Covariance Matrices and Valuations

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Abstract

A complete classification of SL(n) covariant matrix-valued valuations on functions with finite second moments is obtained. It is shown that there is a unique homogeneous such valuation. This valuation turns out to be the moment matrix.

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A function Z defined on a lattice $(\mathcal{L}, \vee, \wedge)$ and taking values in an abelian semigroup is called a valuation if

$$\mathrm{Z}(f \vee g) + \mathrm{Z}(f \wedge g) = \mathrm{Z}(f) + \mathrm{Z}(g) \tag{1}$$

for all $f, g \in \mathcal{L}$ (see, for example, [5]). A function Z defined on some subset $\mathcal{S}$ of $\mathcal{L}$ is called a valuation on $\mathcal{S}$ if (1) holds whenever $f, g, f \vee g, f \wedge g \in \mathcal{S}$.

Results on valuations on compact convex sets in $\mathbb{R}^n$ are classical and start with Dehn's solution of Hilbert's Third Problem in 1901. Here the operations $\vee$ and $\wedge$ are union and intersection, respectively. In the 1950s, a systematic study of valuations was initiated by Hadwiger, who was in particular interested in classifying valuations on the set of compact convex sets in $\mathbb{R}^n$. Probably the most celebrated result is Hadwiger's classification of continuous and rigid motion invariant valuations on compact convex sets, which establishes a characterization of the intrinsic volumes (see [14, 17]; see [1–4, 9, 10, 12, 19–22, 33, 36, 37, 41] for some of the more recent results). The systematic study of valuations in a more general setting is of more recent vintage. Here valuations were investigated on star shaped sets [15, 16], on Lebesgue spaces [38, 39], on Orlicz spaces [18], on spaces of functions of bounded variation [40] and on Sobolev spaces [23, 25] (see also [24]).

In numerous applications in statistics and information theory, two matrices associated to functions (in particular, probability densities) play a critical role: the covariance or moment matrix and the Fisher information matrix. The Fisher information matrix, $\mathrm{J}(f)$, of a weakly differentiable

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function $f : \mathbb{R}^n \to [0, \infty)$ is the $n \times n$-matrix with (not necessarily finite) entries

$$\mathrm{J}_{ij}(f) = \int_{\mathbb{R}^n} \frac{\partial \log f(x)}{\partial x_i} \frac{\partial \log f(x)}{\partial x_j} f(x)\, dx. \tag{2}$$

In [23], the Fisher information matrix was characterized as the unique (up to multiplication with a constant) continuous and homogeneous matrix-valued valuation Z on the Sobolev space $W^{1,2}(\mathbb{R}^n)$ such that

$$\mathrm{Z}(f \circ \phi^{-1}) = \phi^{-t}\, \mathrm{Z}(f)\, \phi^{-1}$$

for all $\phi \in \mathrm{SL}(n)$, where ϕ^{-t} denotes the inverse of the transpose of ϕ and Z is called homogeneous if, for some $q \in \mathbb{R}$, we have $\mathrm{Z}(sf) = |s|^q\, \mathrm{Z}(f)$ for all $s \in \mathbb{R}$. The natural lattice structure on $W^{1,2}(\mathbb{R}^n)$ (as well as other function spaces) is given by letting $f \vee g$ denote the pointwise maximum and $f \wedge g$ the pointwise minimum of f and g. The proof of the characterization [23] makes essential use of a characterization [20] of the so-called LYZ ellipsoid introduced by Lutwak, Yang and Zhang [27, 28], which corresponds to a $\mathrm{SL}(n)$ covariant valuation on compact convex sets. Such $\mathrm{SL}(n)$ covariant functions have found important applications and are attracting increased interest (see, e.g., [8–10, 12, 13, 19–21, 26, 29–32]).

In this paper, we obtain a characterization of the moment matrix. For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, the moment matrix, $\mathrm{K}(f)$, is the $n \times n$-matrix with (not necessarily finite) entries,

$$\mathrm{K}_{ij}(f) = \int_{\mathbb{R}^n} f(x)\, x_i\, x_j\, dx.$$

If f is a probability density with mean zero, then $\mathrm{K}(f)$ is the covariance matrix of f. Let $\mathcal{L}_2(\mathbb{R}^n)$ be the space of measurable functions with finite second moments, that is, the space of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} |f(x)|\, |x|^2\, dx < \infty$, where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$. Let $\mathbb{M}^n$ denote the space of real symmetric $n \times n$-matrices. A function $\mathrm{Z} : \mathcal{L}_2(\mathbb{R}^n) \to \mathbb{M}^n$ is called $\mathrm{SL}(n)$ covariant if

$$\mathrm{Z}(f \circ \phi^{-1}) = \phi\, \mathrm{Z}(f)\, \phi^t$$

for all $f \in \mathcal{L}_2(\mathbb{R}^n)$ and $\phi \in \mathrm{SL}(n)$. We obtain the following classification of matrix-valued valuations. Let $n > 2$.

