MOMENT VECTORS OF POLYTOPES

MONIKA LUDWIG

Dedicated to my teacher Prof. Peter M. Gruber
on the occasion of his 60th birthday

Abstract. We give a classification of Borel measurable, $SL(d)$ covariant or contravariant, homogeneous, vector valued valuations on the space of d-dimensional convex polytopes containing the origin in their interiors. The only examples are moment vectors of polytopes and moment vectors of polar polytopes.
MSC 2000: 52B45, 52A38.

1 Introduction and Statement of Results

Let $\mathcal{K}^d$ denote the space of convex bodies in Euclidean d-dimensional space $\mathbb{E}^d$ and let $\mathcal{K}_o^d$ denote the space of convex bodies in $\mathbb{E}^d$ containing the origin o in their interiors. For convex polytopes denote the corresponding spaces by $\mathcal{P}^d$ and $\mathcal{P}_o^d$. Studying these spaces has always been a central subject of convex geometry (see Gruber's survey [3]). A special aspect here is the classification of additive functions on these spaces. Here a function z defined on a space $\mathcal{D}$ and taking values in an abelian semi-group is called additive or a valuation, if

$$z(K_1) + z(K_2) = z(K_1 \cup K_2) + z(K_1 \cap K_2)$$

whenever $K_1, K_2, K_1 \cup K_2, K_1 \cap K_2 \in \mathcal{D}$.

Hadwiger's classical characterization theorem states that every continuous, rigid motion invariant, real valued valuation on $\mathcal{K}^d$ is a linear combination of quermassintegrals (see [4] and also Klain's short proof [6]). This result has important applications in geometric probability (see the books of Hadwiger [4], Schneider and Weil [18], Klain and Rota [7]) and started the systematic study of valuations on these spaces (see the surveys [11], [9]).

Here we are interested in the classification of vector valued valuations. For this question, the fundamental notion is the moment vector

$$m(K) = \int_K x \, dx$$

of $K \in \mathcal{K}^n$, i.e., $m(K)$ is the centroid of K multiplied by the volume $V(K)$ of K. If we replace volume by the moment vector in the definiton of quermassintegrals, we obtain the definition of quermassvectors. Schneider [14] proved the following analogue of Hadwiger's characterization theorem: Every continuous, rotation covariant, vector valued valuation z on $\mathcal{K}^d$ with the property that $z(K + x) - z(K)$

is parallel to x for every $x \in \mathbb{E}^d$ is a linear combination of quermassvectors (see also [5]). Characterizations of further important vector valued valuations like the Steiner point [12] and the centroid [15] are due to Schneider. For detailed information, we refer to [13] and [16], Chapter 5.4.

If we consider convex bodies containing the origin in their interiors, we get additional examples of invariant valuations. To state our results, we fix some notation. Let P^* denote the polar body of P, $P \in \mathcal{P}_o^d$, i.e.,

$$P^* = \{y \in \mathbb{E}^d \mid x \cdot y \leq 1 \text{ for all } x \in P\},$$

where $x \cdot y$ denotes the standard inner product of x and y. Call $z : \mathcal{P}_o^d \to \mathbb{R}$ or $z : \mathcal{P}_o^d \to \mathbb{E}^d$ (Borel) measurable if the pre-image of every open set is a Borel set. Let $GL(d)$ denote the group of general linear transformations, i.e., of linear transformations ϕ with determinant $\det \phi \neq 0$, and let $SL(d)$ denote the group of special linear transformation, i.e., of linear transformations ϕ with $\det \phi = 1$.

Examples of $SL(d)$ invariant, real valued valuations on $\mathcal{P}_o^d$ are volume, volume of the polar body, and the constant. In [8], it was proved that if we consider homogeneous functionals these are already all examples.

Theorem 1 ([8]). *A functional $\mu : \mathcal{P}_o^d \to \mathbb{R}$, $d \geq 2$, is a measurable valuation with the property that*

$$\mu(\phi P) = |\det \phi|^q \, \mu(P)$$

for every $\phi \in GL(d)$ with $q \in \mathbb{R}$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$\mu(P) = c \quad or \quad \mu(P) = c\, V(P) \quad or \quad \mu(P) = c\, V(P^*)$$

for every $P \in \mathcal{P}_o^d$.

Examples of vector valued valuations on $\mathcal{P}_o^d$ are the moment vector $m(P)$ and the moment vector $m^*(P)$ of the polar body of P. It is easy to see that m and m^* are measurable valuations, and that the following transformation rules hold. For every $\phi \in GL(d)$ and $P \in \mathcal{P}_o^d$,

$$m(\phi P) = |\det \phi| \, \phi\, m(P) \quad \text{and } m^*(\phi P) = |\det \phi^{-t}| \, \phi^{-t} m^*(P),$$

where ϕ^{-t} is the inverse of the transpose of ϕ. For $d = 2$ there are additional examples. Let $\psi_{\pi/2}$ denote the rotation by an angle $\pi/2$, let $\tilde{m}(P) = \psi_{\pi/2}^{-1} m^*(P)$ and let $\tilde{m}^*(P) = \tilde{m}(P^*)$. Then

$$\tilde{m}(\phi P) = |\det \phi|^{-2} \, \phi\, \tilde{m}(P) \quad \text{and } \tilde{m}^*(\phi P) = |\det \phi^{-t}|^{-2} \, \phi^{-t} \tilde{m}^*(P),$$

We show that the valuations $m, m^*, \tilde{m}, \tilde{m}^*$ are the only examples of vector valued valuations with these transformation properties.

Theorem 2. *A function $z : \mathcal{P}_o^d \to \mathbb{E}^d$, $d \geq 2$, is a measurable valuation with the property that*

$$z(\phi P) = |\det \phi|^q \, \phi z(P) \tag{1}$$

for every $\phi \in GL(d)$ with $q \in \mathbb{R}$ if and only if $d \geq 3$ and there is a constant $c \in \mathbb{R}$ such that

$$z(P) = c\, m(P)$$

for every $P \in \mathcal{P}_o^d$ or $d = 2$ and there is a constant $c \in \mathbb{R}$ such that

$$z(P) = c\, m(P) \quad \text{or} \quad z(P) = c\, \psi_{\pi/2}^{-1} m^*(P)$$

for every $P \in \mathcal{P}_o^2$.

