P}_n$ and $\mathcal{P}_{(n)}$ denote the sets of polytopes with at most n vertices and n facets, respectively, and define $\delta(C,\mathcal{P}_n)$ and $\delta(C,\mathcal{P}_{(n)})$ as above. It is shown in [7] that there are positive constants ldel_{d-1} and ldiv_{d-1} (depending only on d) such that for a convex body $C \subset \mathbb{E}^d$ of class $\mathcal{C}^2$ and with positive Gaussian curvature,
$$\delta(C,\mathcal{P}_n) \sim \frac{1}{2}\mathrm{ldel}_{d-1} \left(\int_{\mathrm{bd}\, C} \kappa_C(x)^{\frac{1}{d+1}} d\sigma(x) \right)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} \tag{3}$$
and
$$\delta(C,\mathcal{P}_{(n)}) \sim \frac{1}{2}\mathrm{ldiv}_{d-1} \left(\int_{\mathrm{bd}\, C} \kappa_C(x)^{\frac{1}{d+1}} d\sigma(x) \right)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} \tag{4}$$
as $n \to \infty$. These constants are defined by means of *Laguerre* tilings, which are also known as *power diagrams* (cf. [1]). The values of these constants are known only for $d = 2$ ($\mathrm{ldel}_1 = \mathrm{ldiv}_1 = 1/16$, cf. [7]), for $d > 3$ it seems to be difficult to determine the exact value of these constants, cf. [6].

We will determine the values of ldel_2 and ldiv_2. These constants are defined in [7] in the following way. Let $L = \{(a_1,r_1),\ldots,(a_m,r_m)\}$ with $a_1,\ldots,a_m \in \mathbb{E}^2$ and $r_1,\ldots,r_m \geq 0$, and define the sets $V_1,\ldots,V_m$ by
$$V_i = \{x \in [0,1]^2 : (x-a_i)^2 - r_i^2 \leq (x-a_j)^2 - r_j^2, j = 1,\ldots,m\}.$$

Then L is called a *Laguerre tiling* of $[0,1]^2$ with the tiles $V_1, \ldots, V_m$. Set

$$v(L) = \sum_{i=1}^{m} \int_{V_i} |(x - a_i)^2 - r_i^2| dx$$

and define

$$\mathrm{ldiv}_2 = \lim_{n\to\infty} n \inf\{v(L) : L \text{ has at most } n \text{ tiles}\},$$

cf. [7].

Denote by P_k the regular k-gon centered at the origin o and of area $|P_k| = 1$. For a convex domain C, set

$$I(C, r) = \int_C |x^2 - r^2| dx$$

and choose ρ_k such that $I(P_k, \rho_k) \le I(P, r)$ for all $r \ge 0$.

THEOREM 1 *Let* $\{Q_1, \ldots, Q_n\}$ *be a tiling of* $[0,1]^2$ *with convex tiles,* $a_i \in \mathbb{E}^2$ *and* $r_i \ge 0$*,* $i = 1, \ldots, n$*. Then*

$$\sum_{i=1}^{n} \int_{Q_i} |(x - a_i)^2 - r_i^2|\, dx \ge \frac{I(P_6, \rho_6)}{n}.$$

This theorem provides a lower bound for ldiv_2 and taking tilings with regular hexagons then shows that this bound is asymptotically optimal. Thus, calculating $I(P_6, \rho_6)$ gives

Corollary 1

$$\mathrm{ldiv}_2 = \frac{5}{18\sqrt{3}} - \frac{1}{4\pi}.$$

In the definition of ldel_2, we have to count the number of vertices in the tiling of $[0,1]^2$ and set

$$\mathrm{ldel}_2 = \lim_{n\to\infty} n \inf\{v(L) : L \text{ has at most } n \text{ vertices}\},$$

cf. [7].

THEOREM 2 *Let $\{Q_1, \ldots, Q_m\}$ be a tiling of $[0,1]^2$ with convex tiles, $a_i \in \mathbb{E}^2$ and $r_i \geq 0$, $i = 1, \ldots, m$, with no more than n vertices. Then*

$$\sum_{i=1}^{m} \int_{Q_i} |(x - a_i)^2 - r_i^2| \, dx \geq \frac{I(P_3, \rho_3)}{2(n-2)}.$$

This provides a lower bound for ldel_2 and considering tilings with regular triangles then gives

Corollary 2

$$\mathrm{ldel}_2 = \frac{1}{6\sqrt{3}} - \frac{1}{8\pi}.$$

That this is the correct value, was conjectured in [3].