Theorem. *A function* $\mathrm{Z} : \mathcal{L}_2(\mathbb{R}^n) \to \langle \mathbb{M}^n, + \rangle$ *is a continuous and* $\mathrm{SL}(n)$ *covariant valuation if and only if there exists a continuous* $\zeta : \mathbb{R} \to \mathbb{R}$ *with the property that* $|\zeta(t)| \le c\, |t|$ *for all* $t \in \mathbb{R}$ *for some* $c \in \mathbb{R}$*, such that*

$$\mathrm{Z}(f) = \mathrm{K}(\zeta \circ f)$$

for every $f \in \mathcal{L}_2(\mathbb{R}^n)$.

If in addition homogeneity is assumed, the following characterization of the moment matrix is obtained.

Corollary. *A function* $\mathrm{Z} : \mathcal{L}_2(\mathbb{R}^n) \to \langle \mathbb{M}^n, + \rangle$ *is a continuous, homogeneous and* $\mathrm{SL}(n)$ *covariant valuation if and only if there is a constant* $c \in \mathbb{R}$ *such that*

$$\mathrm{Z}(f) = c\,\mathrm{K}(f)$$

for every $f \in \mathcal{L}_2(\mathbb{R}^n)$.

The corollary is dual to the classification of matrix-valued valuations on Sobolev spaces [23]. In the theorem, we do not assume that Z is homogeneous. The proof makes use of ideas from a recent classification of convex body valued valuations by Haberl [11], where no homogeneity was assumed. We note that it is not known whether the homogeneity assumption can also be omitted in the characterization of the Fisher information matrix [23].

1 Valuations on convex polytopes

We work in Euclidean n-space, $\mathbb{R}^n$, and write $x = (x_1, \dots, x_n)$ for $x \in \mathbb{R}^n$. The vectors of the standard basis of $\mathbb{R}^n$ are denoted by $e_1, \dots, e_n$. Let $\mathcal{P}^n$ denote the space of compact convex polytopes in $\mathbb{R}^n$ equipped with the usual topology coming from the Hausdorff metric. For more information on convex sets, we refer to the books by Gardner [6], Gruber [7] and Schneider [34].

For $P \in \mathcal{P}^n$, the moment matrix, $\mathrm{M}\,P$, of P is the $n \times n$-matrix with entries

$$\mathrm{M}_{ij}\,P = \int_P x_i\, x_j\, dx.$$

A function $\mathrm{Y} : \mathcal{P}^n \to \mathbb{M}^n$ is called $\mathrm{SL}(n)$ covariant, if

$$\mathrm{Y}(\phi P) = \phi\;\mathrm{Y}(P)\,\phi^t$$

for all $P \in \mathcal{P}^n$ and $\phi \in \mathrm{SL}(n)$. Note that M is $\mathrm{SL}(n)$ covariant.

Let $I = (s,t)$ with $0 < s \le t$ be an interval and T_I the convex hull of $s\,e_1, \dots, s\,e_n$ and $t\,e_1, \dots, t\,e_n$, that is, T_I is the difference of two scaled standard simplices. Let $\mathcal{T}^n$ be the set of all images of such T_I under the $\mathrm{GL}(n)$ equipped with the Hausdorff metric. A function $\mathrm{Y} : \mathcal{T}^n \to \mathbb{M}$ is called simple if Z vanishes on lower dimensional sets. Let $n > 2$.

Lemma 1. *A function* $\mathrm{Y} : \mathcal{T}^n \to \langle \mathbb{M}^n, + \rangle$ *is a continuous, simple and* $\mathrm{SL}(n)$ *covariant valuation if and only if there is a constant* $c \in \mathbb{R}$ *such that*

$$\mathrm{Y}\,T = c\,\mathrm{M}\,T$$

for every $T \in \mathcal{T}^n$.