Theorem 3. *A function $z : \mathcal{P}_o^d \to \mathbb{E}^d$, $d \geq 2$, is a measurable valuation with the property that*

$$z(\phi P) = |\det \phi^{-t}|^q \, \phi^{-t} z(P) \tag{2}$$

for every $\phi \in GL(d)$ with $q \in \mathbb{R}$ if and only if $d \geq 3$ and there is a constant $c \in \mathbb{R}$ such that

$$z(P) = c\, m^*(P)$$

for every $P \in \mathcal{P}_o^d$ or $d = 2$ and there is a constant $c \in \mathbb{R}$ such that

$$z(P) = c\, m^*(P) \quad \text{or} \quad z(P) = c\, \psi_{\pi/2}^{-1} m(P)$$

for every $P \in \mathcal{P}_o^2$.

In recent years, also tensor valued valuations on the space of convex bodies have attracted much interest (see [10], [1], [2], [17]). In a subsequent paper, we will discuss this question for the space $\mathcal{P}_o^d$.

2 Proofs

Let $z : \mathcal{P}_o^d \to \mathbb{E}^d$ be a measurable valuation such that (1) holds for fixed $q \in \mathbb{R}$. The function z^*, defined by $z^*(P) = z(P^*)$ for $P \in \mathcal{P}_o^d$, is again measurable. For $P, Q, P \cup Q \in \mathcal{P}_o^d$, we have

$$(P \cup Q)^* = P^* \cap Q^* \quad \text{and} \quad (P \cap Q)^* = P^* \cup Q^*.$$

Therefore $z^*(P) + z^*(Q) = z^*(P \cap Q) + z^*(P \cup Q)$, i.e., z^* is a valuation on $\mathcal{P}_o^d$. For $\phi \in GL(d)$ and $P \in \mathcal{P}_o^d$, we have $(\phi\, P)^* = \phi^{-t}\, P^*$ and by (1)

$$z^*(\phi P) = z((\phi P)^*) = z(\phi^{-t} P^*) = |\det \phi^{-t}|^q \phi^{-t} z(P^*) = |\det \phi^{-t}|^q \phi^{-t} z^*(P).$$

Thus $z^* : \mathcal{P}_o^d \to \mathbb{E}^d$ is a measurable valuation with the property that $z^*(\phi P) = |\det \phi^{-t}|^q \phi^{-t} z^*(P)$ for every $\phi \in GL(d)$, and Theorems 2 and 3 are equivalent for fixed $q \in \mathbb{R}$. This enables us to prove both theorems by first proving Theorem 2 for $q > -1$ and then Theorem 3 for $q \leq -1$.

2.1 Proof of Theorem 2 for $q > -1$

1. We begin by proving Theorem 2 for $q > -1$ and $d = 2$. We fix an x_1-x_2-coordinate system and write $x = (x_1, x_2)^t$ for $x \in \mathbb{E}^2$. Denote by $\mathcal{Q}_o(x_1, x_2)$ the set of convex polygons $Q = [I_1, I_2]$ where I_1 and I_2 are closed intervals lying on the x_1-axis and x_2-axis, respectively, and containing the origin in their interiors. Here $[P_1, \dots, P_n]$ denotes the convex hull of $P_1, \dots, P_n$. Let I_1 be fixed and define $w : \mathcal{P}_o^1 \to \mathbb{E}^2$ by $w(I_2) = z([I_1, I_2])$. Then w is a valuation on $\mathcal{P}_o^1$. Since z is measurable, so is w. Let $\phi \in SL(2)$ be the linear transformation that multiplies the x_1-coordinate by a factor $1/r$ and the x_2-coordinate by a factor r. By (1), we have

$$z(\phi[I_1, I_2]) = z([r^{-1} I_1, r\, I_2]) = \phi z([I_1, I_2]). \tag{3}$$

Consequently,

$$\begin{aligned} z_1([r^{-1} I_1, r\, I_2]) &= r^{-2q-1} z_1([I_1, r^2 I_2]) &= r^{-1} z_1([I_1, I_2]), \\ z_2([r^{-1} I_1, r\, I_2]) &= r^{-2q-1} z_2([I_1, r^2 I_2]) &= r\, z_2([I_1, I_2]), \end{aligned}$$

and

$$w_1(r^2 I_2) = r^{2q} w_1(I_2) \quad \text{and} \quad w_2(r^2 I_2) = r^{2q+2} w_2(I_2),$$

i.e., w_1 is homogeneous of degree q and w_2 is homogeneous of degree $q + 1$.

We need the following result (cf. [8], equations (3) and (4)), which is a simple consequence of solving Cauchy's functional equation. Let $\nu : \mathcal{P}_o^1 \to \mathbb{R}$ be measurable valuation that is homogeneous of degree r. If $r = 0$, then there are constants $a, b \in \mathbb{R}$ such that

$$\nu([-s, t]) = a \log(\frac{t}{s}) + b \tag{4}$$

for every $s, t > 0$, and if $r \neq 0$, then there are constants $a, b \in \mathbb{R}$ such that

$$\nu([-s, t]) = a\, s^r + b\, t^r \tag{5}$$

for every $s, t > 0$.

1.1. We consider the case $q > -1$, $q \neq 0$. It follows from (5) that

$$w_1([-s, t]) = a_1\, s^q + b_1\, t^q \quad \text{and} \quad w_2([-s, t]) = a_2\, s^{q+1} + b_2\, t^{q+1},$$

and

$$z_1([I_1, I_2]) = a_1(I_1)\, s_2^q + b_1(I_1)\, t_2^q \quad \text{and} \quad z_2([I_1, I_2]) = a_2(I_1)\, s_2^{q+1} + b_2(I_1)\, t_2^{q+1},$$

where $I_2 = [-s_2, t_2]$. The functionals $a_1, b_1, a_2, b_2 : \mathcal{P}_o^1 \to \mathbb{R}$ are measurable valuations. By (3) we have

$$\begin{aligned} z_1([r^{-1} I_1, r\, I_2]) &= r^{2q+1} z_1([r^{-2} I_1, I_2]) &= r^{-1} z_1([I_1, I_2]), \\ z_2([r^{-1} I_1, r\, I_2]) &= r^{2q+1} z_2([r^{-2} I_1, I_2]) &= r\, z_2([I_1, I_2]). \end{aligned} \tag{6}$$