The classical momentum lemma of L. Fejes Tóth (cf. [2], p. 198) states that for a non–negative monotonous function g, the extremum of the integral

$$\int_P g(|x|) \, dx$$

over k–gons P with given area is attained for the regular k–gon. A simple consequence is that $I(P, 0) \geq I(P_k, 0)|P|^2$, which can be used to prove (2), cf. [4]. In the proof of Theorems 1 and 2, we need the following analogue of the momentum lemma:

THEOREM 3 *If P is a convex k-gon, $k \geq 3$, then $I(P, r) \geq I(P_k, \rho_k)|P|^2$.*

Note that similar arguments with simpler calculations than for general approximation would yield (1).

2 Some general observations

Let $B(\rho)$ be the circle centered at the origin and with radius ρ. Letting r vary and calculating the critical point of $I(C, r)$ shows

LEMMA 1 *Let C be a convex domain and let ρ be chosen such that $I(C, \rho) \leq I(C, r)$ for all $r \geq 0$. Then $|C \cap B(\rho)| = |C \backslash B(\rho)|$.*

Using this, we see that $B(\rho_k) \subset P_k$ for $k = 3, \ldots$. Thus, elementary calculations give

$$I(P_k, \rho_k) = \frac{1}{2k \tan \frac{\pi}{k}} + \frac{\tan \frac{\pi}{k}}{6k} - \frac{1}{4\pi}. \tag{5}$$

LEMMA 2 *Let C be a convex domain and $r \geq 0$. If $o \notin \operatorname{int} C$, then $I(C, r) \geq 1.1 \cdot I(P_3, \rho_3) \cdot |C|^2$.*

Proof: Since C is convex and $o \notin \operatorname{int} C$, we can choose a (closed) half-plane H containing o on its boundary such that $C \subseteq H$. Choose r_0 and r_1 satisfying

$$|B(r_1) \backslash B(r)| = |B(r) \backslash B(r_0)| = |C|$$

and define $G(r, |C|) = (B(r_1) \backslash \operatorname{int} B(r_0)) \cap H$. If $x \in C \backslash B(r)$ is not in $B(r_1) \backslash B(r)$, then

$$x^2 - r^2 \geq \max\{u^2 - r^2 : u \in B(r_1) \backslash B(r)\}$$

and if $x \in C \cap B(r)$ is not in $B(r) \backslash B(r_0)$, then

$$r^2 - x^2 \geq \max\{r^2 - u^2 : u \in B(r) \backslash B(r_0)\}.$$

Thus

$$\int_{C \backslash B(r)} (x^2 - r^2)\, dx \geq \int_{G(r,|C|) \backslash B(r)} (x^2 - r^2)\, dx$$

and

$$\int_{C \cap B(r)} (r^2 - x^2)\, dx \geq \int_{G(r,|C|) \cap B(r)} (r^2 - x^2)\, dx.$$

Therefore

$$I(C, r) \geq I(G(r, |C|)).$$

Combining this with

$$I(G(r, |C|)) = \frac{|C|^2}{2\pi} \geq 1.1 \cdot \left(\frac{1}{3\sqrt{3}} - \frac{1}{4\pi} \right) |C|^2 = 1.1 \cdot I(P_3, \rho_3)|C|^2$$

where (5) was used, proves the lemma. □

3 An auxiliary function

Let $T = T(t)$ be a triangle with a right angle, $|T| = 1$ and an angle t at o. There always exists an optimal $\rho = \rho(t)$, such that

$$I(T(t), \rho(t)) \leq I(T(t), r)$$

for all $r \geq 0$. Define

$$c(t) = I(T(t), \rho(t))$$

for $0 < t < \pi/2$.

With the help of $c(t)$ we can give a sharp lower bound for $I(T, r)$ for general triangles T.