Proof. For a permutation π on $\{1,\dots,n\}$, let ϕ_π be the associated permutation matrix. Hence ϕ_π is orthogonal and $\phi_\pi\phi_\pi^t = \mathrm{id}$, where id is the $n\times n$-identity matrix. Set $I=(s,t)$ with $0<s\le t$ and let T_I be the convex hull of $s\,e_1,\dots,s\,e_n$ and $t\,e_1,\dots,t\,e_n$. Note that

$$\phi_\pi T_I = T_I \tag{3}$$

for every T_I and every permutation π. Since Y and M are both $\mathrm{SL}(n)$ covariant, it suffices to prove that there is a constant $c\in\mathbb{R}$ such that

$$\mathrm{Y}\,T_I = c\,\mathrm{M}\,T_I$$

for every T_I.

Let π be an even permutation. Then ϕ_π has determinant 1 and since Y is $\mathrm{SL}(n)$ covariant, it follows from (3) that

$$\mathrm{Y}\,T_I = \phi_\pi\,\mathrm{Y}\,T_I\,\phi_\pi^t. \tag{4}$$

For given I and $x>0$, set $z_{ij}(x) = (\mathrm{Y}\,T_{\sqrt[n]{x}I})_{ij}$. Since $\mathrm{Y}\,T_I$ is a symmetric matrix, it follows from (3) and (4) that for $i=1,\dots,n$,

$$z_{ii}(x) = z_{\pi i,\pi i}(x) =: z(x) \tag{5}$$

and for $i,j=1,\dots,n$ with $i\ne j$,

$$z_{ij}(x) = z_{\pi i,\pi j}(x) =: w(x) \tag{6}$$

for $x>0$. Set $J=\sqrt[n]{x}I$.

For $0<\lambda<1$, let H_λ be the hyperplane through the origin with normal vector $(1-\lambda)\,e_1-\lambda\,e_2$. The hyperplane H_λ dissects the set T_J into two sets $T_J\cap H_\lambda^+, T_J\cap H_\lambda^-\in\mathcal{T}^n$, where H_λ^+, H_λ^- are the closed halfspaces bounded by H_λ. Since Y is a simple valuation, we have

$$\mathrm{Y}\,T_J = \mathrm{Y}(T_J\cap H_\lambda^+) + \mathrm{Y}(T_J\cap H_\lambda^-). \tag{7}$$

Let $\phi_\lambda\in\mathrm{GL}(n)$ map e_1 to $\lambda e_1+(1-\lambda)e_2$ and e_i to e_i for $i=2,\dots,n$. Let $\psi_\lambda\in\mathrm{GL}(n)$ map e_2 to $\lambda e_1+(1-\lambda)e_2$ and e_i to e_i for $i=1,3,\dots,n$. Note that $\det\phi_\lambda=\lambda$ and $\det\psi_\lambda = 1-\lambda$. Then

$$T_J\cap H_\lambda^+ = \phi_\lambda T_J = \frac{1}{\sqrt[n]{\lambda}}\,\phi_\lambda T_{\sqrt[n]{\lambda}J}$$

and

$$T_I\cap H_\lambda^- = \psi_\lambda T_J = \frac{1}{\sqrt[n]{1-\lambda}}\,\phi_\lambda T_{\sqrt[n]{1-\lambda}J},$$

where $1/\sqrt[n]{\lambda}\,\phi_\lambda, 1/\sqrt[n]{1-\lambda}\,\phi_\lambda\in\mathrm{SL}(n)$. Since Y is $\mathrm{SL}(n)$ covariant, (7) implies that

$$\mathrm{Y}\,T_J = \lambda^{-\frac{2}{n}}\phi_\lambda\,\mathrm{Y}\,T_{\sqrt[n]{\lambda}J}\phi_\lambda^t + (1-\lambda)^{-\frac{2}{n}}\psi_\lambda\,\mathrm{Y}\,T_{\sqrt[n]{1-\lambda}J}\psi_\lambda^t. \tag{8}$$

Looking at the coefficient $(\mathrm{Y}\, T_J)_{nn}$, we obtain

$$z(x) = \lambda^{-\frac{2}{n}} z(\lambda x) + (1-\lambda)^{-\frac{2}{n}} z((1-\lambda)x).$$