Therefore a_1, b_1 are homogeneous of degree $q+1$ and a_2, b_2 are homogeneous of degree q. By (5) there are constants a_i, b_i, c_i, d_i, $i=1,2$, such that

$$\begin{aligned} z_1([I_1,I_2]) &= (a_1\,s_1^{q+1}+b_1\,t_1^{q+1})\,s_2^q+(c_1\,s_1^{q+1}+d_1\,t_1^{q+1})\,t_2^q,\\ z_2([I_1,I_2]) &= (a_2\,s_1^{q}+b_2\,t_1^{q})\,s_2^{q+1}+(c_2\,s_1^{q}+d_2\,t_1^{q})\,t_2^{q+1} \end{aligned} \tag{7}$$

for every $s_1,t_1,s_2,t_2>0$. Let

$$\phi=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix} \quad\text{and}\quad \psi=\begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}.$$

By (1), we have

$$\begin{aligned} z_1(\phi[I_1,I_2]) &= z_1([-I_2,I_1]) &= -z_2([I_1,I_2]),\\ z_2(\phi[I_1,I_2]) &= z_2([-I_2,I_1]) &= z_1([I_1,I_2]), \end{aligned} \tag{8}$$

and

$$\begin{aligned} z_1(\psi[I_1,I_2]) &= z_1([-I_1,I_2]) &= -z_1([I_1,I_2]),\\ z_2(\psi[I_1,I_2]) &= z_2([-I_1,I_2]) &= z_2([I_1,I_2]). \end{aligned} \tag{9}$$

We use (7), compare coefficients in (8) and (9), and obtain

$$\begin{aligned} z_1([I_1,I_2]) &= a\,(s_1^{q+1}-t_1^{q+1})\,(s_2^q+t_2^q),\\ z_2([I_1,I_2]) &= a\,(s_1^{q}+t_1^{q})\,(s_2^{q+1}-t_2^{q+1}) \end{aligned} \tag{10}$$

for every $s_1,t_1,s_2,t_2>0$ with $a\in\mathbb{R}$.

Let $\mathcal{R}_o^2(x_1)$ be the set of convex polygons $[I_1,u,v]$ where I_1 is a closed interval on the x_1-axis containing the origin in its interior and u,v are points in the open lower and upper halfplane, respectively. Denote by $\mathcal{Q}_o^2$ the set of $SL(2)$-images of $Q\in\mathcal{Q}_o(x_1,x_2)$ and by $\mathcal{R}_o^2$ the set of $SL(2)$-images of $R\in\mathcal{R}_o^2(x_1)$. We need the following result.

Lemma 1. *Let $z:\mathcal{P}_o^2\to\mathbb{R}$ be a measurable valuation such that (1) and (10) hold. If $q>-1$ and $q\neq 0,1$, then $z(Q)=o$ for every $Q\in\mathcal{Q}_o^2$.*

Proof. Let $R=[I_1,s\,u,t\,v]$ where $I_1=[-s_1,t_1]$ lies on the x_1-axis, $u=\binom{x}{-1}$, $v=\binom{y}{1}$ with $x,y\in\mathbb{R}$, $s_1,t_1,s,t>0$. First we show that

$$\lim_{s,t\to 0} z_2([I_1,s\,u,t\,v]) \tag{11}$$

exists. Since z_2 is a valuation, we have for $0<t'<t$ and $t''>0$ suitably large

$$z_2([I_1,s\,u,t\,v])+z_2([I_1,-t''\,v,t'\,v])=z_2([I_1,s\,u,t'\,v])+z_2([I_1,-t''\,v,t\,v]).$$

Since $[I_1,-t''\,v,t'\,v],[I_1,-t''\,v,t\,v]\in\mathcal{Q}_o^2$, we obtain by (1) and (10)

$$z_2([I_1,s\,u,t\,v])-z_2([I_1,s\,u,t'\,v])=a\,(s_1^q+t_1^q)(t'^{q+1}-t^{q+1}).$$

Similarly, we have for $0 < s' < s$ and $s'' > 0$ suitably large

$$z_2([I_1, s\,u, t'\,v]) - z_2([I_1, s'\,u, t'\,v]) = a\,(s_1^q + t_1^q)(s^{q+1} - s'^{q+1}).$$

Since $q > -1$, this implies that the limit (11) exists. Note that

$$z_2([I_1, s\,u, t\,v]) = z_2([I_1, s'\,u, t'\,v]) + a\,(s_1^q + t_1^q)(s^{q+1} - s'^{q+1} - t^{q+1} + t'^{q+1}). \quad (12)$$

Next, we show that the limit in (11) is equal to 0. For I_1 fixed, set $f(x,y) = \lim_{s,t\to 0} z_2([I_1, s\,u, t\,v])$, where $u = \binom{x}{-1}$ and $v = \binom{y}{1}$. Since z_2 is a valuation, we have for $r > 0$ suitably small

$$z_2([I_1, s\,u, t\,v]) + z_2([I_1, -s\,r\,e, t\,r\,e]) = z_2([I_1, s\,u, t\,r\,e]) + z_2([I_1, -s\,r\,e, t\,v])$$

where $e = \binom{0}{1}$. This implies that

$$f(x,y) + f(0,0) = f(x,0) + f(0,y). \quad (13)$$

Note that $f(0,0) = 0$, since $[I_1, -s\,r\,e, t\,r\,e] \in \mathcal{Q}_o^2$ and since (10) holds. Set

$$\phi = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

By (1) we have $z_2([I_1, s\,u, t\,v]) = z_2([I_1, s\,\phi u, t\,\phi v]) = z_2([I_1, -s\,e, t\,w])$ where $w = \binom{x+y}{1}$. Therefore

$$f(x,y) = f(0, x+y). \quad (14)$$

Set $g(x) = f(0,x)$. Then it follows from (13) and (14) that

$$g(x+y) = g(x) + g(y).$$

This is one of Cauchy's functional equations. Since z_2 is measurable, so is g. Therefore there is a $w_2(I_1) \in \mathbb{R}$ such that

$$\lim_{s,t\to 0} z_2([I_1, s\,u, t\,v]) = g(x+y) = w_2(I_1)(x+y). \quad (15)$$

Using this we obtain the following. By (1) z_2 is homogeneous of degree $2q+1$. Therefore

$$w_2(r\,I_1) = r^{2q+1} w_2(I_1). \quad (16)$$

On the other hand, let $\psi \in GL(2)$ be the linear transformation that multiplies the x_1-coordinate by r and the x_2-coordinate by 1. Then $z_2(\psi R) = r^q\, z_2(R)$ and by (15), $w_2(r\,I_1) = r^{q-1}\, w_2(I_1)$. Since $q > -1$, this combined with (16) shows that $w_2(I_1) = 0$. Thus we obtain by (12) that