LEMMA 3 *Let T be a triangle with an angle $2t$ at o. Then*

$$I(T, r) \geq \frac{1}{2} c(t) |T|^2.$$

Proof: First, we show that among all such triangles T and $r \geq 0$, there is a triangle S and a $\rho \geq 0$ such that

$$\frac{I(T, r)}{|T|^2} \geq \frac{I(S, \rho)}{|S|^2}, \tag{6}$$

i.e., that the infimum of $I(T, r)/|T|^2$ is attained for S and ρ. By Lemma 1, we may always assume that

$$|T \cap B(r)| = |T \backslash B(r)|. \tag{7}$$

Consider a sequence $\{T_i, r_i\}$ such that $|T_i|$ is a given value and $I(T_i, r_i)$ approaches the infimum. If $\{T_i\}$ is unbounded then $\{r_i\}$ approaches infinity by (7). For large r_i, (7) yields that the area of the part of T_i outside of $B(r_i + 1)$ is at least $\frac{1}{4}|T|$. If x is chosen from that part, then $x^2 - r_i^2 > 2r_i + 1$, and hence $I(T_i, r_i)$ tends to infinity. We conclude that $\{T_i\}$ and $\{r_i\}$ are bounded, and hence the infimum of $I(T, r)/|T|^2$ is attained.

Second, let S and ρ be chosen such that (6) holds. Denote by H the side of S not containing o and let m be the midpoint of H. Then

$$S \text{ is symmetric} \tag{8}$$

with respect to the line connecting o and m. To show this, keep ρ fixed and rotate the side H around m by an angle φ. Let S_φ be the triangle obtained in this way. Then

$$|S_\varphi| = |S| + O(\varphi^2)$$

and

$$I(S_\varphi, \rho) = I(S, \rho) + \left.\frac{\partial I(S_\varphi, \rho)}{\partial \varphi}\right|_{\varphi=0} \cdot \varphi + O(\varphi^2).$$

Consequently, the minimality property of S yields

$$\left.\frac{\partial I(S_\varphi, \rho)}{\partial \varphi}\right|_{\varphi=0} = 0. \tag{9}$$

This can be written as

$$\int_0^l |(\tau + a)^2 - s^2|\tau \, d\tau - \int_0^l |(\tau - a)^2 - s^2|\tau \, d\tau = 0 \tag{10}$$

where $2l$ the length of H, a is the distance of m and the orthogonal projection of o to the affine hull $\operatorname{aff} H$, and $2s$ is the length of $\operatorname{aff} H \cap B(\rho)$. It follows from Lemma 1 that

$$|\operatorname{aff} H \cap B(\rho)| < l. \tag{11}$$

We have to distinguish three cases.

(i) H does not intersect $B(\rho)$. Then, evaluating (10) gives $4al^3/3 = 0$ and $a = 0$.

(ii) H intersects $B(\rho)$ exactly once. If $a \geq s$, then

$$(\tau + a)^2 - s^2 > |(\tau - a)^2 - s^2|$$

holds for $\tau > 0$. Thus (10) does not hold in this case. Therefore $a < s$ or equivalently $m \in \operatorname{int} B(\rho)$. But this implies $|H \cap B(\rho)| > l$ which contradicts (11). So this case cannot occur.

(iii) H intersects $B(\rho)$ twice, and hence $|H \cap B(\rho)| = 2s$. Then evaluating (10) gives $a(l^3 - 2s^3) = 0$. By (11), we know that $l^3 - 2s^3 \neq 0$. Therefore $a = 0$.

Thus in each case, $a = 0$ holds, which is in turn equivalent to (8).

Finally, it follows from (8) that

$$I(S, \rho) = \frac{1}{2} c(t) |S|^2.$$

Combined with (6) this proves the lemma. □

Let t_1 be the unique t, $0 < t < \pi/2$, satisfying $\tan t = 2\,t$. Then for $t < t_1$ the third side of T does not intersect $B(\rho)$ and

$$c(t) = \frac{1}{\tan t} + \frac{\tan t}{3} - \frac{1}{2t}.$$

$c(t)$ attains a unique minimum at t_0, $\pi/4 < t_0 < \pi/3$. We use the following properties of $1/c(t)$.