Setting $f(x) = x^{-2/n} z(x)$, we get $f(x) = f(\lambda x) + f((1-\lambda)x)$. Using the fact that every continuous solution of the Cauchy functional equation, $f(x+y) = f(x) + f(y)$, is linear, we conclude that

$$z(x) = a\, x^{\frac{n+2}{n}}$$

with a suitable constant $a \in \mathbb{R}$. Looking at the coefficient $(\mathrm{Y}\, T_J)_{11}$, we obtain from (8) that

$$z(x) = \lambda^{-\frac{2}{n}} \lambda^2 z(\lambda x) + (1-\lambda)^{-\frac{2}{n}} \big((1+\lambda^2) z((1-\lambda)x) + 2\lambda\, w((1-\lambda)x)\big).$$

Hence

$$w(x) = \frac{a}{2}\, x^{\frac{n+2}{n}}.$$

Since M is also a continuous, simple and $\mathrm{SL}(n)$ covariant valuations, we conclude that there is a function $c(s,t)$ defined for $0 \le s \le t$ such that

$$\mathrm{Y}\, T_{(s,t)} = c(s,t)\ \mathrm{M}\, T_{(s,t)}$$

and

$$c(r\, s, r\, t) = r^{n+2} c(s,t)$$

for $r > 0$. Since Y is a simple valuation, we have for $1 < r < s$

$$\mathrm{Y}\, T(1,r) + \mathrm{Y}\, T(r,s) = \mathrm{Y}\, T(1,s).$$

Hence, setting $g(r) = \mathrm{Y}\, T(1,r)$, we have

$$g(r) + r^{n+2} g(\frac{s}{r}) = g(s)$$

or equivalently,

$$g(rx) = g(r) + r^{n+2} g(x).$$

Using that

$$g(r) + r^{n+2} g(x) = g(rx) = g(xr) = g(x) + x^{n+2} g(r),$$

we obtain that there is a constant b such that

$$g(x) = b(1 - x^{n+2}).$$

We conclude that there is a constant c such that

$$\mathrm{Y}\, T_{(s,t)} = c\ \mathrm{M}\, T_{(s,t)}.$$

This completes the proof of the lemma. □

2 Background material on functions with finite second moments

Set $\|f\| = \int_{\mathbb{R}^n} |f(x)|\,|x|^2\,dx$. We say that $f_k \to f$ as $k \to \infty$ in $\mathcal{L}_2(\mathbb{R}^n)$, if $\|f_k - f\| \to 0$ as $k \to \infty$. Note that it follows immediately from the definition that $(\mathcal{L}_2(\mathbb{R}^n), \vee, \wedge)$ is a lattice. Let $\mathbb{1}_C$ be the indicator function of $C \subset \mathbb{R}^n$, that is, $\mathbb{1}_C(x) = 1$ for $x \in C$ and $\mathbb{1}_C(x) = 0$ for $x \notin C$.

In the following lemma, we prove some well known properties of the function $f \mapsto \mathrm{K}(f)$. Let $A(\mathbb{R})$ be the set of continuous $\alpha : \mathbb{R} \to \mathbb{R}$ such that there is $a \in \mathbb{R}$ with $|\alpha(t)| \le a\,|t|$ for all $t \in \mathbb{R}$.

Lemma 2. *The function* $\mathrm{Z} : \mathcal{L}_2(\mathbb{R}^n) \to \langle \mathbb{M}^n, + \rangle$, *defined by* $\mathrm{Z}(f) = \mathrm{K}(\alpha \circ f)$ *with* $\alpha \in A(\mathbb{R})$, *is a continuous and* $\mathrm{SL}(n)$ *covariant valuation. The function* $\mathrm{Z} : \mathcal{L}_2(\mathbb{R}^n) \to \langle \mathbb{M}^n, + \rangle$, *defined by* $\mathrm{Z}(f) = c\,\mathrm{K}(f)$ *with* $c \in \mathbb{R}$, *is a continuous, homogeneous and* $\mathrm{SL}(n)$ *covariant valuation.*

Proof. Let $\alpha(t) \le a\,|t|$ for $t \in \mathbb{R}$. Since

$$\Big| \int_{\mathbb{R}^n} x_i\, x_j\, |\alpha(f(x))|\, dx \Big| \le a \int_{\mathbb{R}^n} |f(x)|\,|x|^2\,dx,$$

we have $\mathrm{K}_{ij}(\alpha \circ f) < \infty$ for $f \in \mathcal{L}_2(\mathbb{R}^n)$. It follows immediately from the definition that $f \mapsto \mathrm{K}(\alpha \circ f)$ is a valuation on $\mathcal{L}_2(\mathbb{R}^n)$. Suppose that $f_k \to f$ in $\mathcal{L}_2(\mathbb{R}^n)$. Then

$$|\,\mathrm{K}_{ij}(\alpha \circ f_k) - \mathrm{K}_{ij}(\alpha \circ f)| \le a \int_{\mathbb{R}^n} |f_k(x) - f(x)|\,|x|^2\,dx.$$

Thus the function $f \mapsto \mathrm{K}(\alpha \circ f)$ is continuous on $\mathcal{L}_2(\mathbb{R}^n)$.