$$z_2([I_1, s\,u, t\,v]) = a\,(s_1^q + t_1^q)\,(s^{q+1} - t^{q+1}). \quad (17)$$

Let T_r^s be the triangle with vertices $\binom{1}{0}$, $\binom{0}{1}$, $\binom{-s}{-s\,r}$, $r, s > 0$. Then $T_r^s = [I_1, s\,r\,u, v]$ with $I_1 = [-s_1, 1]$, $s_1 = s/(1 + s\,r)$, $u = \binom{x}{-1}$, $x = -1/r$, $v = \binom{y}{1}$, $y = 0$. By (17) we have

$$z_2(T_r^s) = a\,((\frac{s}{1 + s\,r})^q + 1)((s\,r)^{q+1} - 1). \tag{18}$$

To determine $z_1(T_r^s)$, note that $T_r^s = \phi T_{1/r}^{s\,r}$ where

$$\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By (1) this implies that

$$z_1(T_r^s) = z_2(T_{1/r}^{s\,r}) = a\,(s^{q+1} - 1)((\frac{s\,r}{1 + s})^q + 1). \tag{19}$$

Let the triangle $T^s(x, y)$ be the convex hull of $\binom{y}{1-y}$, $\binom{x}{1-x}$, $\binom{-s}{-s}$. For $0 \le x < 1/2 < y \le 1$, we have $T^s(x, y) \in \mathcal{P}_o^2$, $T_r^s = T^s(0, 1) = T^s(0, y) \cup T^s(x, 1)$ with $r = 1$, and $T^s(x, y) = T^s(0, y) \cap T^s(x, 1)$. Since z is a valution, this implies that

$$z(T^s(0, y)) + z(T^s(x, 1)) = z(T^s(0, 1)) + z(T^s(x, y)). \tag{20}$$

Let

$$\phi = \begin{pmatrix} y & x \\ 1 - y & 1 - x \end{pmatrix}.$$

Then $T^s(x, y) = \phi T_r^{s\,t}$ with $r = (2y - 1)/(1 - 2x)$ and $t = (1 - 2x)/(y - x)$. Therefore we get by (1)

$$z(T^s(x, y)) = (y - x)^q\,\phi z(T_r^{s\,t}).$$

If $q > 0$, then by (19) and (18) $\lim_{s\to 0} z_1(T_r^{s\,t}) = \lim_{s\to 0} z_2(T_r^{s\,t}) = -a$. Therefore it follows from (20) that

$$a\left(y^{q+1} + (1 - x)^q(1 + x)\right) = a\,(1 + (y - x)^q(y + x))\,. \tag{21}$$

Taking the limit as $x, y \to 1/2$ in (21), we obtain $a\,2\,(1/2)^q = a$. This shows that $a = 0$ for $q \ne 1$. If $-1 < q < 0$, we set $y = 1 - x$ and $s = 1$. Then the right hand sight of (20) vanishes. We multiply (20) by $(1 - 2\,x)^{q+1}$ and take the limit as $x \to 1/2$, and obtain $a\,(1/2)^{q+1} = 0$. This shows that $a = 0$ for $-1 < q < 0$ and completes the proof of the lemma. □

For $q > -1$, $q \ne 0, 1$, we apply Lemma 1 and obtain $z(Q) = o$ for every $Q \in \mathcal{Q}_o^2$. Using Lemmas 3 and 4 (stated and proved below) shows that Theorem 2 holds for $q > -1, q \ne 0, 1$, and $d = 2$. If $q = 1$, then (10) implies that $z(Q) = -6\,a\,m(Q)$ for every $Q \in \mathcal{Q}_o^2$. Applying Lemmas 3 and 4 to $w(P) = z(P) + 6\,a\,m(P)$ shows that $w(P) = o$ for every $P \in \mathcal{P}_o^2$. This implies that Theorem 2 holds for $q = 1$ and $d = 2$.

1.2. We consider the case $q = 0$. Then we have by (4) and (5)

$$\begin{aligned} z_1([I_1, I_2]) &= a_1(I_1)\, \log(\frac{t_2}{s_2}) + b_2(I_1), \\ z_2([I_1, I_2]) &= a_2(I_1)\, s_2 + b_2(I_1)\, t_2, \end{aligned}$$

where $I_2 = [-s_2, t_2]$. The functionals $a_1, b_1, a_2, b_2 : \mathcal{P}_o^1 \to \mathbb{R}$ are measurable valuations. Equations (6) imply that a_1, b_1 are homogeneous of degree 1 and that a_2, b_2 are homogeneous of degree 0. Thus, by (4) and (5) there are constants a_i, b_i, c_i, d_i, $i = 1, 2$, such that

$$\begin{aligned} z_1([I_1, I_2]) &= (a_1\, s_1 + b_1\, t_1)\, \log(\frac{t_2}{s_2}) + (c_1\, s_1 + d_1\, t_1), \\ z_2([I_1, I_2]) &= (a_2\, \log(\frac{t_1}{s_1}) + b_2)\, s_2 + (c_2\, \log(\frac{t_1}{s_1}) + d_2)\, t_2. \end{aligned}$$

Comparing coefficients in (8) and (9) shows that

$$z_1([I_1, I_2]) = a(s_1 - t_1) \quad \text{and} \quad z_2([I_1, I_2]) = a(s_2 - t_2) \tag{22}$$

for every $s_1, t_1, s_2, t_2 > 0$ with $a \in \mathbb{R}$.

We need the following result.

Lemma 2. *Let $z : \mathcal{P}_o^2 \to \mathbb{R}$ be a measurable valuation such that (1) and (22) hold. Then $z(R) = o$ for every $R \in \mathcal{R}_o^2$.*

Proof. Let $R = [I_1, s\, u, t\, v]$ where $I_1 = [-s_1, t_1]$ lies on the x_1-axis, $u = \binom{x}{-1}$, $v = \binom{y}{1}$ with $x, y \in \mathbb{R}$, $s_1, t_1, s, t > 0$. First, note that for $q = 0$ (22) implies that (10) holds. Therefore we can proceed as in Lemma 1 and obtain by (17)

$$z_2([I_1, s\, u, t\, v]) = a\,(s - t). \tag{23}$$

Next we show that

$$\lim_{s,t \to 0} z_1([I_1, s\, u, t\, v]) \tag{24}$$

exists. Since z_1 is a valuation, we have for $0 < t' < t$ and $t'' > 0$ suitably large

$$z_1([I_1, s\, u, t\, v]) + z_1([I_1, -t''\, v, t'\, v]) = z_1([I_1, s\, u, t'\, v]) + z_1([I_1, -t''\, v, t\, v]).$$