LEMMA 4 $1/c(t)$ *is concave for* $t \leq t_1$*, increasing for* $0 < t \leq t_0$ *and decreasing for* $t_0 \leq t \leq t_1$*.*

Proof: Derivating $c(t)$ yields

$$\begin{aligned} c'(t) &= -\frac{1}{\tan^2 t} + \frac{\tan^2 t}{3} + \frac{1}{2t^2} - \frac{2}{3} \\ c''(t) &= \frac{2}{\tan^3 t} + \frac{2}{\tan t} + \frac{2}{3}\tan t + \frac{2}{3}\tan^3 t - \frac{1}{t^3}. \end{aligned}$$

To show that $1/c(t)$ is concave, is equivalent to prove that $c(t)\,c''(t) - 2c'(t)^2 > 0$. We have

$$\begin{aligned} c(t)\,c''(t) - 2c'(t)^2 &= \frac{(\tan t - t)(3t - 3\tan t + 3t\tan^2 t - \tan^3 t)}{3t^3 \tan^3 t} \\ &\quad + \frac{16}{9}(1 + \tan^2 t) - \frac{\tan t}{3t^2}(t + 2\tan t + t\,\tan^2 t) \end{aligned}$$

It is not difficult to see that

$$3t - 3\tan t + 3t\tan^2 t - \tan^3 t \geq 0$$

for $0 \le t \le t_1$. Thus, using $\tan t \le 2t$, gives

$$\begin{aligned} c(t)\, c''(t) - 2c'(t)^2 &\ge \frac{16}{9}(1+\tan^2 t) - \frac{\tan t}{3\,t}(5+\tan^2 t) \\ &= \frac{1}{9\,t}(16\,t + 16\,t\tan^2 t - 15\,\tan t - 3\,t\,\tan^2 t) > 0. \end{aligned}$$

□

LEMMA 5 *Let $f(t;\pi/k)$ be the linear function representing the tangent to $1/c(t)$ at π/k. Then*

$$\frac{1}{c(t)} \le f\left(t; \frac{\pi}{k}\right)$$

for $0 < t < \pi/2$ and $k = 3, 4, \ldots$.

Proof: By Lemma 4, this holds for $t \le t_1$ and it remains to be shown that

$$c(t) \ge \frac{1}{f(t;\frac{\pi}{3})} \qquad (12)$$

for $t_1 \le t < \pi/2$. Let o, $(h,0) = (h(t),0)$, and $(h,l) = (h(t),l(t))$ be the vertices of $T = T(t)$ and denote by s the length of the intersection of $B(\rho)$ and the side of T not containing o. Then Lemma 1 yields that

$$\frac{s}{l} \le 1 - \frac{1}{\sqrt{2}}. \qquad (13)$$

For small $\varepsilon > 0$, we have

$$I(T,\rho) - I(T(t-\varepsilon),\rho(t-\varepsilon)) \ge I(T,\rho) - I(T(t-\varepsilon),\rho). \qquad (14)$$

Note that as $\varepsilon \to 0$,

$$\int_{T\setminus T(t-\varepsilon)} |x^2-\rho^2|\, dx = \int_0^{\sqrt{h^2+l^2}} |\tau^2-\rho^2|\tau\, d\tau \cdot \varepsilon + o(\varepsilon)$$

and

$$\begin{aligned} \int_{T(t-\varepsilon)\setminus T} |x^2-\rho^2|\, dx &= \int_0^l |h^2+u^2-\rho^2|\, du \cdot (h(t-\varepsilon)-h) + o(\varepsilon) \\ &= \int_0^l |h^2+u^2-\rho^2|\, du \left(\frac{h^2+l^2}{2l}\right) \cdot \varepsilon + o(\varepsilon). \end{aligned}$$

Thus the coefficient of ε in the left hand side of (14) is

$$\int\limits_0^{\sqrt{h^2+l^2}} |\tau^2-\rho^2|\tau\, d\tau - \int_0^l |h^2+u^2-\rho^2|\, du \left(\frac{h^2+l^2}{2l}\right) =$$

$$\begin{aligned} &= -\frac{2}{3}+\frac{4}{l^4}+\underbrace{\left(\frac{4\,s^2}{l^2}-\frac{8\,s^3}{3\,l^3}\right)}_{\geq 0}+l^4\underbrace{\left(\frac{1}{12}-\frac{2}{3}\left(\frac{s}{l}\right)^3+\frac{1}{2}\left(\frac{s}{l}\right)^4\right)}_{\geq 13/24-\sqrt{2}/3} \\ &\geq -\frac{2}{3}+\frac{4}{l^4}+\left(\frac{13}{24}-\frac{\sqrt{2}}{3}\right)l^4 \geq 1 \end{aligned}$$