Let $s \in \mathbb{R}$ and $\phi \in \mathrm{SL}(n)$. Since

$$\mathrm{K}(f) = \int_{\mathbb{R}^n} x\, x^t\, |f(x)|\, dx,$$

we have

$$\mathrm{K}(s\,f) = |s|\,\mathrm{K}(f) \quad \text{and} \quad \mathrm{K}(f \circ \phi^{-1}) = \phi\,\mathrm{K}(f)\,\phi^t.$$

Consequently, the function $f \mapsto \mathrm{K}(\alpha \circ f)$ is $\mathrm{SL}(n)$ covariant and the function $f \mapsto c\,\mathrm{K}(f)$ is $\mathrm{SL}(n)$ covariant and homogeneous. □

The following lemma, which follows immediately from the definitions, describes an important connection between functions on $\mathcal{L}_2(\mathbb{R}^n)$ and on $\mathcal{T}$.

Lemma 3. *For* $T \in \mathcal{T}^n$ *and* $\alpha \in \mathbb{R}$, *we have* $\mathrm{K}(\alpha\,\mathbb{1}_T) = \alpha\,\mathrm{M}\,T$.

3 Proof of the Theorem

In Lemma 2, it was shown that for $\alpha \in A(\mathbb{R})$, the function $f \mapsto \mathrm{K}(\alpha \circ f)$ is a continuous and $\mathrm{SL}(n)$ covariant valuation on $\mathcal{L}_2(\mathbb{R}^n)$. Suppose that Z is a continuous and $\mathrm{SL}(n)$ covariant valuation. The following lemmas establish that there is function $\zeta \in A(\mathbb{R})$ such that $\mathrm{Z}(f) = \mathrm{K}(\zeta \circ f)$ for all $f \in \mathcal{L}_2(\mathbb{R}^n)$.

Lemma 4. *If* $\mathrm{Z} : \mathcal{L}_2(\mathbb{R}^n) \to \langle \mathbb{M}^n, + \rangle$ *is a continuous and* $\mathrm{SL}(n)$ *covariant valuation, then there is a continuous function* $\zeta : \mathbb{R} \to \mathbb{R}$ *such that*

$$\mathrm{Z}(\alpha \, \mathbb{1}_T) = \zeta(\alpha)\, \mathrm{K}(\mathbb{1}_T)$$

for every $T \in \mathcal{T}^n$.

Proof. For $\alpha \in \mathbb{R}$, define the function $\mathrm{Y}_\alpha : \mathcal{T}^n \to \langle \mathbb{M}^n, + \rangle$ by setting

$$\mathrm{Y}_\alpha T = \mathrm{Z}(\alpha \, \mathbb{1}_T).$$

Since Z is a valuation on $\mathcal{L}_2(\mathbb{R}^n)$, it follows for $S, T, S \cap T, S \cup T \in \mathcal{T}^n$ that

$$\begin{aligned} \mathrm{Y}_\alpha S + \mathrm{Y}_\alpha T &= \mathrm{Z}(\alpha \, \mathbb{1}_S) + \mathrm{Z}(\alpha \, \mathbb{1}_T) \\ &= \mathrm{Z}(\alpha(\mathbb{1}_S \vee \mathbb{1}_T)) + \mathrm{Z}(\alpha(\mathbb{1}_S \wedge \mathbb{1}_T)) \\ &= \mathrm{Y}_\alpha(S \cup T) + \mathrm{Y}_\alpha(S \cap T). \end{aligned}$$