Since $[I_1, -t''\, v, t'\, v], [I_1, -t''\, v, t\, v] \in \mathcal{Q}_o^2$, we obtain by (1) and (22)

$$z_1([I_1, s\, u, t\, v]) - z_1([I_1, s\, u, t'\, v]) = y\, a\,(t' - t).$$

In a similar way, we see that

$$z_1([I_1, s\, u, t'\, v]) - z_1([I_1, s'\, u, t'\, v]) = -x\, a\,(s - s').$$

This implies that the limit (24) exists. Note that

$$z_1([I_1, s\,u, t\,v]) = z_1([I_1, s'\,u, t'\,v]) + y\,a\,(t' - t) - x\,a\,(s - s'). \tag{25}$$

For I_1 fixed, set $f(x, y) = \lim_{s,t\to 0} z_1([I_1, s\,u, t\,v])$, where $u = \binom{x}{-1}$ and $v = \binom{y}{1}$. Since z_1 is a valuation, we have for $r > 0$ suitably small

$$z_1([I_1, s\,u, t\,v]) + z_1([I_1, -s\,r\,e, t\,r\,e]) = z_1([I_1, s\,u, t\,r\,e]) + z_1([I_1, -s\,r\,e, t\,v])$$

where $e = \binom{0}{1}$. This implies that

$$f(x, y) + f(0, 0) = f(x, 0) + f(0, y). \tag{26}$$

Note that $f(0, 0) = a\,(s_1 - t_1)$, since $[I_1, -s\,r\,e, t\,r\,e] \in \mathcal{Q}^2_o$. Set

$$\phi = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Then we have

$$z_1(\phi[I_1, s\,u, t\,v]) = z_1([\phi I_1, s\,\phi u, t\,\phi v]) = z_1([I_1, -s\,e, t\,w])$$

and by (1) and (23)

$$z_1(\phi[I_1, s\,u, t\,v]) = z_1([I_1, s\,u, t\,v]) + x\,z_2([I_1, s\,u, t\,v])$$

where $w = \binom{x+y}{1}$. Consequently

$$f(x, y) = f(0, x + y). \tag{27}$$

Set $g(x) = f(0, x) - f(0, 0)$. Then it follows from (26) and (27) that

$$g(x + y) = g(x) + g(y).$$

This is one of Cauchy's functional equations. Since z_1 is measurable, so is g. Therefore there is a $w_1(I_1) \in \mathbb{R}$ such that $g(x) = w_1(I_1)\,x$ and

$$\lim_{s,t\to 0} z_1([I_1, s\,u, t\,v]) = g(x + y) + f(0, 0) = w_1(I_1)(x + y) + a\,(s_1 - t_1). \tag{28}$$

Using this we obtain the following. By (1), z_1 is homogeneous of degree 1. Therefore

$$w_1(r\,I_1) = r\,w_1(I_1). \tag{29}$$

On the other hand, let $\phi \in GL(2)$ be the linear transformation that multiplies the x_1-coordinate by r and the x_2-coordinate by 1. Then $z_1(\phi R) = r\,z_1(R)$ and by (28), $w_1(r\,I_1) = w_1(I_1)$. Combined with (29) this shows that $w_1(I_1) = 0$. Therefore we obtain by (28) and (25) that

$$z_1([I_1, s\,u, t\,v]) = a\,(s_1 - t_1) - y\,a\,t - x\,a\,s. \tag{30}$$

Let T_r^s be the triangle with vertices $\binom{1}{0}$, $\binom{0}{1}$, $\binom{-s}{-s\,r}$, $r, s > 0$. Then $T_r^s = [I_1, s\,r\,u, v]$ with $I_1 = [-s_1, 1]$, $s_1 = s/(1 + s\,r)$, $u = \binom{x}{-1}$, $x = -1/r$, $v = \binom{y}{1}$, $y = 0$. By (30) and (23) we have

$$z_1(T_r^s) = a\,(\frac{s}{1+s\,r} - 1) + a\,s \quad \text{and} \quad z_2(T_r^s) = a\,(s\,r - 1). \tag{31}$$

We can determine $z_1(T_r^s)$ also in the following way. Since $T_r^s = \phi T_{1/r}^{s\,r}$ with

$$\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we have by (1), $z_1(T_r^s) = z_2(T_{1/r}^{s\,r}) = a\,(s-1)$. Combined with (31) this shows that $a = 0$. Because of (23) and (30) this completes the proof of the lemma. $\square$

Since $z(R) = o$ for every $R \in \mathcal{R}_o^2$, using Lemma 4 we obtain that Theorem 2 holds for $q = 0$ and $d = 2$.

2. Now let $d \geq 3$. We use induction on the dimension d. Suppose that Theorem 2 is true for $q > -1$ in dimension $(d-1)$.

We fix an x_1-...-x_d-coordinate system, identify the x_1-...-x_{d-1}-coordinate hyperplane with $\mathbb{E}^{d-1}$, and write $x = (x_1, \dots, x_d)^t = (x', x_d)^t$ with $x' \in \mathbb{E}^{d-1}$ for $x \in \mathbb{E}^d$. Let $\mathcal{Q}_o(x_d)$ be the set of convex polytopes $Q = [P', I]$ where $P' \in \mathcal{P}_o^{d-1}$ and I is a closed interval lying on the x_d-axis and containing the origin in its interior. For I fixed, define $z' : \mathcal{P}_o^{d-1} \to \mathbb{E}^{d-1}$ and $\mu : \mathcal{P}_o^{d-1} \to \mathbb{R}$ by

$$z'(P') = \begin{pmatrix} z_1([P', I]) \\ \vdots \\ z_{d-1}([P', I]) \end{pmatrix} \tag{32}$$

and

$$\mu(P') = z_d([P', I]). \tag{33}$$

Then z' and μ are measurable valuations on $\mathcal{P}_o^{d-1}$. For every $\phi' \in GL(d-1)$ we have

$$z'(\phi' P') = |\det \phi'|^q \phi' z'(P') \quad \text{and} \quad \mu(\phi' P') = |\det \phi'|^q \mu(P'). \tag{34}$$

This can be seen in the following way. Let $\phi \in GL(d)$ with coefficients ϕ_{ij} be such that $\phi_{ij} = \phi'_{ij}$ for $i, j = 1, \dots, d-1$, $\phi_{dj} = \phi_{id} = 0$ for $i, j = 1, \dots, d-1$, and $\phi_{dd} = 1$. Then $\det \phi = \det \phi'$ and (1) shows that (34) holds.