where we used (13) and $l^2(t)=2\tan t \geq 2\tan t_1 = 4t_1$. We deduce by (14) that

$$I(T,\rho)-I(T(t-\varepsilon),\rho(t-\varepsilon)) \geq \varepsilon + o(\varepsilon),$$

and hence

$$c(t) \geq c(t_1)+(t-t_1).$$

Finally, some simple calculations yield (12). □

LEMMA 6 *$t\,c(t)$ is monotonously increasing for $t \leq \pi/3$.*

Proof: We have

$$(tc(t))' = \frac{3\tan t+\tan^3 t-2t\tan^2 t+t\tan^4 t-3t}{3\tan^2 t}.$$

Since $\tan t \geq t$ and the enumerator $E(t)$ satisfies $E(0)=0$ and

$$E'(t) = 4\tan^2 t - 4t\tan t + 4\tan^4 t + 4t\tan^5 t,$$

we deduce that $E(t) > 0$ for $0 < t < \pi/3$. □

4 Proof of Theorem 3

Since, by the definition of $c(t)$ and (5), $I(P_k, \rho_k) = c(\pi/k)/(2\,k)$, it follows from Lemma 6 that

$$I(P_3, \rho_3) > I(P_4, \rho_4) > I(P_5, \rho_5) > \dots . \tag{15}$$

Therefore, if $o \notin \operatorname{int} P$, we have by Lemma 2

$$I(P, r) > 1.1 \cdot I(P_3, \rho_3)\,|P|^2 > I(P_k, \rho_k)\,|P|^2,$$

i.e., the theorem holds in this case. So, let $o \in \operatorname{int} P$ and dissect P into triangles $T_1, \dots, T_k$ with a common vertex o, and let $2\,t_j$ be the angle of T_j at o. By Lemma 3, we have

$$I(P, r) = \sum_{i=1}^{k} I(T_i, r) \geq \frac{1}{2} \sum_{i=1}^{k} c(t_i)\,|T_i|^2.$$

The Cauchy-Schwarz inequality yields

$$\sum_{i=1}^{k} c(t_i)|T_i|^2 \geq \left(\sum_{i=1}^{k} \frac{1}{c(t_i)} \right)^{-1} \left(\sum_{i=1}^{k} |T_i| \right)^2 . \tag{16}$$

By Lemma 5,

$$\sum_{k=1}^{k} \frac{1}{c(t_i)} \leq \sum_{k=1}^{k} f\left(t_i; \frac{\pi}{k}\right) = \frac{k}{c(\frac{\pi}{k})}.$$

Therefore,

$$I(P, r) \geq \frac{1}{2} \left(\sum_{i=1}^{k} \frac{1}{c(t_i)} \right)^{-1} |P|^2 \geq \frac{1}{2k} c\left(\frac{\pi}{k}\right)\,|P|^2 = I(P_k, \rho_k)\,|P|^2,$$

which proves the theorem.

5 Proof of Theorem 2

We can dissect every tile Q_i into triangles such that we obtain a simplicial tiling with tiles $T_1, \dots, T_k$ and at most n vertices. If we double each tile, we may think of this as a polytope with $f_2 = 2k$ facets, f_1 edges and $f_0 < 2n$ vertices. By Euler's formula $f_2 - f_1 + f_0 = 2$ and $f_2 \le 2f_0 - 4$ which implies

$$k \le 2(n-2). \tag{17}$$

Therefore, by Theorem 3, the inequality of quadratic and arithmetic means, and (17) we obtain

$$\begin{aligned}\sum_{i=1}^{m}\int_{Q_i} |(x-a_i)^2 - r_i^2|\,dx &\ge I(P_3,\rho_3)\sum_{i=1}^{k}|T_i|^2\\ &\ge I(P_3,\rho_3)\left(\sum_{i=1}^{k}|T_i|\right)^2\frac{1}{k} \ge \frac{I(P_3,\rho_3)}{2(n-2)},\end{aligned}$$

which proves the theorem.