Thus $\mathrm{Y}_\alpha : \mathcal{T}^n \to \langle \mathbb{M}^n, + \rangle$ is a valuation. Since for $\phi \in \mathrm{SL}(n)$

$$\mathrm{Y}_\alpha(\phi T) = \mathrm{Z}(\alpha \, \mathbb{1}_{\phi T}) = \mathrm{Z}(\alpha \, \mathbb{1}_T \circ \phi^{-1}) = \phi \, \mathrm{Z}(\alpha \, \mathbb{1}_T) \phi^t = \phi \, \mathrm{Y}_\alpha T \, \phi^t,$$

the function Y_α is $\mathrm{SL}(n)$ covariant. Thus we obtain from Lemma 1 that for $n > 2$ there exists a continuous function $\zeta : \mathbb{R} \to \mathbb{R}$ such that

$$\mathrm{Z}(\alpha \, \mathbb{1}_T) = \zeta(\alpha) \, \mathrm{M}\, T$$

for all $T \in \mathcal{T}^n$. The statement now follows from Lemma 3. □

The following lemma is very similar to a result by Tsang [38, Lemma 3.6] and therefore the proof is omitted.

Lemma 5. *Let* $\zeta : \mathbb{R} \to \mathbb{R}$ *be continuous and* $\zeta \not\equiv 0$. *If* $\mathrm{K}(\zeta \circ f)$ *is finite for all* $f \in \mathcal{L}_2(\mathbb{R}^n)$, *then* $\zeta \in A(\mathbb{R})$.

Note that a continuous and $\mathrm{SL}(n)$ covariant valuation maps the zero function to the zero matrix. Hence the following lemma concludes the proof the theorem.

Lemma 6. *Let* $\mathrm{Z}_1, \mathrm{Z}_2 : \mathcal{L}_2(\mathbb{R}^n) \to \langle \mathbb{M}^n, + \rangle$ *be continuous valuations that map the zero function to the zero matrix. If* $\mathrm{Z}_1(\alpha \, \mathbb{1}_T) = \mathrm{Z}_2(\alpha \, \mathbb{1}_T)$ *for all* $T \in \mathcal{T}^n$ *and* $\alpha \in \mathbb{R}$, *then*

$$\mathrm{Z}_1(f) = \mathrm{Z}_2(f) \tag{9}$$

for all $f \in \mathcal{L}_2(\mathbb{R}^n)$.

Proof. Since Z_1 and Z_2 are valuations and $Z_1(0) = Z_2(0) = 0$, we have for $k = 1, 2$,

$$Z_k(f \vee 0) + Z_k(f \wedge 0) = Z_k(f) + Z_k(0) = Z_k(f).$$

Thus it suffices to show that (9) holds for all $f \in \mathcal{L}_2(\mathbb{R}^n)$ with $f \geq 0$ and with $f \leq 0$.

Since simple functions of compact support, whose support does not contain the origin, are dense in $\mathcal{L}_2(\mathbb{R}^n)$, it is not difficult to see that also simple functions of the form $\sum_{i=1}^m \alpha_i \mathbb{1}_{T_i}$, where α_i are reals and $T_i \in \mathcal{T}^n$ have pairwise disjoint interiors, are dense in $\mathcal{L}_2(\mathbb{R}^n)$. Since Z_1 and Z_2 are continuous, it suffices to prove (9) for a simple functions f of the form $\sum_{i=1}^m \alpha_i \mathbb{1}_{T_i}$, where $\alpha_i \geq 0$ and $T_i \in \mathcal{T}^n$ have pairwise disjoint interiors. First, let $f \geq 0$. Since the coefficients α_i are non-negative, we have for $k = 1, 2$,

$$Z_k(f) = Z_k(\alpha_1 \mathbb{1}_{T_1} \vee \cdots \vee \alpha_m \mathbb{1}_{T_m}) = Z_k(\alpha_1 \mathbb{1}_{T_1}) + \cdots + Z_k(\alpha_m \mathbb{1}_{T_m}).$$

If $f \leq 0$, then the coefficients α_i are non-positive and for $k = 1, 2$,

$$Z_k(f) = Z_k(\alpha_1 \mathbb{1}_{T_1} \wedge \cdots \wedge \alpha_m \mathbb{1}_{T_m}) = Z_k(\alpha_1 \mathbb{1}_{T_1}) + \cdots + Z_k(\alpha_m \mathbb{1}_{T_m}).$$

In both case, we have

$$Z_1(f) = \sum_{i=1}^m Z_1(\alpha_i \mathbb{1}_{T_i}) = \sum_{i=1}^m Z_2(\alpha_i \mathbb{1}_{T_i}) = Z_2(f).$$

This concludes the proof of the lemma. □

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