Let $q \neq 1$, $q > -1$. We apply Theorem 2 for $q > -1$ in dimension $(d-1)$ and obtain that $z'(P') = o'$. If $q \neq 0$, then Theorem 1 implies that $\mu(P') = 0$. If $q = 0$, then we obtain that $\mu(P') = c$ and $z_d([P', I]) = c(I)$. We take $Q = [I_1, \dots, I_d]$, where I_j is an interval on the x_j-axis containing the origin in its interior, and $\phi \in SL(d)$ that interchanges the first and last coordinates, and obtain from (1) that we have $c(I) = 0$. Thus for $q \neq 1$, $q > -1$,

$$z(Q) = o \tag{35}$$

for $Q \in \mathcal{Q}_o(x_d)$.

Now let $q = 1$. Then Theorem 2 in dimension $(d-1)$ and Theorem 1 imply that

$$z'(P') = a\, m'(P') \quad \text{and} \quad \mu(P') = b\, V_{d-1}(P') \tag{36}$$

where m' is the moment vector in $\mathbb{E}^{d-1}$ and V_{d-1} is volume in $\mathbb{E}^{d-1}$. Thus we have

$$z([P', I]) = \begin{pmatrix} a(I)\, m'(P') \\ b(I)\, V_{d-1}(P') \end{pmatrix}$$

where $a, b : \mathcal{P}_o^1 \to \mathbb{R}$ are measurable valuations. Let $\phi \in SL(d)$ be the transformation that multiplies the first $(d-1)$ coordinates by r and the last coordinate by $r^{-(d-1)}$. By (1) we have

$$z(\phi[P', I]) = z([r\, P', r^{-(d-1)} I]) = \phi z([P', I])$$

and

$$z(\phi[P', I]) = z(r\, [P', r^{-d} I]) = r^{d+1} z([P', r^{-d} I]).$$

Therefore a is homogeneous of degree 1 and by (5) there are constants $a_1, a_2 \in \mathbb{R}$ such that

$$a([-s, t]) = a_1\, s + a_2\, t.$$

The functional b is homogeneous of degree 2 and by (5) there are constants $b_1, b_2 \in \mathbb{R}$ such that

$$b([-s, t]) = b_1\, s^2 + b_2\, t^2.$$

Now let ϕ be the orthogonal reflection on the hyperplane $\mathbb{E}^{d-1}$. Then

$$z(\phi[P', I]) = z([P', -I]) = \phi z([P', I]).$$

Consequently, $a_1 = a_2$ and $b_1 = -b_2$. To determine a_1 and b_1, let $P = [I_1, \dots, I_d]$ where I_j is an interval on the x_j-axis containing the origin in its interior, $I_1 = I_d$, and let ϕ be a linear transformation that interchanges the first and last coordinates. Then $\phi P = P$ and by (1) $z_d(\phi P) = z_1(P)$. By calculating $m(Q)$, we obtain that

$$z(Q) = a\, m(Q) \tag{37}$$

for $Q \in \mathcal{Q}_o(x^d)$ with $a \in \mathbb{R}$.

Let $\mathcal{R}_o^d(x_d)$ be the set of convex polytopes $[P', u, v]$ where $P' \in \mathcal{P}_o^{d-1}$ and u, v are points in the halfspace $x_d < 0$ and $x_d > 0$, respectively. Denote by $\mathcal{Q}_o^d$ the set of $SL(d)$-images of $Q \in \mathcal{Q}_o(x_d)$ and by $\mathcal{R}_o^d$ the set of $SL(d)$-images of $R \in \mathcal{R}_o^d(x_d)$. We need the following results.

Lemma 3. *Let $z : \mathcal{P}_o^d \to \mathbb{E}^d$ be a measurable valuation such that (1) holds. If z vanishes on $\mathcal{Q}_o^d$ and $q > -1$, then $z = o$ for every $R \in \mathcal{R}_o^d$.*

Proof. Let $R = [P', s\,u, t\,v]$ where $P' \in \mathcal{P}_o^{d-1}$, $u = \binom{u'}{-1}$ and $v = \binom{v'}{1}$ with $u', v' \in \mathbb{E}^{d-1}$ and $s, t > 0$. Since z is a valuation, we have for $0 < t < t'$ and $t'' > 0$ suitably small

$$z([P', s\,u, t\,v]) + z([P', -t''\,v, t'\,v]) = z([P', s\,u, t'\,v]) + z([P', -t''\,v, t\,v]).$$

Since $[P', -t''\,v, t'\,v], [P', -t''\,v, t\,v] \in \mathcal{Q}_o^d$ and since z vanishes on $\mathcal{Q}_o^d$, this implies that $z([P', s\,u, t\,v])$ does not depend on $t > 0$. A similar argument shows that it does not depend on $s > 0$. Thus

$$z([P', s\,u, t\,v]) = z([P', u, v]) \tag{38}$$

for $s, t > 0$.

For P' fixed, set $f(u', v') = z([P', u, v])$. Since z is a valuation, we have for $r > 0$ suitably small

$$z([P', u, v]) + z([P', -r\,e, r\,e]) = z([P', u, r\,e]) + z([P', -r\,e, v])$$

where $e = \binom{o'}{1}$. By (38) this implies that

$$f(u', v') + f(o', o') = f(u', o') + f(o', v'). \tag{39}$$

Note that since $[P', -r\,e, r\,e] \in \mathcal{Q}_o^d$, we have $f(o', o') = 0$. Let

$$\phi = \begin{pmatrix} 1 & \dots & 0 & u_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & u_{d-1} \\ 0 & \dots & 0 & 1 \end{pmatrix}. \tag{40}$$

Then $\phi\binom{u'}{-1} = \binom{o'}{-1}$ and $\phi\binom{v'}{1} = \binom{u'+v'}{1} = w$. Since (1) holds, this implies that

$$z_d([P', u, v]) = z_d([\phi P', \phi u, \phi v]) = z_d([P', -e, w])$$

and

$$f_d(u', v') = f_d(o', u' + v'). \tag{41}$$

Note that

$$z_i([P', -e, w]) = z_i([P, u, v]) + u_i\, z_d([P', u, v]) \tag{42}$$

for $i = 1, \dots, d-1$. Set $g_d(u') = f_d(o', u')$. Then we get by (39) and (41) that

$$g_d(u' + v') = g_d(u') + g_d(v').$$

This is one of Cauchy's functional equations. Since z is measurable, there is a $w'(P') \in \mathbb{E}^{d-1}$ such that

$$z_d(R) = z_d([P', u, v]) = w'(P') \cdot (u' + v') \tag{43}$$

for every $u', v' \in \mathbb{E}^{d-1}$.