To obtain the corollary, cover $[0,1]^2$ with k non-overlapping regular triangles of equal area $|T|$. Then, we obtain a Laguerre-tiling L with, say, n vertices by setting $r^2 = |T|/(2\pi)$ for each tile. We have

$$v(L) \le k\, I(P_3,\rho_3)|T|^2.$$

Since we may choose the triangles such that $k|T| \to 1$ and $k/n \to 2$ as $k \to \infty$,

$$\limsup_{n\to\infty} n\, v(L_k) \le \frac{I(P_3,\rho_3)}{2},$$

and by Theorem 2, we have $\mathrm{ldel}_2 = I(P_3,\rho_3)/2$.

6 Proof of Theorem 1

To every tile Q_i with l_i sides we assign $2l_i$ rectangular triangles of area $|Q_i|/(2l_i)$ and with angle π/l_i at the vertex o. Let $k = 2\sum_{i=1}^{n} l_i$, let $T_1,\dots,T_k$ be these triangles, and let t_j denote the angle of T_j at o. Then $\sum_{j=1}^{k} t_j = 2\pi n$. By Theorem 3,

$$\int_{Q_i} |(x-a_i)^2 - r_i^2|\,dx \ge I(P_{l_i},\rho_{l_i})|Q_i|^2$$

and

$$\sum_{i=1}^{n}\int_{Q_i} |(x-a_i)^2 - r_i^2|\,dx \ge \sum_{j=1}^{k} c(t_j)|T_j|^2.$$

By (16), we obtain from this

$$\sum_{i=1}^{n} \int_{Q_i} |(x-a_i)^2 - r_i^2|\, dx \geq \sum_{j=1}^{k} c(t_j)|T_j|^2 \geq \left(\sum_{j=1}^{k} \frac{1}{c(t_j)} \right)^{-1} \left(\sum_{j=1}^{k} |T_j| \right)^2 .$$

We have

$$k \leq 12(n-1). \tag{18}$$

This can be seen in the following way. If we double each tile Q_i, we may think of this as a polytope with f_0 vertices, $f_1 = 1/2\,k$ edges and $f_2 = 2n$ facets. By Euler's formula $f_2 - f_1 + f_0 = 2$ and $f_1 \leq 3f_2 - 6$, which implies (18).

So, we obtain by Lemma 4, Jensen's inequality, Lemma 6 and (18)

$$\sum_{j=1}^{k} \frac{1}{c(t_j)} \leq \frac{k}{c(\frac{1}{k}\sum_{j=1}^{k} t_j)} = \frac{2\pi n}{\frac{2\pi n}{k} c(\frac{2\pi n}{k})} \leq \frac{12\,n}{c(\frac{\pi}{6})}.$$

Thus

$$\sum_{i=1}^{n} \int_{Q_i} |(x-a_i)^2 - r_i^2|\, dx \geq \frac{c(\frac{\pi}{6})}{12\,n} = \frac{I(P_6, \rho_6)}{n},$$

which proves the theorem.

Corollary 1 follows as Corollary 2, except that the triangular tiling is replaced by the hexagonal tiling.

References

[1] Aurenhammer,F.: Power diagrams: properties, algorithms and applications. SIAM J. Comput. **16** (1987), 78 – 96

[2] Fejes Tóth,L.: Lagerungen in der Ebene, auf der Kugel und im Raum. 2nd ed. Springer-Verlag, Berlin 1972

[3] Gruber,P.M.: Volume approximation of convex bodies by inscribed polytopes. Math. Ann. **281** (1988), 229 – 245

[4] Gruber,P.M.: Volume approximation of convex bodies by circumscribed polytopes. The "Victor Klee Festschrift", 309 – 317. DIMACS Ser. **4**, Amer. Math. Soc. 1991

[5] Gruber,P.M.: Aspects of approximation of convex bodies. In: Handbook of convex geometry **A**, 319 – 345, North-Holland, Amsterdam 1993

[6] Gruber,P.M.: Asymptotic estimates for best and stepwise approximation of convex bodies II. Forum Math. **5** (1993), 521 – 538

[7] Ludwig,M.: Asymptotic approximation of smooth convex bodies by general polytopes. Manuscript

[8] McClure,D.E., Vitale,R.A.: Polygonal approximation of plane convex bodies. J. Math. Anal. Appl. **51** (1975), 326 – 358

Károly Böröczky, Jr.
Mathematical Institute of the
Hungarian Academy of Sciences
P.O. Box 127
H-1364 Budapest

Monika Ludwig
Technische Universität Wien
Abteilung für Analysis
Wiedner Hauptstr. 8-10/1142
A-1040 Wien