Using this we obtain the following. By (1), z_d is homogeneous of degree $d\,q+1$. Since we know by (38) that $z([r\,P', r\,u, r\,v]) = z([t\,P', u, v])$ for $r > 0$, this and (43) imply that

$$w'(r\,P') = r^{d\,q+1} w'(P'). \tag{44}$$

On the other hand, let $\psi \in GL(d)$ be the map that multiplies the first $(d-1)$ coordinates with r and the last coordinate with 1. Then $z_d(\psi R) = r^{(d-1)q}\, z_d(R)$ and by (43) this implies that

$$w'(r\,P') = r^{(d-1)q-1}\, w'(P').$$

Since $q > -1$, this combined with (44) shows that $w'(P') = o'$. Thus by (43), $z_d(R) = 0$.

Using this and (42) we obtain by the same arguments as for $i = d$ that there are $w'_{(i)}(P') \in \mathbb{E}^{d-1}$ such that

$$z_i(R) = z_i([P', u, v]) = w'_{(i)}(P') \cdot (u' + v')$$

for $i = 1, \dots, d-1$. As in (44) we have

$$w'_{(i)}(r\,P') = r^{d\,q+1} w'_{(i)}(P')$$

and using ψ shows that

$$w'_{(i)}(r\,P') = r^{(d-1)q}\, w'(i)(P').$$

Since $q > -1$, this shows that $w'_{(i)}(P') = o'$ and $z_i(R) = 0$ for $i = 1, \dots, d-1$. This completes the proof of the lemma. □

Lemma 4 ([8]). *Let $\mu : \mathcal{P}_o^d \to \mathbb{R}$ be a valuation. If μ vanishes on $\mathcal{R}_o^d$, then $\mu(P) = 0$ for every $P \in \mathcal{P}_o^d$.*

If $q \neq 1$, (35) holds. Therefore by Lemmas 3 and 4 we obtain $z(P) = o$ for every $P \in \mathcal{P}_o^d$. This proves Theorem 2 in this case. If $q = 1$, (37) holds. We apply Lemmas 3 and 4 to $w(P) = z(P) - a\,m(P)$ and obtain that $w(P) = o$ for every $P \in \mathcal{P}_o^d$. Thus $z(P) = a\,m(P)$ for every $P \in \mathcal{P}_o^d$. This completes the proof of Theorem 2 for $q > -1$.

2.2 Proof of Theorem 3 for $q \leq -1$

1. We begin by proving Theorem 3 for $q \leq -1$ and $d = 2$. Define

$$w(P) = \psi_{\pi/2}\, z(P),$$

where

$$\psi_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $w : \mathcal{P}_o^2 \to \mathbb{R}$ is a measurable valuation. Since z transforms according to (2), we have

$$w(\phi P) = |\det \phi|^{-q} \psi_{\pi/2}\, \phi^{-t} \psi_{\pi/2}^{-1}\, w(P) = |\det \phi|^{-q-1} \phi\, w(P)$$

for every $\phi \in GL(2)$. Thus w transforms according to (1) with $p = -q-1$. Applying Theorem 1 for $p \geq 0$ and $d = 2$ gives the following. For $q \neq -2$, we have $w(P) = o$ and

$$z(P) = o$$

for every $P \in \mathcal{P}_o^2$. For $q = -2$, there is a constant $c \in \mathbb{R}$ such that $w(P) = c\, m(P)$ and

$$z(P) = c\, \psi_{\pi/2}^{-1}\, m(P)$$

for every $P \in \mathcal{P}_o^2$. This proves Thereom 3 for $q \leq -1$ and $d = 2$.

2. Now let $d \geq 3$. We use induction on the dimension d. Suppose that Theorem 3 is true for $q \leq -1$ in dimension $(d-1)$.

For I fixed, define $z' : \mathcal{P}_o^{d-1} \to \mathbb{E}^{d-1}$ and $\mu : \mathcal{P}_o^{d-1} \to \mathbb{R}$ by (32) and (33). Then z' and μ are measurable valuations on $\mathcal{P}_o^{d-1}$. As in the proof of Theorem 2 we have

$$z'(\phi' P') = |\det \phi'^{-t}|^q \phi'^{-t} z'(P') \quad \text{and} \quad \mu(\phi' P') = |\det \phi'^{-t}|^q \mu(P') \tag{45}$$

for every $\phi' \in GL(d-1)$.

Let $q \leq -1$, $q \neq -2$. Theorem 3 for $q \leq -1$ in dimension $(d-1)$ implies that $z'(P') = o'$. If $q < -1$, then Theorem 1 implies that $\mu(P') = 0$. If $q = -1$, then we obtain that $\mu(P') = c\, V_{d-1}(P'^*)$ and $z_d([P', I]) = c(I)\, V_{d-1}(P'^*)$. We take $Q = [I_1, \dots, I_d]$, where I_j is an interval on the x_j-axis containing the origin in its interior, and $\phi \in SL(d)$ that interchanges the first and last coordinates, and obtain from (2) that we have $c(I) = 0$. Thus we get for $q \leq -1$, $q \neq -2$

$$z(Q) = o \tag{46}$$

for $Q \in \mathcal{Q}_o(x_d)$.

Let $q = -2$. If $d = 3$, then $z'(P') = c\, \psi_{\pi/2}^{-1}\, m(P')$ and $\mu(P') = 0$. Let Q and ϕ be defined as before. Then (2) shows that $c = 0$. Therefore (46) holds. The same argument as for $q \neq -2$ now implies that (46) holds for $d \geq 3$.

We need the following result.

Lemma 5. *Let $z : \mathcal{P}_o^d \to \mathbb{E}^d$ be a measurable valuation such that (2) holds. If z vanishes on $\mathcal{Q}_o^d$ and $q \leq -1$, then $z(R) = o$ for every $R \in \mathcal{R}_o^d$.*

Proof. Let $R = [P', s\,u, t\,v]$ where $P' \in \mathcal{P}_o^{d-1}$, $u = \binom{u'}{-1}$ and $v = \binom{v'}{1}$ with $u', v' \in \mathbb{E}^{d-1}$ and $s, t > 0$. We use notation and results from Lemma 3. We have by (38) that

$$z([P', s\,u, t\,v]) = z([P', u, v]) \tag{47}$$

for $s,t>0$, and by (39)

$$f(u',v')=f(u',o')+f(o',v'). \tag{48}$$

where P' is fixed and $f(u',v')=z([P',u,v])$. Let ϕ be as in (40). Then $\phi\binom{u'}{-1}=\binom{o'}{-1}$ and $\phi\binom{v'}{1}=\binom{u'+v'}{1}=w$, and by (2),

$$z_i([P',u,v])=z_i([\phi P',\phi u,\phi v])=z_i([P',-e,w])$$

and

$$f_i(u',v')=f_i(o',u'+v') \tag{49}$$

for $i=1,\dots,d-1$. Note that

$$z_d([P',-e,w])=-u_1\,z_1([P',u,v])-\ldots-u_{d-1}\,z_{d-1}([P',u,v])+z_d([P',u,v]). \tag{50}$$

Set $g_i(u')=f_i(o',u')$. Then we get by (48) and (49) that

$$g_i(u'+v')=g_i(u')+g_i(v').$$

These are equations of Cauchy's type. Since z is measurable, there are $w'_{(i)}(P')\in\mathbb{E}^{d-1}$ such that

$$z_i(R)=z_i([P',u,v])=w'_{(i)}(P')\cdot(u'+v') \tag{51}$$

for every $u',v'\in\mathbb{E}^{d-1}$ and $i=1,\dots,d-1$.

Using this we obtain for every i, $1\le i\le d-1$, the following. By (2), z_i is homogeneous of degree $-(d\,q+1)$. Since we know by (47) that $z([r\,P',r\,u,r\,v])=z([t\,P',u,v])$ for $r>0$, this and (51) imply that

$$w'_{(i)}(r\,P')=r^{-(d\,q+1)}w'(P'). \tag{52}$$

On the other hand, let $\psi\in GL(d)$ be the map that multiplies the first $(d-1)$ coordinates by r and the last coordinate by 1. Then $z_i(\psi R)=r^{-((d-1)\,q+1)}\,z_i(R)$, and by (51)

$$w'_{(i)}(r\,P')=r^{-((d-1)\,q+2)}\,w'_{(i)}(P').$$

Since $q\le -1$, this combined with (52) shows that $w'_{(i)}(P')=o'$. Thus by (51), $z_i(R)=0$.

Using this and (50), we get $z_d([P',u,v])=z_d([P',-e,w])$. The same argument as for $1\le i\le d-1$ shows that there is a $w'(P')\in\mathbb{E}^{d-1}$ such that

$$z_d(R)=z_d([P',u,v])=w'(P')\cdot(u'+v') \tag{53}$$

for every $u',v'\in\mathbb{E}^{d-1}$. Note that (52) hold for $i=d$. Let ψ be defined as before. Then $z_d(\psi R)=r^{-(d-1)\,q}\,z_i(R)$, and by (53)

$$w'_{(i)}(r\,P')=r^{-((d-1)\,q+1)}\,w'_{(i)}(P').$$

Since $q\le -1$, this combined with (52) shows that $z_d(R)=0$. This completes the proof of the lemma. □

We apply Lemmas 5 and 4 and obtain that $z(P)=o$ for every $P\in\mathcal{P}_o^d$. This proves Theorem 3 for $q\le -1$.

References

[1] S. Alesker, *Continuous valuations on convex sets*, Geom. Funct. Anal. **8** (1998), 402–409.

[2] S. Alesker, *Description of continuous isometry covariant valuations on convex sets*, Geom. Dedicata **74** (1999), 241–248.

[3] P.M. Gruber, *The space of convex bodies*, Handbook of Convex Geometry, Vol. A (P.M. Gruber and J. Wills, eds.), North-Holland, Amsterdam, 1993, 301–318.

[4] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, Berlin, 1957.

[5] H. Hadwiger and R. Schneider, *Vektorielle Integralgeometrie*, Elem. Math. **26** (1971), 49–57.

[6] D. Klain, *A short proof of Hadwiger's characterization theorem*, Mathematika **42** (1995), 329–339.

[7] D. Klain and G. Rota, *Introduction to geometric probability*, Cambridge University Press, Cambridge, 1997.

[8] M. Ludwig, *Valuations on polytopes containing the origin in their interiors*, Adv. Math., to appear.

[9] P. McMullen, *Valuations and dissections*, Handbook of Convex Geometry, Vol. B (P.M. Gruber and J. Wills, eds.), North-Holland, Amsterdam, 1993, 933–990.

[10] P. McMullen, *Isometry covariant valuations on convex bodies*, Rend. Circ. Mat. Palermo (2) Suppl. (1997), no. 50, 259–271.

[11] P. McMullen and R. Schneider, *Valuations on convex bodies*, Convexity and its applications (P.M. Gruber and J. Wills, eds.), Birkhäuser, 1983, 170–247.

[12] R. Schneider, *On Steiner points of convex bodies*, Israel J. Math. **9** (1971), 241–249.

[13] R. Schneider, *Krümmungsschwerpunkte konvexer Körper. I*, Abh. Math. Sem. Univ. Hamburg **37** (1972), 112–132.

[14] R. Schneider, *Krümmungsschwerpunkte konvexer Körper. II*, Abh. Math. Sem. Univ. Hamburg **37** (1972), 204–217.

[15] R. Schneider, *Volumen und Schwerpunkt von Polyedern*, Elem. Math. **28** (1973), 137–141.

[16] R. Schneider, *Convex bodies: the Brunn–Minkowski theory*, Cambridge Univ. Press, Cambridge, 1993.

[17] R. Schneider, *Tensor valuations on convex bodies and integral geometry*, Rend. Circ. Mat. Palermo (2) Suppl. (2000), no. 65, part I, 295–316.

[18] R. Schneider and W. Weil, *Integralgeometrie*, B. G. Teubner, Stuttgart, 1992.

Current address:
Mathematisches Institut
Universität Bern
Sidlerstrasse 5
3012 Bern, Switzerland
monika.ludwig@math-stat.unibe.ch

Permanent address:
Abteilung für Analysis
Technische Universität Wien
Wiedner Hauptstraße 8-10/1142
1040 Wien, Austria
monika.ludwig@tuwien.ac.at